Teachers' Understanding and Use of Mathematical Structure

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In this paper, we examine junior secondary mathematics teachers' understanding of mathematical structure, and how they promote structural thinking in their teaching. Five teachers were surveyed, and three were interviewed and the observed teaching a junior secondary mathematics class. Results showed that teachers have conflicting understandings of structure, and their perceived understandings, obtained from survey data, were not reinforced by their interview responses or observations of their teaching. Analysis of connections, recognising patterns, identifying similarities and differences, and generalising (CRIG) components from observation data showed a lack of attention to structural thinking.

Recent results on the Trends in Mathematics and Science Study (TIMSS; Mullis, Martin, Goh, & Cotter, 2016), and OECD Programme for International Student Assessment (PISA; Thomson, De Bortoli, & Underwood, 2016) reveal Australia's decline on international mathematics tests rankings, which has caused concern about the state of mathematics teaching and learning. Lokan, McRae, and Hollingsworth (2003) and Vincent and Stacey (2008) asserted that Australian mathematics teaching was dominated by procedural pedagogical practices, reinforcing that Australian mathematics teaching concentrated on the use of textbook exercises and worksheets, which resulted in instrumental learning (Skemp, 1976) instead of conceptual understanding (Hiebert, 1986).

Recently, there has been an increasing research interest in mathematical structure and structural thinking. Mason, Stephens, and Watson (2009) defined mathematical structure, referred to here as structure, as "the identification of general properties which are instantiated in particular situations as relationships between elements" (p. 10). Mason et al. (2009) argued that students involved in structural thinking receive an intrinsic reward, and that teachers' awareness of structural relationships transforms students' thinking and disposition to engage. They claimed that structure is essential to mathematics teaching and learning as it relates procedures and concepts to promote structural thinking.

Recent studies have focused on describing mathematical structure across early childhood and secondary students (Mulligan & Mitchelmore, 2009; Stephens, 2008) and out-of-field mathematics teachers (Vale, McAndrew, & Krishnan, 2011), supporting a need for further research in mathematics teachers' understanding of structure.

Research Questions

Three main questions were addressed in this study:

1. How do mathematics teachers demonstrate an awareness of mathematical structure?
2. How do mathematics teachers promote structural thinking when teaching mathematics?
3. Is there a discrepancy between what teachers say and do concerning mathematical structure?

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Literature Review and Theoretical Framework

The notion of structure is found in the Australian Curriculum: Mathematics (ACM; Australian Curriculum, Assessment and Reporting Authority, 2015) through the proficiency strands of understanding, fluency, problem solving and reasoning. In the NSW K-10 Mathematics Syllabus (NSW Board of Studies, 2012), produced in response to the ACM, structure can be identified in the working mathematically processes of communicating, problem solving, reasoning, understanding, and fluency.

The concept of mathematical structure has a long history in mathematics education research, but it is not a term very familiar to teachers. Taylor and Wade (1965) acknowledged that "structure" occurred frequently in mathematics education literature. Fischbein and Muzicant (2002), Stephens (2008), and Mason et al. (2009) all identified it as synonymous with relational thinking (Skemp, 1976). Barnard (1996) described it in terms of cognitive units or blocks of information, while Mulligan and Mitchelmore (2009) associated it with young children's ability to recognise patterns and relationships.

Effective mathematics teaching must include attention to structure. Pedagogical Content Knowledge (PCK; Shulman, 1987) and Mathematical Content Knowledge (MCK; Ball, Thames, & Phelps, 2008) both demonstrate the importance of structure in pedagogy and content knowledge in mathematics teaching. Structure connects mathematical procedures and concepts that are integral to PCK and MCK. Teachers who embed structure in their lessons develop students' mathematical understandings more coherently and with depth. Vale et al. (2011) educated practising teachers to appreciate structure. In a study of secondary mathematics teachers, Cavanagh (2006) found that they did not identify with working mathematically, but did use components of working mathematically to develop students' structural thinking.

The present study draws on the theoretical framework developed by John Mason and colleagues. Mason (2003) was "interested in the lived experience of mathematical thinking" (p. 17). His personal awareness in thinking mathematically made him attentive towards the form and structure of learners' mathematical thinking. Mason et al. (2009) later noted that structural thinking occurs on a continuum, and that it is difficult to identify student mathematical thinking as it exists between a single idea, or an idea related to a set of properties. Mason et al. (2009) identified five forms of structure that allow structural thinking to be observed: holding wholes (gazing), recognising relationships, discerning details, perceiving properties, and reasoning.

From these forms, the first author developed four observable components of structure to identify teachers' understanding of structure from their utterances when teaching. The first form of holding wholes (gazing) is interpreted as *connections* to other mathematics learning because gazing makes connections to the whole. The second form of *recognising* deals with mathematical relationships in patterns. The third, discerning details, is acknowledged as *identifying* similarities and differences. The final forms, perceiving properties and reasoning, were combined to form the single component of *generalising,* which relates mathematical ideas to a whole. Overall, the generalising component was considered to have greater relevance to this research.

These components of *connections* (C), *recognising patterns* (R), *identifying similarities and differences* (I), and *generalising* (G) will be referred to by the acronym CRIG. The connections component considers how we think about mathematics and making connections between present, prior, and future learning. Connections require recalling mathematical knowledge and adapting it to new knowledge. The NSW K-10 Mathematics Syllabus recognises that "students develop understanding and fluency in mathematics through inquiry and exploring and connecting mathematical concepts" (NSW Board of Studies, 2012). This syllabus has as one of its outcomes for working mathematically that a student "communicates and connects mathematical ideas" (NSW Board of Studies, 2012, outcome MA4-1WM).

Recognising patterns is identified extensively throughout mathematics education as critical to mathematics learning and the development of relationships. In the NSW mathematics syllabus (NSW Board of Studies, 2012), patterns are associated with the Number and Algebra strand as: "Develop efficient strategies for numerical calculation, recognise patterns, describe relationships". In Stage 4, a Number and Algebra content strand outcome is: "Create and displays number patterns" (NSW Board of Studies, 2012, outcome MA4-11NA). It also states: "Students develop efficient strategies for numerical calculation, recognise patterns, and describe relationships" (NSW Board of Studies, 2012, p. 18).

Learning mathematics includes developing skills in identifying similarities and differences; differences could be equal or unequal, bigger or smaller. In the NSW mathematics syllabus (NSW Board of Studies, 2012), Stage 4, Number and Algebra outcome MA4-4NA states, "compares, orders and calculates with integers".

Generalising was identified by Mason et al. (2009), who wrote that an appreciation of structure is supported by experiencing generality. The NSW mathematics syllabus K–10 (NSW Board of Studies, 2012) Number and Algebra strand includes experiencing generalisation. Concept formation is a process that involves generalising, and the movement of the concrete towards abstraction is associated with structure.

Analysing teachers' awareness of structural relationships is difficult, but recognising structure can be considered through teachers' talk about structure and where it occurs in their utterances when teaching mathematics. The CRIG components form a basis for identifying explicit characteristics, in analysing teachers' awareness of structure and the promotion of structural thinking in their teaching.

Methodology

Context and Participants

The study took place in a comprehensive secondary Catholic boys' school of approximately 300 students in metropolitan Sydney. The principal gave permission to conduct the research, and the head teacher of mathematics confirmed that all eight mathematics teachers at the school would participate. These teachers were invited to join the study, but only five teachers participated in the survey, and three of the five were subsequently selected as case studies, interviewed, and then observed teaching mathematics. Of the three teachers, two were women and one was a man, and their teaching experience ranged from 3 to 17 years. The least experienced teacher had mathematics as her first subject, the male teacher was head of mathematics with a PDHPE background, and the third teacher was science trained.

Instruments

The first instrument was a survey, hosted on SurveyMonkey, designed to identify the teachers' understanding of structure. The survey contained 22 statements that were anonymously answered using a 5-point Likert scale (Disagree, Partially Disagree, Neither Agree Nor Disagree, Partially Agree, Agree). The survey statements were grouped into four categories: mathematical pedagogy and content, mathematical structure, CRIG components of mathematical structure, and structural thinking.

The first author conducted individual semi-structured interviews with each of the three teachers as the second form of data collection, and these lasted about 10 minutes. The interview questions were intended to glean further information that expanded on the survey responses. For consistency purposes, it was necessary to give teachers a definition of structure. Each teacher read the following passage before the interview began:

> Some authors describe mathematical structure as the building blocks of mathematical learning. Mathematical structure can be found in connecting mathematical concepts, recognising and reproducing patterns, identifying similarities and differences, and generalising results. Students who perform structural thinking use these skills without always considering them when solving problems. Many students need to be taught these skills when introduced to concepts as a reminder of how to think mathematically.

The interviews were recorded on a mobile phone, and transcribed by the first author to a Word document and copied to NVivo. Next, the three teachers were each observed teaching three consecutive 50-minute junior secondary mathematics lessons over a one-week period. The first author identified and recorded each of the teachers' utterances that referred to a CRIG component. These were entered into an observation template in a Word document and then copied to an Excel spreadsheet file.

Analysis

The survey data were analysed with percentage breakdowns of the Likert scale scores for each statement (scores ranged 1 for Disagree to 5 for Agree).

Interview responses were first analysed to identify how teachers' comments demonstrated levels of awareness of structure. The first author categorised words and phrases made by the teachers as being either specific or nonspecific. A specific statement was one that related directly to mathematics, such as "Doing series and sequences, I took them back to tables of values", and a nonspecific statement had no direct impact on the mathematics teaching, such as, "Recognising similarities and differences, I do that". Subsequently, the transcripts were re-coded to categorise teachers' responses according to the four CRIG components. A content analysis, to categorise concepts through words and phrases that referenced structure, was then conducted to identify three major themes recognising students' structural thinking, student engagement when thinking structurally, and the benefits of structural thinking.

Data from the lesson observation templates were entered into an Excel spreadsheet to allow for allocating and filtering of teachers' utterances into the four CRIG components. Next, two sub-categories were created. The first subcategory identified an utterance as a high or low level of structural thinking. Any utterance that promoted a higher level of structural thinking was coded as analytical. Utterances that were weak in structural thinking were coded as superficial. For example, "What does the denominator tell us about the fraction?" was coded as analytical, while "We never add the denominators" was coded as superficial.

The second subcategory of utterances identified whether teachers focused on concepts or procedures when attempting to promote structural thinking. Two new codes were introduced. The first was the concept domain, where utterances explained or questioned the way something was done, such as: "Dividing by a quarter is the same as multiplying by…". The second code was the content domain in which teachers' utterances were procedural or topic-oriented, such as what to do to solve a problem: "What you do to the bottom you do to the top".

Results and Discussion

Survey results shown in Table 1 indicate that teachers believed that they possess a high level of awareness of structure, as averages for all the statements were close to the maximum.

Table 1
Survey Question Group Averages from Likert Scale Responses

Group	Questions	Survey classification	Average
1	1–6, 20	Mathematical structure	4.56
2	7–11	CRIG components of mathematical structure	4.56
3	12–19	Structural thinking	4.18
4	21–22	Mathematical pedagogy and content	4.60

The teachers' interview responses reflect varied and individual interpretations of the meaning of structure, such as the building blocks of knowledge, organisational features of a lesson or curriculum, or as the structure of a solution.

Table 2 contains the frequencies of specific and nonspecific responses made in each of the four CRIG components from the interview questions. Statements related to connections were more frequent than any of the other components; however, all these statements were nonspecific. No assumption was made that a CRIG response represents a teacher's awareness of structure; in fact, the high number of nonspecific connections statements indicates a lack of structural awareness.

Table 2
Frequency of Teachers' CRIG Specific/Nonspecific Responses to Interview Questions

CRIG component	Specific/nonspecific example	Frequency
Connections	Specific	0
	Nonspecific	13
Recognising patterns	Specific	1
	Nonspecific	8
Identifying similarities and differences	Specific	1
	Nonspecific	2
Generalising	Specific	2
	Nonspecific	5

Lesson observation data in Table 3 show the frequencies of utterances. CRIG components are first identified, then the first subcategory (analytical or superficial), and then the second subcategory, domain (concept or content). Except for generalising, across the CRIG categories, there were fewer analytical/concept utterances compared to the dominance of superficial/content utterances. This indicates that reference to, or the promotion of, structural thinking was infrequent. Analytical/content utterances had the lowest frequency across all CRIG categories, indicating teachers' references to procedures were fewer when structural features are identified.

Table 3
Teacher Utterances as Combined Categories Frequency

CRIG component	Name of utterance	Domain	Frequency
Connecting	Analytical	Concept	14
		Content	3
	Superficial	Concept	4
		Content	17
Recognising	Analytical	Concept	6
		Content	2
	Superficial	Concept	4
		Content	21
Identifying	Analytical	Concept	3
		Content	3
	Superficial	Concept	14
		Content	32
Generalising	Analytical	Concept	37
		Content	11
	Superficial	Concept	22
		Content	34

Table 4 presents the lesson observation data without the CRIG components. To identify teachers' awareness of structure, the frequency of analytical/concept utterances was considered. This was close to one quarter of the total number, indicating some attempts by teachers to promote structural thinking and suggests that the teachers in the study were not structurally aware. To see if teachers were promoting structural thinking, responses during the interviews were reviewed. Only one teacher showed an awareness of structure in the interview. The attempts to promote structural thinking observed during the lessons reflect a discrepancy between teachers' understanding of structure and how they used it when teaching mathematics.

Table 4
Frequency of Concept/Content Statement as Analytical/Superficial

Analytical/superficial	Concept/content	Frequency
Analytical	Concept	60
	Content	19
Superficial	Concept	44
	Content	104

The research questions focused on what teachers say they know about structure and how their actual teaching promotes structural thinking identified through an analysis of their utterances. The inconsistency between the results in the survey, the interview, and classroom observation provides incongruous answers to these questions. Data from teacher observation responses shows that their descriptions used to identify structural thinking was inconsistent with the survey and interview data.

Survey results indicated that teachers felt they were aware of what structure means, but interview descriptions of what they described as structure contradicted this. The interviews gave no conclusive evidence that teachers understood what was intended by mathematical structure. Observation data revealed limited attention to mathematical structure. Procedural understanding, indicated by utterances that inhibited structural thinking, occurred more often than conceptual understanding utterances, possibly impeding the promotion of structural thinking in teaching and learning.

A comparison between teachers' interview responses with utterances made when teaching showed that the interview comments were weak in structural awareness, yet about a quarter of their classroom utterances promoted conceptual understanding. This suggests that the teachers may unknowingly promote structural thinking.

Further comparison between the interview and observational data showed a shift in the teachers' attention to individual CRIG components from the interview to the classroom observation. In the interviews, teachers identified connections and recognising patterns, but in the classroom, their attention was aligned with generalising. This creates ambiguity in teachers' awareness of structure when expressing themselves in an interview as compared to what they say when teaching mathematics. Overall, these data indicate that teachers may not have a deep understanding of structure. The benefits of structural thinking are acknowledged, but are not substantial when teaching mathematics.

In Cavanagh's (2006) study, it was found that mathematics teachers did not have a deep understanding of working mathematically processes when teaching mathematics, even though they taught some working mathematically processes. Working mathematically processes have components of structure embedded throughout, so some similarities can be drawn between these studies. When it comes to teaching components of structure, teachers believe that they are utilising structure, but they are not aware of the complex components of structure.

Mason et al. (2009) espoused the importance of students engaging in structural thinking to be able think deeply about mathematics and argued that this will happen when the teacher is structurally aware. All teachers of mathematics need to be aware of the role of structure in developing their mathematical knowledge and pedagogical practices. Effective delivery of mathematical content, either procedural or conceptual, can increase structural thinking when the teacher is aware of this structure. Teachers' awareness of structure can support all stages of the structural thinking continuum and overcome what Mason et al. (2009) called the "mythical chasm" between procedural and conceptual understandings of mathematics learning.

Conclusions and Further Research

This study has shown that there is a critical need to improve teachers' understanding of structure to encourage structural thinking skills and this needs to occur early in their career, and primarily in teacher education programs. The research reported here has formed a basis for the first author's broader doctoral study into pre-service teachers' noticing of structural thinking. This new research will attempt to develop pre-service teachers' noticing of structural thinking and awareness of structure through a community of inquiry. The pre-service teachers will be involved in teaching Years 5 to 8 mathematics. How these pre-service teachers' notice structural thinking will be explored to see if their understanding of structure improves and how their pedagogical practices change to promote structural thinking.

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