Symposium: Stimulating Proportional Reasoning through Engaging Contexts

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Proportional reasoning is fundamental to successful operation with many topics in the primary school curriculum, including fractions, decimals, place value, ratio, proportions and percentages. The literature continually documents students' difficulties with these topics and by extension, their limited proportional reasoning capabilities. Research into proportional reasoning has a long history and continues to generate strong interest. Why is this type of reasoning so elusive and why is it so difficult to develop?

In this symposium, our aim is to continue to emphasise the importance of proportional reasoning and its pervasiveness throughout the school curriculum and to share alternative ways to promoting students' proportional reasoning capabilities. The development of proportional reasoning is underpinned by multiplicative thinking. Our concern is that multiplicative thinking in primary schools is too often thought of in terms of repeated addition leading to "equal groups", multiplication facts, and algorithms. Many teachers are not aware of the potential to support students' proportional reasoning in terms of rate and multiplicative comparison. To address the theme of this each researcher has critically reflected on how meaningful problems can serve to build conceptual understanding of proportionality.

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Creating Opportunities for Multiplicative Reasoning Using "Elastics" as a Context

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The representations, tools and task discussed were designed as a response to a pedagogical challenge: How can the "times as many" idea of multiplication be investigated meaningfully? The task involves experimenting with different types of elastic to test their "stretchiness". Reasoning during the task, with the affordances offered by the tools, learners as young as 11 years old were able to reason in terms of the multiplicative comparison. We discuss what mathematical insight, activity, and understanding is available to learners via engagement with the task.

Here we report the trialling of a task intended to stimulate proportional reasoning, particularly the idea of multiplicative comparison. Concepts underpinning multiplication are complex and the *times as many* aspect of multiplication is challenging for students to learn and for teachers to teach. However, the idea is important for students' understanding of ratio, proportion, fractions and scale. Usually students are expected to demonstrate an understanding of these topics in the secondary years. We believe that the foundational thinking for these mathematical concepts can be laid in the upper primary school.

Background

The central importance of exemplification in mathematics is at the heart of this study. We agree with Sfard (1991) in connecting the genesis of mathematical knowledge with the process of coming to know. Sfard saw examples as raw material for generalizing processes and conceptualizing new objects. The mathematical example we offer students is intended to illustrate the concept of *times as many* and uses an "investigative approach in which learners experience the mathematisation of situations as a practice, and with guidance, abstract and re-construct general principles themselves" (Bills et al., 2006, p.1-128).

Multiplicative reasoning is vital for children's mathematical development. It is not simply a generalisation of additive reasoning; multiplicative reasoning requires a qualitative shift in understanding (Vergnaud, 1983). Research indicates that many students rely on additive reasoning when the problems require multiplicative reasoning (Anghileri, 2001). Current teaching practices may be unintentionally reinforcing this additive thinking rather than challenging it (Downton & Sullivan, 2013; 2017). The *times as many* idea, first described by Greer (1989) is a multiplicative comparison. It is considered difficult for students to learn because the idea is linguistically and conceptually hard. However, this type of comparison is present in everyday life.

The research question was: Does measuring the stretch of elastic provide an investigative context that leads students to conceptualise the situation as a multiplicative relationship?

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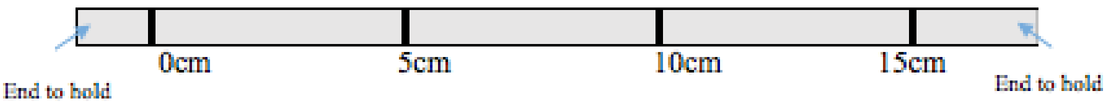
Method

We characterised the study as design research because it was: interventionist; iterative; process oriented; utility oriented, practical in a real context; and theory oriented (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006).

Fifty-six 11-12 year-old students in Years 5 and 6 in a Victorian primary school participated in two separate mathematics classes. Each one hour lesson was taught by the second author. The two classroom teachers were briefed in advance, and were present when the lesson was conducted. Thus, three experienced teachers listened and observed as the students experimented and searched for patterns in their results.

The problem: *How stretchy is elastic?* was posed introducing four types of elastic: shirring elastic, hat elastic, 5mm wide elastic, and rubber elastic, as we called them. We displayed the names and a 20-25cm sample of each to give students the vocabulary necessary to distinguish between them. Students were invited to form groups of two or three and collect their materials: each elastic sample length of approximately 20 centimetres, a metre rulers/tape measure, pencils, and strips of paper. This gave students a chance to handle the elastic samples and think a little about them. We then set up the "test" method demonstrating marking the one elastic in 5 centimetre sections as shown:



We demonstrated stretching the elastic as far as possible, taking care not to over-stretch it so that it could not spring back into shape. We asked the students to experiment and use a metre ruler to find out where the original marks stretch to. We encouraged students to keep records of their results, to swap with their partner and repeat the experiment. Students were expected to consider their findings and explain what they noticed. The lesson concluded with verbal reports from three teacher-selected groups of students.

Data were collected using classroom observations, work samples and video. Mathematical conversations (Cheeseman, 2009) were held with students as they worked, photos were taken of work in progress and finished reports. In addition, video was taken of verbal reports. We looked in detail at the students' finished written work to analyse student responses to the task. We treated the written work as representing the group's thinking.

Results

We began with the evidence on paper and using a grounded theory approach, put the formed three broad categories: work showing clear evidence of *times as many* thinking; recordings that presented raw data in systematic ways with experimental results potentially showing a multiplicative comparison but with no evidence of *times as many* thinking; and recordings showing only final lengths. Each of these categories will be illustrated in turn.

Category 1. Clear evidence of times as many thinking

Times as many thinking is evident in Figure 1. Although symbols are invented and not entirely consistent, the top left quadrant reveals that these students noticed that the mark they made at 5 cm was equal to 15 cm when the hat elastic was stretched, was a factor of three. The 10 cm mark became 28 cm, which was approximately or about three times the original (A 3x). Throughout the recording a capital A is consistently used to denote the approximate nature of the multiplicative relationship.

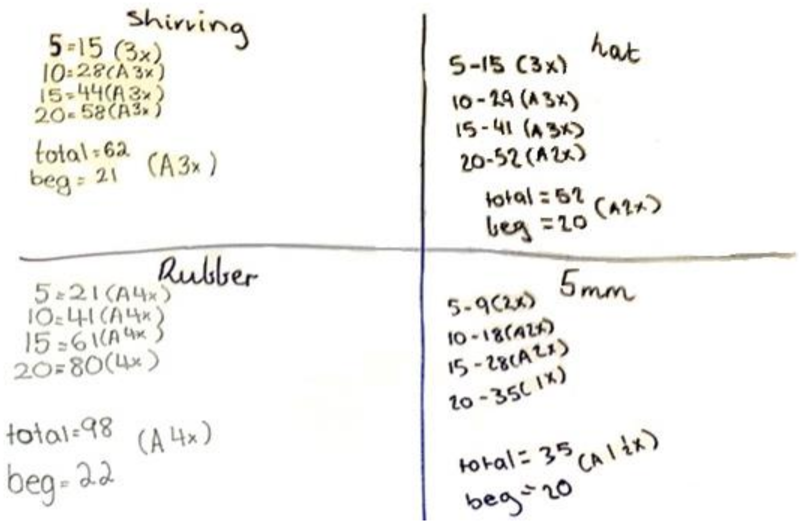


Figure 1. An example of Category 1. Times as many thinking

Some students conducted a very careful and precise test of each of the elastics and recorded their results in a clear and logical manner (Fig 2). Whether they could not see any patterns in the data due to the error margins in the measurement, or whether they did not look for patterns is unknown.

Category 2. Systematic experimental results with no evidence of identified patterns.

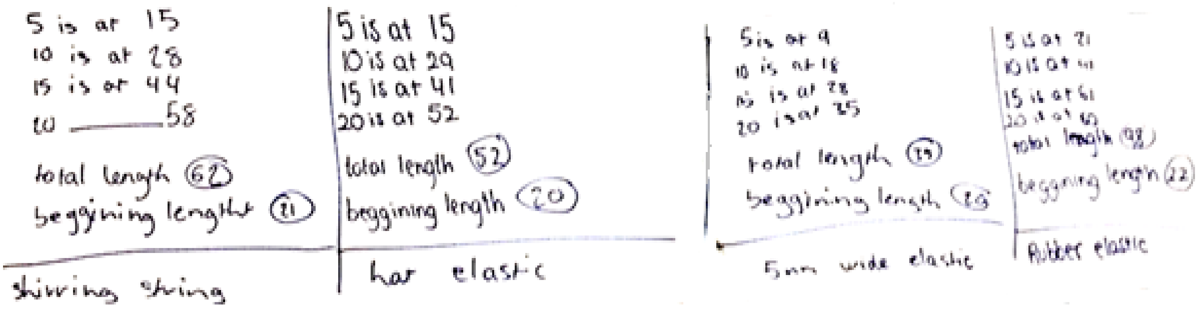


Figure 2. Category 2. Evidence of experimental results but no multiplicative thinking.

Category 3. Results focused on a comparison of maximum length stretched

Work samples in category 3 showed a student focus on the maximum length to which each elastics sample could each be stretched. These students had apparently transformed the problem from, “How stretchy is the elastic?” to “How far does the elastic stretch?” This seemingly small change of wording changed the focus from the features of the elastic to the greatest length that can be attained. This thinking is illustrated by the following report:

> We measured the elastics one at a time and marked them each multiple of 5 … on the elastic. The next step was to stretch the elastic and see how far it went. Then we saw what was the last mark we did on the elastic and saw how far the elastic stretched and we worked out the difference.

Some work samples in this category saw the stretch of the elastic as an additive action where their calculation involved subtracting the original length from the final length.

Discussion and Implications

We cannot say exactly how many students have begun to develop emerging concepts of multiplicative comparison because in groups of students it is sometimes not clear which of the individuals has which concept. What we can definitely say, based on this experiment, is that some students 10-12 years of age can deduce *times as many* relationships in sets of data. Our observations suggest 13 (23%) of the 56 students in the classes we observed looked for multiplicative patterns in their results.

This was the first time, as far as we could ascertain, that the students were offered the opportunity to consider times as many relationships at school. We think that with follow-up learning opportunities, early *times as many* concepts could be established and possibly initiated for other students. We are keen to experiment with potential ideas for further learning. For example, because results the students collected were meaningful and accurate we would use them with the whole class by displaying them and challenging the students to search for patterns in the figures. In this way the finding of *times as many* ideas could be made explicit. Another possible follow-up lesson could be devised using different elastics samples or rubber bands to test the “stretchiness” using the same experimental methods.

The *times as many* aspect of multiplicative reasoning is not presented to students as often as it might be. We wonder whether contexts that exemplify the concept are difficult to find. We encourage teachers to think of other everyday situations that might serve to help students to conceptualise multiplicative comparison ideas of multiplication.

We recommend a sequence of lessons on the idea of “stretch factors”. As we think that several mathematical examples would likely serve to establish multiplicative relationships for some of the students who were thinking additively and a sequence of investigations would consolidate the learning for students who have developing times as many concepts.

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Sharing the Cost of a Taxi Ride

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We aimed to explore the extent to which a challenging yet accessible real world financial context where two people stand to gain from sharing a taxi ride might stimulate students' mathematical exploration and discovery related to multiplicative thinking and proportional reasoning. Data were collected from 37 Year 5 and 6 students (10-12 years of age) in a Catholic primary school in suburban Melbourne. The findings reveal that the majority of students had some intuitive understanding of how to solve a financial problem that involved rates, and at least half of them used either proportional reasoning or multiplicative thinking. We argue that, given the right problem context, upper primary school students can be encouraged to engage in proportional reasoning earlier than the Australian Curriculum requires.

Given the increasingly challenging economic conditions and financial products and services we face, the need to prepare young people to make informed financial decisions is a topical priority for schools and teachers. Money and financial mathematics features explicitly in the Australian Curriculum (AC) Mathematics (M). There are a number of everyday financial contexts that require multiplicative thinking and proportional reasoning that might be meaningfully explored in the upper primary years of schooling. These include sharing costs like a restaurant bill, transport and accommodation in ways that are fair, and accounting for fluctuating monthly expenses over the course of an annual budget.

Multiplicative thinking is conceptually complex and yet the intended curriculum does not reflect this complexity. While problem solving and reasoning are two of the four AC:M proficiency strands, ratio and proportional reasoning are not suggested within the AC:M until Years 7 and 8. The actual term proportional reasoning is not stated until Years 9 and 10, where a need to "interpret proportional reasoning" is specified (ACARA, 2015, npn).

Meanwhile, teachers seem to have difficulty finding productive approaches to teaching all but the simplest multiplicative "equal groups" ideas (Downton, 2010). Related to this, various studies have found that Year 7 and 8 students' difficulties in solving problems involving fractions, decimals, ratio and proportion are attributable to a reliance on additive reasoning when multiplicative reasoning is required (e.g., Hilton, Hilton, Dole, Goos & O' Brien, 2012). Others have argued that students' lack of proportional reasoning is directly related to their limited experience with different multiplicative situations, including rate and ratio (see Greer, 1988).

With this situation in mind, we aimed to examine the extent to which a challenging yet accessible real world financial problem might stimulate students' exploration and discovery related to multiplicative thinking and proportional reasoning. Our research question was: In what ways do 10-12 year old students formulate and employ mathematics when solving a real world financial problem that involves sharing costs?

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The Research Design

We will present an aspect of ongoing classroom research in a small Catholic primary school in suburban Melbourne, Australia. Data were collected in the school's two Year 5 and 6 composite classrooms. The first author presented a 60 minute modelled lesson exploring a task where two people share a taxi ride. The task deals with ideas of rate and was presented as follows:

> Catching a taxi
>
> The taxi fare is $3 flagfall (what you pay when you get into the taxi) and then $1.50 per km after that. It does not matter how many people are in the taxi.
>
> Mike and Matt decide to share a taxi because they are going in the same direction but to different houses. The journey to Mike's house is 20 km, then a further 30 km to Matt's house.
>
> How much should each of them pay for the taxi? Explain why your suggestion is fair for both people.

The fact that the characters Mike and Matt are travelling different distances means that sharing the cost of the trip evenly may not be the fairest solution. We considered the task an appropriate choice for examining further the ways and means by which real world problem contexts can stimulate student mathematical exploration and discovery, particularly in terms of proportional reasoning.

The two lessons were audio and video recorded. Students' worksheets, were collected at the end of each lesson. Across the two classes, there were 37 student participants.

The OECD PISA 2012 mathematical literacy assessment framework (OECD, 2013) served as useful framing for data collection and analysis. The framework depicts a modelling cycle involving three mathematical processes that students apply as they attempt to solve problems - formulate, employ and interpret (OECD, 2013). These mathematical processes might be understood as stages of realistic or applied modelling through which a real-world problem is solved (Stacey, 2015). First, the problem solver identifies or formulates the problem context mathematically. This involves making various assumptions to simplify the situation. In so doing, the problem solver shifts the problem from the real world to the mathematical world (OECD, 2006). Next, the problem solver employs mathematical knowledge, skills and reasoning to obtain mathematical results. This usually involves mathematical manipulation, transformation and computation, with and without physical and digital tools. Finally, the problem solver interprets the mathematical results against the problem context. This involves the problem solver evaluating the adequacy and reasonableness of the mathematical results, shifting them back to the real world (OECD, 2006).

Student worksheets were analysed for the purpose of categorising how students formulated and employed mathematics. Hence, we examined the thinking evident in the response, but also the mathematical strategies used. We were also interested in the explanations students gave about why their suggestion was fair for both people, as these explanations revealed insights into how students interpreted their solutions against the problem context. Using a grounded theory approach (Strauss & Corbin, 1990) four categories for formulating the problem emerged. These are presented below, from most to least sophisticated:

A. Some students perceived the journey as taking place in two parts, but suggested that as the men shared the first 20km of the distance, they should also share the cost of that leg of the journey. In this scenario, Mike and Matt would pay

$16.50 each to travel the first 20km and Matt would pay an additional $45 to travel the next 30km alone, meaning a total of $16.50 for Mike and $61.50 for Matt. This approach showed a sophisticated grasp of the problem context, as well as proportional reasoning.

B. Some students perceived the journey as taking place in two parts, with Mike paying $31.50 to travel 20km and Matt paying $46.50 to travel the additional 30km. In this case, there is no benefit for Mike in sharing a taxi, but there is a saving for Matt.

C. Some students perceived that Mike and Matt should pay separately based on the distance travelled. In this scenario, students typically suggested that the flagfall should be shared evenly. Here, Mike would pay $31.50 to travel 20km; and Matt would pay $76.50 to travel 50km. Unchecked, such a misconception would result in a windfall for the taxi driver.

D. Some students calculated the total cost of the journey ($78) and divided this by two. In this scenario, Mike and Matt would share the cost evenly, paying $39 each. Here, there is no benefit for Mike in sharing a taxi – in fact he would pay more than if he was to travel home alone.

Within each of the above categories, three categories of strategies for employing mathematics were readily able to be identified: additive thinking; multiplicative thinking; and proportional reasoning. A fourth category - no documented strategy - was applied to student worksheets where there was an answer, but no mathematical working. It is important to note that this category signals the possibility of mental computation.

Results and Findings

Table 1 presents the two levels of categorisation described above: how students formulated the problem (rows); and how they employed mathematics (columns). Of the 37 student participants, three students noted more than one solution. As each of these solutions was considered separately, a total of 39 responses were categorised and tabled. Four incidences of student mathematical error were noted.

Table 1
Analysis of the Way the Problem was Formulated and the Mathematical Thinking Employed

	Way mathematical thinking was employed				
Way of formulating the problem	Proportional reasoning	Multiplicative thinking	Additive thinking	No documented strategy	Total
A	3	4	0	0	7
B	2	6	3	5**	16
C	1	4*	3	4	12
D	0	1	1*	2	4
Total	6	15	7	11	39

* indicates one error in computation

The student worksheets revealed that 90% of students were able to correctly calculate an answer based on how they mathematised the problem. It was evident that the multiplicative nature of the reasoning required to find a solution was clear to the students. Twenty-one students (54%) used proportional reasoning (6) or multiplicative thinking (15), with a total of seven using repeated addition as a way of finding a solution. It is possible that those with no documented strategy (11) also used these methods, but their records were unclear. We can say that the problem was largely understood as a multiplicative situation.

During our presentation, these results, supported by examples of student worksheets, will be presented and discussed with the intention of arguing that the task, lesson structure and pedagogical architecture encouraged proportional reasoning at a younger age than it appears the Australian Curriculum: Mathematics.

Conclusion and Implications

The findings suggest that the majority of students could formulate a real world financial problem that involved proportional reasoning and employ mathematics in a way that reflected the approach they selected. Further, at least half of the students used proportional reasoning or multiplicative thinking, which suggests that Years 5 and 6 students can not only attend to rate tasks such as this, but some can appropriately apply proportional reasoning. Catching a taxi seemed to provide an appropriate hook to introduce proportional reasoning – a concept that does not appear in the Australian Curriculum: Mathematics until Years 7 & 8 – to upper primary school students. The findings suggest that, given the right challenging yet accessible real-world problem context, upper primary students can explore and discover more complex mathematical tasks than curriculum writers, task designers and teachers might assume.

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Authentic Numeracy Contexts for Proportional Reasoning – the Case of the Seven Summits

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This paper presents a case study of one primary school teacher's journey of realisation about the importance of proportional reasoning for numeracy. Through immersing students in a rich numeracy investigation to meaningfully compare the world's tallest mountains, this teacher reflects on authentic contexts and hands-on experiences for promoting and enhancing students' multiplicative thinking. This study included analysis of interview data, classroom observations and student artefacts against a rich model of numeracy that served to emphasise the power of meaningful contexts for promoting multiplicative comparison.

Introduction

Proportional reasoning is one of the most commonly applied mathematics concepts in the real world (Lanius & Williams, 2003). Adjusting measures of ingredients in a recipe, adding sugar for the perfect cup of coffee, estimating the time to travel when found in a traffic jam, choosing the right food storage container when saving left-overs, calculating percentage discounts on sale items, are some everyday tasks that require proportional reasoning. As proportional reasoning is required in so many everyday situations, it is essential to numeracy (Dole, Goos, Hilton & Hilton, 2015). However, students' difficulties with proportional reasoning tasks are well-documented (e.g., Lamon, 2007).

In the absence of knowledge of ways to promote proportional reasoning, teachers may revert to skill-based approaches that may serve to hamper students' proportional reasoning development and capacity to use proportional reasoning in complex and unfamiliar situations. Tasks requiring proportional reasoning are a continual stumbling block for so many students in many areas of the curriculum, which suggests the need for a broad-spectrum, multi-pronged strategy for action.

Theoretical Framework

The theoretical framework that guided the research reported in this paper draws from two fields of maths education research: (1) proportional reasoning, and (2) numeracy.

The essence of proportional reasoning is an awareness of how two quantities are related in a multiplicative sense. The American Association for the Advancement of Science (AAAS) (2001) identified two key components of proportional reasoning: Ratios and Proportion (parts and wholes, descriptions and comparisons, and computation) and Describing Change (related changes, kinds of change, and invariance). Lamon (2007) outlined central core ideas for proportional reasoning as rational number interpretation, measurement, quantities and covariation, relative thinking, unitizing, sharing and comparing, and reasoning up and down. These two sources highlight the encompassing

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nature of proportional reasoning and the fact that it is more extensive than simple rules or calculation procedures, and certainly more than promoting multiplication as repeated addition. This theoretical framework has underpinned the design of tools for assessing middle school students' proportional reasoning (see Hilton, Hilton, Dole & Goos, 2016).

A rich model of numeracy has been proposed by Goos (2007). The model highlights five elements of numeracy as comprising mathematics knowledge, use of tools, positive dispositions, a critical orientation, and grounded in context. The model has been found to support teachers in designing rich learning tasks to promote numeracy (Goos, Geiger & Dole, 2013). Drawing on this theoretical framework, this project aimed to answer the following research question: to what extent can a rich model of numeracy and a broad conceptualisation of proportional reasoning support teachers in building curriculum knowledge for proportional reasoning?

Design and Approach

This paper reports on a single case study of a teacher who participated in a large project that involved middle school teachers from five school clusters over an extended period of three years. In this project, we designed a professional development (PD) program to build teachers' awareness of the pervasiveness of proportional reasoning throughout the curriculum. Teachers tailored and trialled teaching sequences on ideas and suggestions presented at the PD. The researchers visited project teachers' classrooms in-between the PD sessions and offered support, advice, and encouragement. The case study reported here draws from interview data (ID), classroom observations (CO) and student artefacts (SA) to describe one teacher's journey of developing awareness of the pervasiveness of proportional reasoning and how engaging learning experiences can support all learners in developing proportional reasoning capabilities.

Results

Luke (pseudonym) is a teacher of a composite upper primary school class of 27 students. The school is located in a rural community. Prior to commencement in this project, Luke had planned to teach a unit based around the seven summits (the highest mountain peaks in each of the seven continents) drawing on his personal interest in mountain climbing. He commenced this unit with students "undertaking some basic mapping and activities involving coordinates". Initially he felt that his students had a "pretty good understanding of how to use scale, but ratio, they didn't understand" (ID). Luke elaborated that he had provided students with some mathematical exercises where the scale was indicated as 1cm:1km. The students successfully completed conversion exercises to determine the length between particular places based on this scale. Luke's comment was in relation to students' conceptualisation of magnitude of the scale in which they were working. This was evidenced when he referred to a map of Australia and Oceania that was located on the classroom wall. The scale was presented as 1:15 000 000 (CO). Luke reflected on how he attempted to make this scale meaningful to the students: "I explained to them in a very poor way, that according to the scale on that map that Australia is 15 000 000 times bigger than the image of it on the map. Of course no one can visualize that." At the end of the lesson, Luke pondered how he might assist students to "visualise" the ratio.

Luke's next (Art) lesson focused on scale drawings of the human body. At the beginning of the lesson, Luke used the word "proportion" and drew students' attention to the structure of the human body. Students paired with a partner and compared the length of their arms, noting where their arms finished, and whether the arms were longer or shorter than other parts of the body. They were then instructed to sit with their partner and draw each other as life-like as they could. Many students expressed frustration with their drawing as the “proportions were wrong” (CO). At the end of the lesson, the drawings were in a rather crude form. Noting students’ frustration with their drawings, Luke asked the students to measure the height of the person on their drawing and to make a calculation of the actual size of the person. Students readily determined that the size of the paper was approximately 20 cm. One student stated that the picture would need to be enlarged 20 times to be lifesize. Many other students readily agreed until there was growing realisation that “twenty times twenty - they’re not that big” was not an appropriate scale-factor. The mathematical behaviour exhibited by the students was exciting to Luke: “they were estimating a ratio, then calculating it and then refining it…Some kids in my class are not into estimating at all, they won’t do it, they just feel that there is too much room for going wrong” (ID). Students then began to spontaneously engage in undertaking multiplicative comparisons: “And from there we looked at their drawings and they actually worked out just looking at the height from head to toe, they worked out an actual correct scale for that drawing. We picked one part of the body and that part of the body was 1cm on the paper, so then it must therefore equal a certain amount in real life” (ID). There was a new sense of industry in the classroom from this point. Students used rulers, tape measures and calculators to take measures of body parts and to then draw them on the page. In efforts to increase the realism of their drawings, many students were seen to sketch and then to erase sections of their work, and then to redraw elements after making further measurements or observations of their partner (CO). Luke reflected on how students would measure each other’s noses and then compare this length to the nose drawn on the paper. They quickly saw “for example, the arms might be in proportion but then the nose was almost as big as an arm in real life.” Luke recounted how he saw students “slapping their heads” and exclaiming “oh no, the eyes would be this big in real life”. Luke described how he saw students taking measurements of different parts of their partner and selecting a scale of 1 to 7. Luke noted the pride in which the students viewed their second sketch compared to their first: “they compared their drawings and talked about their first sketch against their refined drawing and they were quite proud. They were telling me how terrible they are at drawing, but it was almost like the scale and ratio had given them a formula for drawing more accurately.” The students were very keen to pin their refined drawings around the room.

It was from this experience that Luke directed the learning to the end-goal he had from the start: the seven summits. Luke found that the students had little trouble in representing the mountains to scale. After introducing and discussing the seven summits (and students exploring further via the internet), Luke instructed students to create a triangle from an A4 sheet of paper that would represent Mt Everest. The students measured the height of the triangle as 18.5 cm. “Then they got the height and divided it by the scale to give them the measurement of the height of each mountain. They reasoned that if Mt Everest at 8848 m scaled to 18.5cm, then the scale was 1 cm: 478 m.” The result was a graphical representation of the seven summits, each mountain a different coloured triangle, all neatly line up as a sequence of triangles of descending heights.

Luke could see the rich numeracy experiences that he had created for the students and how the activities built on one another to continue to fuel students’ interests and learning:

> yeah so initially it started out as our SOSE and Science, but then health because we started talking about pulmonary oedema and hypoxia the kids had no understanding of what these meant but just this week the kids finished off a dictionary of all of the different mountaineering terminology,

equipment and the conditions or ailments that affect them. So it ended up going into our health curriculum because the kids would then argue about whether hypoxia and pulmonary oedema were diseases, because you can't catch it, so how are they diseases? So it's been a really good term, when I first walked in and told them that we were going to be doing it for a term, none of them cared.

Discussion and Conclusion

In analysing the lesson sequence from a numeracy perspective (Goos, 2007), we can see the richness of these experiences. The sequence was grounded in *context*, with Luke's personal interest in mountain climbing fuelling and generating continued student interest in natural phenomena. The students applied their *mathematical knowledge* to the context, making mathematical estimations and reasoning and justifying their calculations. They used *tools*, not only via the use of measuring instruments and calculating devices, but also through the creation of the visual representational tool of the seven summits. They were developing positive *dispositions*. They were clearly enjoying the learning in which they were engaged. They also took risks in calculating and estimating and sharing their ideas, rather than seeking confirmation from the teacher at every step. They were developing a *critical orientation*, not only through their growing awareness of health issues and mountain climbing, but on reviewing the reasonableness of their results. The lesson sequence was also a multi-directional approach to developing ratio and scale and proportional reasoning. The purpose for scale and ratio emanated from the task, and all students were seen to build their confidence in relation to dealing with scale and large numbers. When discussing their mountain representation pictures, the students would readily discuss how they approached the calculations and could confidently discuss the magnitude of the mountains. Of most interest was the complex scale factor of 1 cm : 478 m that was confidently discussed by all students (CO).

This case study has presented one teachers' journey of realisation about the power of a multi-dimensional approach to proportional reasoning through a rich numeracy investigation. With the end product in mind - the seven summits - the teacher did not revert to a skill and drill lesson of scale and ratio. In fact, early lessons of this type were regarded to be of minimal value for the end-goal: an appreciation of the size of the largest mountains in each continent. This case study serves to remind us of the value of non-sequential approaches to developing multiplicative thinking and proportional reasoning, through learning experiences that are inclusive of all learners.

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Teachers' Perceptions of Students' Development of Multiplicative Thinking

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Having an understanding of the key ideas underpinning multiplicative thinking is critical for future learning beyond the primary school years. The shift to multiplicative thinking can be challenging for both students and teachers due to its multifaceted nature. This paper reports on a pilot study of professional learning in schools that identified multiplicative thinking, an area of concern. We sought to explore in situ professional learning (school-based) within 14 primary schools across a six-month period. Our findings suggest that in situ professional learning had a positive impact on teachers' mathematical content knowledge and pedagogical content knowledge.

In the current political climate, there is increased pressure on teachers to improve student-learning outcomes in mathematics education. In particular, there is a concern regarding the number of students in Years 5 to 8 who rely on additive thinking to solve proportional reasoning problems when multiplicative thinking is required and those who cannot distinguish whether a task requires additive thinking or multiplicative thinking (Van Doreen, De Bock, & Verschaffel, 2010). This may be attributed to an emphasis in the early and middle primary years on multiplication as repeated addition, equal groups and arrays. Alternatively teachers' limited understanding of the complexity associated with the development of multiplicative thinking and their knowledge of the different multiplicative structures may be contributing factors.

Theoretical Framework

A recurring theme in the literature is that multiplicative thinking is a crucial stage in students' mathematical understanding, the basis of proportional reasoning, and a necessary pre-requisite for understanding algebra, ratio, rate, scale, and interpreting statistical and probability situations (e.g., Hilton, Hilton, Dole, Goos, & O'Brien, 2012). Some scholars argue that the difficulties associated with students' lack of proportional reasoning are related to their limited experiences of different multiplicative situations such as multiplicative comparison (times-as-many) and rate/ratio (e.g., Greer, 1988) or to their reliance on additive thinking when multiplicative thinking is required (e.g., Van Doreen et al., 2010). Greer (1988) suggests that students need to engage in multi-step contextual problems that include more complex numbers so that the appropriate operation cannot be intuitively grasped.

In relation to professional learning models, research suggests that professional learning for teachers needs to be situated in realistic contexts as part of the on-going work in schools, in contrast to one-off models of professional development (Bruce, Esmonde, Ross, Dookie, & Beatty, 2010). Teachers are seen as learners and schools as learning communities (Clarke & Hollingworth, 2002). Bruce et al., (2010) support Clarke and

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Holllingworth’s notion of professional learning (PL) being embedded in classroom experiences and practices within the school context, and argue that such professional learning is characterised as occurring in sustained and iterative cycles of planning, practice and reflecting. Dole, Clarke, Wright, and Roche (2008) engaged teachers in a focused professional learning program on teachers’ understanding of proportional reasoning. They found that although there were marginal differences in teachers’ proportional reasoning, teachers had the language to discuss proportional reasoning, and could articulate the difference between additive and multiplicative thinking.

Informed by the research literature, a pilot PL program focused on developing teachers’ knowledge of multiplicative thinking was situated within each participating school. The study aimed to address the research question: What is the impact of an in situ, spaced, professional learning on teachers’ pedagogical content knowledge for developing multiplicative thinking in their students?

Method

The purpose of this mixed methods study was to examine the perceived impact of an in situ PL program on teachers’ pedagogical content knowledge related to multiplicative thinking. We characterised this pilot study as an effectiveness study (Bruce et al., 2010) as it studied PL opportunities for classroom teachers within their own setting and measured their pedagogical content knowledge (PCK) through the use of an online survey, administered pre and post the PL.

The structure of the professional learning (PL) was informed by the abovementioned research. The research team, led by the first author, developed five 90-minute PL modules with co-researchers (Teaching Educators) from a New South Wales Catholic Education System. Each module focused on an aspect of multiplicative thinking and pedagogy, and included challenging tasks, professional readings and between session classroom tasks. The co-researchers facilitated the PL at participating schools across terms two to four, and provided in classroom support in Years 3 and 4, due to the identified need and high proportion of students still reliant on counting based strategies.

Fourteen primary schools (approximately 230 participants: classroom teachers, specialists, lead teachers, and leadership teams) across the diocese agreed to participate in this research, as multiplicative thinking was their PL priority. The data collection instruments included teacher online surveys, focus group interviews and teacher reflective journals. The data reported here pertain to one open response question from the online teacher survey: How do you believe students develop multiplicative thinking?

All responses were entered into a spreadsheet, coded and categorised through the analysis of data using a grounded theory approach (Strauss & Corbin, 1990). If a teacher wrote multiple ideas, each was coded as a separate response. The first two authors independently coded the teachers’ responses using open coding to identify key themes.

Results and Discussion

Table 1 shows seven themes developed from the data analysis and teachers’ illustrative responses to the aforementioned question. Pre PL 37% of respondents and post 25% believed that students develop multiplicative thinking by using some form of representation that leads to the development of abstract thinking. There is a noticeable shift in responses post the PL from a focus on aspects of general pedagogy (themes 1, 4, and 7) to focussing on aspects relating to multiplicative thinking (themes 2, 5, and 6).

Table 1

Percentage of Responses Relating to How Students Develop Multiplicative Thinking

Theme	Pre (n= 244)	Post (n= 236)	Illustrative of comments written by teachers
1.Materials and representations moving to abstract thinking	37	25	Pre: By working with concrete materials, partial models, to abstract thinking. Post: Build up multiplicative foundation, move from visualising arrays to abstract thinking and reasoning.
2. Moving from additive to multiplicative thinking	12	22	Pre: From additive thinking to applying known facts. Post: Use of arrays, and times as many that encourage multiplicative thinking and reasoning strategies such as known and derived facts that shift their thinking.
3. Relationship: multiplication and division	4	10	Pre: Knowing link between multiplication & division Post: When engaging with problems/tasks that require thinking about inverse operations.
4. Engage in real life problems and open tasks	30	12	Pre: Being exposed to real life problems. Post: Engage in real life multiplicative tasks and multi step word problems and open tasks that encourage MT
5. Use of multiplicative language	4	11	Pre: Experience the language of 'groups of', 'arrays' Post: Opportunities that expose them to multiplicative language such as commutativity, times as many.
6. Experiencing multiplicative structures	0	13	Pre: Provide 'groups of' and 'arrays' activities. Post: Regular experience with challenging problems relating to arrays, times-as-many, allocation and rate.
7. Teacher demonstration and practice	13	7	Pre: Teacher modelling strategies, and practice times tables. Post: Having strategies shared by students and reinforced by teachers and through practice of a variety of questions.

Prior to the PL 80% of responses related to general pedagogical approaches to mathematics, compare to 44% post the PL. In contrast, 56% of responses related to multiplicative thinking post the PL, which was more than double that of the pre PL (20%). This appears to suggest that the program challenged existing ideas about students' development of multiplicative thinking and resulted in a shift in teachers' perceptions.

We anticipated a reduction in a procedural approach to learning (Theme 7) and using materials (Theme 1). While there was some reduction as a result of the PL it is evident that these views are strongly held, particularly in relation to use of materials to support students' shift to abstract thinking and teachers wanting to do explicit demonstration.

Teachers became increasingly aware that students' development of multiplicative thinking is linked to shifting from additive thinking and counting based strategies (Theme 2, Table 1). Many responses indicated that some powerful and engaging tasks facilitated the transition from additive to multiplicative thinking. Nick, a Year 4 teacher, recorded the following in his reflective diary after exploring the carrot patch task with his students.

> Having to imagine the missing carrots in the array was powerful and the kids were using distributive property and the language of arrays, partitioning, factors and multiples.

The biggest shifts related to themes four and six. While we were initially surprised that there was a decline in teachers' focus on the importance of engaging students in real life problems and open tasks, we realised that teachers' experience of the different multiplicative structures (rectangular array, rate, ratio, and times-as-many) had a major impact on their own learning. Sophie, a Year 3 teacher, recorded this entry in her diary.

> The language of times-as-many was challenging for students initially but once they had more experience with tasks like this, I saw a shift in the strategies they used and they were using multiplicative language and making connections between multiplication and division.

Concluding Comments

The PL provided teachers with a range of rich and challenging tasks using everyday relevant content related to arrays, rate/ratio, and times-as-many that teachers then explored with their students in the classroom. Making links to proportional reasoning in the modules when exploring teachers' and students' solution strategies to rate/ratio was critical. Teachers realised that primary school students can engage in tasks such as these and do so using proportional reasoning and multiplicative thinking. They saw the tasks as a major source of their learning and understanding of the complexity of developing multiplicative thinking. Teachers also recognised that developing such tasks was their greatest challenge when planning for learning, and indicated they need further support in this area. The findings suggest that providing in situ targeted professional learning over a sustained period of time that requires teachers to implement the learning with their students improves their knowledge of multiplicative thinking and proportional reasoning. It equipped these teachers with ways to support their students' development of multiplicative thinking using rich learning experiences relating arrays, rate/ratio and times-as-many. However, they still require on-going support and PL to embed the practices and deepen their understanding.

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