In Search of Mathematical Structure: Looking Back, Beneath and Beyond — 40 Years On

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This presentation reflects on over three decades of research focused on the development of mathematical structure. 'Looking back', it traces the key theoretical influences that informed a series of studies on children's imagery, patterns and relationships including multiplicative reasoning and spatial structuring. 'Looking beneath', the development of Awareness of Mathematical Pattern and Structure is highlighted through development of an interview-based assessment and pedagogical program. 'Looking beyond', it raises key questions about the importance of developing deep mathematical thinking leading to generalisation, and realising this possibility for young children.

This annual Clements/Foyster lecture provides a unique opportunity to be both "reflective and forward thinking" about our research and its impact. Under the theme, *40 years on: We are still learning,* we can celebrate our research strengths, our collegiality, and Australasia's place in mathematics education research internationally. While this presentation provides a reflection on my contribution to mathematics education research for more than three decades, it traces the significant impact of those colleagues whose ideas shaped my ever-developing theoretical perspectives and the many research questions that I sought to answer.

'Looking back': Mathematical thinking

While my interest in how children developed mathematical ideas stemmed from my initial study of educational psychology and years of primary teaching it was not until I studied under the guidance of Professor Brian Low from 1984-1990 that my search for the origins of mathematical thinking in children took hold. As one of the foundational members Brian emanated MERGA's collegial spirit of MERGA and was pivotal in forming a strong group of mathematics education researchers at Macquarie University in the early 1980s. This was a time when scholars such as Richard Skemp, John Mason and Alan Bishop influenced the research direction of many Australian mathematics education researchers. My attempts to narrow down a purposeful research investigation always led to a more fundamental question— *What is mathematical thinking and how does it develop?*

The ICME-5 conference in Adelaide 1984, was a critical opportunity to discuss firsthand the cutting edge research and various theoretical perspectives of eminent scholars such as Alan Bell, Kath Hart, John Mason, Tom Romberg, Tom Carpenter, Jeremy Kilpatrick, Gerard Vergnaud, Efraim Fischbein and Les Steffe to name just a few. The Working Group on Primary Mathematics provided different perspectives on how to investigate such a broad and complex question. But a key message was the need to research how children's informal mathematics can develop prior to formal instruction—studies that describe and explain informal mathematical thinking and the strategies that children develop to solve mathematical problems.

An investigation of children's development of multiplication and division concepts seemed a focused and logical extension of the work by Steffe and other studies for example, on additive word problems and counting strategies. My longitudinal study of children aged 6 or 7 years that ensued would hopefully yield some new evidence of the informal mathematical strategies and children's representations of multiplicative situations.

At a deeper level, I was also searching for some clues about how recognising patterns and relationships, and the processes of modeling, representing, visualising, symbolising, abstracting and generalising were central to mathematical thinking. From the outset I considered that any investigation of domain specific concepts, skills or strategies would need to look more deeply at these processes. *While a Piagetian view of developmental stages still prevailed I questioned whether the processes of abstraction and generalisation could be developed in young children, even prior to formal instruction.*

I will show later in his paper how I have returned many times to these origins and how my present research investigations still emulate the convergence of these ideas.

My initial work on multiplication and division problems provided an opportunity to look more deeply into underlying mathematical processes. The research focused on children's multiplicative structures and their representations which could be traced to a number of theoretical perspectives: the Structure of Observed Learning Outcomes model (SOLO) (Biggs & Collis, 1982), intuitive models' theory (Fischbein, 1977), conceptual fields (Mulligan & Vergnaud, 2006), and multiplicative reasoning (Steffe, 1994).

Taking on multiple perspectives encouraged the integration of different but seemingly complementary ideas about mathematical structure. Ideally the longitudinal study of children's multiplication and division concepts was one conceptual domain that could allow further exploration into the application of mathematical structure. On one hand, the analysis of semantic structure of word problems enabled an investigation of mathematical structure in terms of 'theorems in action'. From another perspective, the initial analysis of children's solutions to word problems applied the SOLO taxonomy, represented as response maps to multiplicative word problems. Adapting the SOLO model allowed 'structure' to be described and so this informed the direction of subsequent analyses.

Fishbein's notion of 'implicit primitive models' directed my attention to exploring the underlying influences of these on children's solution strategies. Children's intuitive models for multiplication and division were analysed through their solutions to a variety of semantic structures of word problems (Mulligan & Mitchelmore, 1997). It was found that instructional approaches were not necessarily the basis for children's implicit models—multiplicative concepts were found already well developed prior to formal instruction. Robust formation of these concepts were essentially based on an equal-groups structure and strategies that reflected this structure such as multiple and double counting, grouping, partitioning, and patterning processes. These were represented by children's inscriptions and articulated through verbal and written explanations. However, children often chose to impose their own, often inappropriate structures, such as additive rather than equal groups, based on their imagistic representations of the problem situation.

While the findings of this study advanced our understanding of children's developing strategies for solving multiplicative problems it raised a much more fundamental question of how children's imagistic representations influenced the structural development of mathematical concepts.

In collaboration with Jane Watson (Mulligan & Watson, 1998), we embarked on a secondary analysis of students' representations (drawn recordings, notations and verbal explanations). Using a "more powerful lens" to look more closely at these data we aimed to identify and describe *structural* characteristics using the Structure of Observed Learning Outcomes model (SOLO) (Biggs & Collis, 1982). Modes of functioning such as ikonic or concrete-symbolic modeling were aligned with increasing levels of structural development (pre-structural, uni-structural, multi-structural, relational). Children's internal representations at pre-structural and uni-structural levels in the ikonic mode reflected the equal-grouping structure of multiplication. From longitudinal tracking of individual children's images, it was found that pre-structural images became more organised mathematically in the ikonic mode, i.e. random inscriptions were developed into groupings. Children's pre-structural responses became less reliant on physical models, and idiosyncratic images were replaced by numerical and symbolic features.

However, we found that analysing students' responses according to SOLO did not provide sufficiently fine-grained categories to find relationships between mathematical structures across mathematical concepts. Working with young children proved to be more challenging because there had been no systematic in depth studies applying the SOLO model to early concept development.

'Looking within': Children's internal images of mathematics

My attention was then turned to children's representations and how they used imagery in various ways to construct and interpret mathematical ideas. Internal, imagistic representation is essential to virtually all mathematical insight and understanding …interactions with external, imagistic representations are important to facilitating the construction of powerful internal imagistic systems in students (Goldin, 1996).

I questioned how these systems were fundamental to developing abstraction and generalisation in mathematics. Features of imagistic systems included visual, verbal and non-notational inscriptions, and kinaesthetic and tactile strategies for encoding mathematical meanings. Children's images could be classified were viewed as either *static* or *dynamic* in nature. I began to take this perspective seriously as a different way of accessing children's underlying development and representation of concepts.

Several new studies on counting and estimation, subitising, the number line, the number system, fractions and decimals were formulated.

Two new research questions were raised:

> *If children's internal imagistic representations are closely linked to the structural development of mathematical concepts, how should these be integrated with assessment and instruction?*
> *What if there is a mismatch between the child's individual and informal mathematical structures and those imposed by instruction and curricula?*

The study of mathematical structure was central to the work of Noel Thomas who explored the relationship between children's counting, grouping and place-value knowledge and their conceptual development of the base ten numeration system. By analysing children's recordings for features of structural development, it was found that children's internal representations of numbers were highly imagistic and that their imagistic configurations embody structural features of the number system to widely varying extents and often in unconventional ways. Close analysis of these structural features provided new evidence that counting and place value knowledge were influenced to a large extent by the way children imagined the counting sequence. We were able to describe several mathematical structural features in their representations: counting and symbols, number patterns and sequences, groupings by tens, use of ten as an iterable unit, recursive grouping, and multiplicative structure supporting place value knowledge. We found a wider use of structure than we had anticipated. What was more powerful was the evidence that children's representational systems were subject to change and they could eventually become powerful autonomous systems (Thomas, Mulligan & Goldin, 2002). However, the structural development of the number system did not closely resemble the curriculum sequences that were typical of instructional programs.

Second grader's representations and conceptual development of number: a longitudinal study

In a new investigation, a 3-year longitudinal study investigated 120 second graders' representations of number involving counting, grouping, base ten structure, multiplicative and proportional reasoning (Mulligan, Mitchelmore, Outhred & Russell, 1997). Although many studies were focussed on early numeracy programs at that time, this study investigated the role of imagery in children's representations of a range of numerical situations. The study was considered by some colleagues at the time, as a departure from mainstream studies. Goldin's model was adapted to analyse representations across an alternative range of tasks such as visualising the counting sequence, and imaging and drawing "what do you see between 0 and 1". Analysis of children's visualisations, drawings, ikons, symbols and explanations of their representations identified how they imposed structure, or lack thereof, on numerical situations. Low achievers were more likely to produce poorly organised, pictorial and ikonic representations that were lacking in structure. These children lacked flexibility in their thinking; they were only able to copy recordings produced by others. Essentially these children lacked a grasp of the number system, of an underlying equal-groups structure, and believed that unitary counting could be used to solve any mathematical problem. Difficulties faced with simple ratio tasks was also linked to children's inability to visualise unit fractions. High achievers, however, used abstract notational representations with well-developed structures from the outset.

A follow-up study of 24 of these children tracked to Grade 5 indicated that low achievers consistently lacked mathematical structure; pictorial and ikonic representations dominated responses with little evidence of meaningful notational systems being developed (Mulligan, 2002).

These studies were consistent with the literature on the differential effects of imagery use in the development of elementary arithmetic and the finding that students who recognise the structure of mathematical processes and representations tend to acquire deep conceptual understanding (Gray, Pitta &Tall, 2000). We formed the hypothesis that:

> *the more a student's internal representational system has developed structurally, the more coherent, well organized, and stable in its structural aspects will be their external representations and the more mathematically competent the student will be.*

The studies that followed this focused on the relationships between structural features and the formation of mathematical concepts.

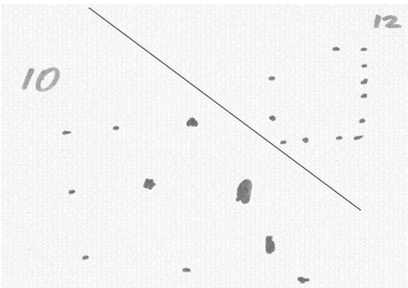
There were other studies that influenced the direction of the larger suite of studies that followed. Students' representations were essentially spatial in nature and these features could not be separated from the process of structuring. The study of two- and three-dimensional structures (Battista, Clements, Arnoff, Battista & Borrow, 1998), and measurement concepts (Outhred & Mitchelmore, 2000) focused on 'spatial because it involved the process of constructing an organisational form to the mathematical ideas. The depictions of groups, arrays, grids, equal-sized units and graphs all relied on some aspects of spatial structuring.

'Looking beneath': Awareness of Mathematical Pattern and Structure

Building on the studies on imagery and multiplicative structures, a suite of related studies with 4 to 8 year olds were designed with the aim to describe as explicitly as possible the structural characteristics in children's mathematical development. It was postulated that there was an underlying common feature that was critical to developing mathematical patterns and relationships and ultimately form simple generalisations. *Awareness of Mathematical Pattern and Structure* (AMPS) was thought to comprise two interdependent components: one cognitive — knowledge of structure, and one meta-cognitive — a tendency to seek and analyse patterns (Mulligan & Mitchelmore, 2009).

Another aim was to develop a reliable assessment that could give qualitative and possibly quantitative indicators of structural development. This assessment would inform the development of a classroom pedagogical program that could potentially promote structural thinking with a broader goal of developing generalisation in early mathematics learning.

Early signs of the development of AMPS were gleaned from the investigation of children's early formation, for example, of subitising and other patterns, representations of shapes, and arrays and girds. Figures 2 and 3 depict a 7-year old child's drawn image of the numbers 10, 11 and 12. There is some indication that the child draws on some emerging features of spatial structuring such as the outline of a square, rows and columns but the structure of the numbers is somewhat random and does reflect equal grouping. In contrast. Figure 4 depicts a sequence of highly structured representations based on a 3 x 3 array, extended to 4 x 3 for 12. Spatial structuring is utilised in the construction of the array.



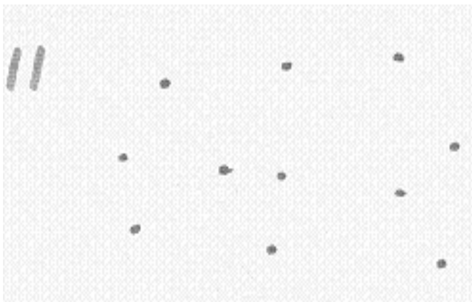


Figure 1. Kindergarten child's image of 10 and 12.

Figure 2. Kindergarten child's image of 11

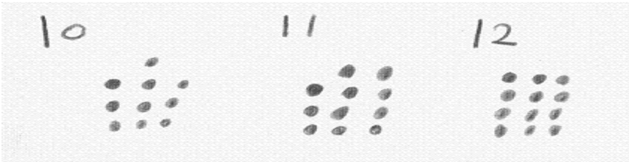


Figure 3. Kindergarten child's image of 10, 11 and 12 (Structural)

The focus on AMPS provided crucial information in the assessment of the child's mathematical concepts. While these examples (Figures 1 - 3) show that both children have learned to count, represent and symbolize number correctly the underlying lack of structure for the child (Figures 1 and 2) may not be visible using traditional forms of numeracy assessment.

Another key question was raised: Why do some children naturally develop and represent pattern and structure in their mathematical representations and others do not?

Preschoolers' representations of patterning: An intervention study

Papic framed a new study focused on the early representational development of patterning with preschoolers (Papic, Mulligan & Mitchelmore, 2011). The development of patterning strategies during the year prior to formal schooling was studied in 53 children from two similar preschools. One preschool implemented a 6-month intervention focusing on repeating and spatial patterns. An interview-based Early Mathematical Patterning Assessment (EMPA) was developed and administered pre- and post-intervention, and again following the first year of formal schooling. Assessment tasks comprised identification, representation, extension, transformation and justification of simple repetitions, and growing patterns. The intervention group outperformed the comparison group across a wide range of patterning tasks at the post and follow-up assessments. Intervention children demonstrated greater understanding of *unit of repeat* and spatial relationships, and most were also able to extend patterns. The notion of unit of repeat informed the subsequent studies on pattern and structure.

Studies on Pattern and Structure

An interview-based assessment, the Pattern and Structure Assessment (PASA) was developed and trialed with 109 Grade 1 students with follow up case studies in Grade 2 (Mulligan & Mitchelmore, 2009).

Three research questions were formulated:

1. *Can the structure of young students' responses to a wide variety of mathematical tasks be reliably classified into categories that are consistent across the range of tasks?*
2. *Do individuals demonstrate consistency in the structural categories shown in their responses?*
3. *If so, is the individual student's general level of structural development related to their mathematical achievement?*

Thirty-nine tasks covered many mathematical concepts and processes such as multiple counting, unitizing, subitising, partitioning, simple repetition, spatial structuring, multiplication and division, and proportional reasoning and transformation. All the tasks required the child to identify, draw and explain their visualisations, representations and aspects of pattern and structure. Responses to these tasks were coded dichotomously (correct or incorrect) but moreover each response was categorised for features of pattern and structure. These responses were later reliably assigned to one of five levels of structure as follows:

- *Pre-structural*. Students pick on particular features that appeal to them but are often irrelevant to the underlying mathematical concept.
- *Emergent.* Students recognise some relevant features, but are unable to organise them appropriately.
- *Partial structural.* Students recognise most relevant features of the structure, but their representations are inaccurate or incomplete.
- *Structural.* Students correctly represent the given structure.
- *Advanced Structural.* Students provide accurate, efficient and generalised use of underlying structure.

The findings showed that there was a high level of consistency in individuals' structural level across tasks. An extremely high correlation between students' structural level and the total number of correct PASA responses was also evident, considered a measure of their mathematical achievement level. Classroom teachers also identified the pre-structural students as low achievers and the structural students as high achievers.

A follow-up study investigated structural development among the eight lowest-achieving students and the eight highest-achieving students over the subsequent 18 months Consistent with earlier results, substantial differences were found between the two groups of students. The high achievers made significant progress over the 18 months and many of their responses fell into the advanced structural level. Low achievers made little progress and their representations became more disorganized and incoherent over time. They had not developed an initial awareness of patterns and structure so their ongoing mathematical learning became meaningless and more 'crowded' over time.

Further development and trailing of the PASA continued as well as the development of a pedagogical program to promote pattern and structure across mathematical concepts. This was supported by a year long whole school intervention which resulted in significant advances for students on PASA and numeracy assessments particularly in the first three years of schooling. The program was further developed through a study of Kindergarten students over a 15-week period. Students' showed rapid and sustained development of simple and complex repetitions, growing patterns, spatial structuring, base ten and multiplicative reasoning was central to the program; measurement and geometry tasks were developed as a vehicle to develop number concepts.

The elements of the program were further refined and extended, and subject to an intensive longitudinal evaluation study from Kindergarten to Grade 1 (see Mulligan, English, Mitchelmore & Crevensten, 2013). The work of English had a significant influence on the development of the study and a review of the PASA (English, 2004). This evaluation study of 316 Kindergartners employed a new form of the PASA and a standardised measure of mathematical achievement (I Can Do Maths). A PASMAP intervention program was trialled with an experimental group over the entire first year of schooling. Analysis indicated highly significant differences on the PASA between intervention students and the 'regular' group at the retention point (p < 0.002) and higher levels of structural development for the intervention students. The study validated the instrument (PASA) and constructed a Rasch scale indicating item fit.

Following the longitudinal evaluation study, a new validation study developed the PASA instrument was re-constructed, administered to a reference sample of 618, 5 to 6-year olds, and subjected to a Rasch analysis (Mulligan, Mitchelmore, & Stephanou, 2015). Three forms of PASA provided a reliable and valid measure of AMPS and it was found highly correlated with a test of mathematical achievement (PATMaths). The important outcome of this aspect of the research was that a measure of AMPS could be provided as well as reliable indicators of structural features that were effective for teacher interpretation.

The analysis also enabled the PASA to be categorised into five structural groupings: sequences, shape and alignment, equal spacing, structured counting and partitioning,

The Pattern and Structure Mathematics Awareness Program (PASMAP)

The PASMAP provides teachers with exemplars and explanations of core structural features gleaned from the research (Mulligan & Mitchelmore, 2016). An emphasis is placed on developing mathematical structures such as equal grouping, equivalence and commutativity, the relationship between metric units, transformations and pattern, and structuring data. The pedagogical approach takes what might seem to be a collection of inquiry-based tasks to a different level. What's critical is moving beyond the modelling and representing processes to visualising and generalising. The pedagogical approach focused on promoting and connecting concepts and relationships, and ultimately generating simple mathematical generalisation directs learning sequences to particular AMPS levels in particular structures, giving the teacher explicit descriptors and examples to inform their pedagogical choices. The challenge for the teacher is to recognise and then capitalise on opportunities for developing pattern and structure, i.e., can you show the same pattern (structure) in a different mode? A more critical question is how we develop teacher content knowledge and pedagogical content knowledge to support the type of thinking that leads to generalisation.

'Looking Beyond': Pattern and structure and spatial reasoning

Pattern and structure 'meets' spatial reasoning: Connecting mathematics learning with spatial reasoning 2017-2020.

Adopting a transdisciplinary perspective has raised new questions about how an *Awareness of Mathematical Pattern and Structure* is inextricably linked with spatial reasoning (Mulligan, Woolcott, Mitchelmore & Davis). The *Knowledge Synthesis of Spatial Reasoning* (Bruce et al., 2016), and the studies on pattern and structure (Mulligan and colleagues), had gained impetus in creating transformative pedagogies that will promote spatial reasoning as integral to mathematics learning for the future. A *Spatial Reasoning Mathematics Program* will be created for Grades 3 to 5 engaging students in spatial problem solving, and encouraging them to generalise their solutions by looking for similarities, differences and structural connections. The project aims to provide a more challenging and integrated view of mathematics learning by leveraging the recent progress made by the international *Spatial Reasoning Study Group.*

An integrated conceptual frame underpins this proposed study, which will allow analyses, of the complex conceptual connectivity involved in learning mathematics, using visual maps created through network analytic tools. Network mapping will provide a representation of how students' learning of mathematical and spatial concepts are interconnected, rather than as a linear, compartmentalised view. This may demonstrate that there are many different pathways that individuals adopt through a complex system of mathematics learning.

The issues discussed at the recent Topic Group in Early Childhood Mathematics at the ICME-13, (2016) supported greater consensus about the need for studies focused on the big ideas, or the study of underlying mathematical processes. It seemed that 30 years since ICME-5 we had come along way—a more holistic and integrated perspective on mathematics learning. While studies on domain-specific concepts and traditional aspects of early numeracy were still represented, the common aim of the group was to explain and describe the wide variation in early mathematical competence. Some new questions were proposed about the importance of mathematical structural development and whether simple forms of mathematical generalisation could be promoted much earlier than traditionally expected. Participants questioned whether the long-term influence of the early development of mathematical structure could result in more effective but very different learning outcomes for older students. Another approach discussed the impact of technological toys and tools on mathematical structural development and how this would provide a more coherent picture of how children's mathematical development may be changing and adapting to dynamic learning environments.

'Looking forward' we can aim to explore further aspects of AMPS: the possibility that low AMPS in early childhood could predict poor performance in mathematics throughout schooling, particularly in relation to algebraic thinking. Extending the AMPS construct to the later years of schooling will involve studies of learning trajectories of students beyond the early years of schooling whose mathematical and scientific reasoning is enhanced by a structural approach. My interest also lies in the application of the PASMAP approach to assisting those students with special needs, students with low levels of AMPS who may be prone to difficulties in learning mathematics, and students with advanced AMPS who are gifted at mathematics. This presentation has raised many questions about the way that we might view early mathematics learning and the development of deep mathematical thinking. I will raise just one more critical question as my concluding remark —What are the consequences for those children who do not develop mathematical structures at an early age, and how can we as researchers ensure that we make positive impact on teaching and learning?

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