Symposium: Multiplicative Thinking: Enhancing the Capacity of Teachers to Teach and Students to Learn

Multiplicative thinking is a key aspect of primary and middle school mathematics and is considered to be a predictor of students' capacity to progress beyond basic mathematical learning. It is characterised by a complex set of connecting ideas about which teachers need to have a broad and deep understanding. The study on which this symposium is based began in 2014 in Western Australia. It has involved over 1900 primary school students of ages 9 to 11 years, approximately 120 teachers, and 16 schools. This symposium presents an overview of the project and then focuses on the New Zealand phase of the project. Assessment of students' multiplicative thinking in the form of a written quiz and semi-structured interviews enabled teachers and researchers to identify students' knowledge and understanding of multiplicative concepts and led to the structuring of a targeted teaching program over several months. Parallel pre and post quizzes were used to investigate the extent of student learning that occurred. A highly significant increase in student attainment was noted. The use of manipulative materials to identify the extent of students' multiplicative thinking was also investigated through semi-structured interviews. Teachers' content knowledge was explored with particular emphasis on the use of student tasks targeting specific aspects of multiplicative thinking. It was found that teachers became more confident in teaching multiplicative concepts, showed a greater awareness of connections between ideas, and demonstrated a growing awareness of the importance of explicit mathematical language.

Keywords

NCON – Number Concepts

PRIM – Primary

PRDE – Professional Development

TPER – Teacher Perceptions

KNBO – Knowledge Building and Organization

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Multiplicative Thinking: Developing a Model for Research and Professional Learning

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The study on which this symposium is based began in 2014 in Western Australia. It has involved over 1900 primary school students of ages 9 to 11 years, approximately 120 teachers, and 16 schools. It began as a research project with a semi-structured interview as the initial data gathering instrument. A written multiplicative thinking quiz (MTQ) was developed to gather large data sets more efficiently. From the data a model for multiplicative thinking was developed and the MTQ was refined. During 2017 data were gathered from schools in Plymouth (UK), Dunedin (NZ), and Victoria. A professional learning program was also developed and papers in this symposium report on the development and use of the instruments, the multiplicative model, data gathered from students, and the professional learning program.

Background

Multiplicative thinking is a critically important aspect of mathematics. It underpins much of the mathematics learned beyond the middle primary years, informs the understanding of proportional reasoning, ratio, and statistical sampling, and is an important component of algebraic reasoning (Siemon, Breed, Virgona, Dole, & Izzard, 2006). It is well documented (Clark & Kamii, 1996; Siemon et al., 2006) that students who do not develop the ability to think multiplicatively find it difficult to move beyond primary school mathematics. It is also of concern that much mathematics teaching is procedural in nature whereas students need assistance to develop a conceptual understanding of the structure of the mathematics (Warren & English, 2000; Young-Loveridge & Mills, 2009).

Effective teaching of multiplicative concepts lies at the heart of success in terms of student learning. There are several aspects to this, one being a broad and deep mathematical content knowledge and a varied pedagogical content knowledge on the part of the teacher. Second, pedagogies need to be explicitly focused on specific aspects of concepts designed to develop a connected understanding of key content. These pedagogies involve questioning, demonstration, discussion, reasoning, investigation, and interaction between all students and the teacher (Askew, 2016). Third, procedural fluency and conceptual understanding must be developed alongside one another with conceptual understanding informing the use of procedures (Hiebert & Grouws, 2007). Finally, an understanding of the structure of the mathematics involved in the development of multiplicative thinking is seen as being critical (Warren & English, 2000).

The following definition of multiplicative thinking is based on the work of Siemon, Breed, Dole, Izard, & Virgona (2006) and Siemon, Bleckley, & Neal (2012). Multiplicative thinking is demonstrated by an ability to:

- Work flexibly with a wide range of numbers including very large and small whole numbers, decimals, fractions, ratio and percentage;
- Work conceptually with the relative magnitude of whole and decimal numbers in a range of representations, demonstrating an understanding of the notions of 'times bigger' and 'times as many';

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- Demonstrate a conceptual understanding of the multiplicative situation, the relationship between multiplication and division, numbers of equal groups, factors and multiples, and the various properties of multiplication; and
- Articulate a conceptual understanding of a range of multiplicative ideas in a connected way with explicit language and terminology (Hurst, 2017).

Development of the project

This project on children's multiplicative thinking began in Western Australia in 2014. To date, researchers have worked with teachers and students from 16 schools in Perth, Western Australia; Dunedin, New Zealand; and Plymouth in the United Kingdom. Approximately 120 teachers and 1900 students have been involved. The key finding from the project so far is that primary school students have the capacity to think multiplicatively and make multiplicative connections. However, whilst all students have demonstrated an understanding of some aspects of multiplicative thinking, only a handful have shown an understanding of how different aspects relate to and inform each other. To this end, the researchers developed a suite of nearly two hundred multiplicative thinking tasks which form part of a professional learning package. Data gathering instruments have also been developed and progressively refined over four year life span of the project. These comprise a Multiplicative Thinking Quiz (written) and a series of semi-structured interviews.

The model for multiplicative thinking and the written instrument were informed by one another. Initially, a comprehensive semi-structured interview was used to gather data. However, while the interview generated very rich data, it was time consuming. Hence, the written Multiplicative Thinking Quiz (MTQ) was developed. This could be administered to a whole class in less time than it took to interview one student. The original model was based on several ideas – numbers of equal groups and the inverse relationship between multiplication and division, the notion of 'times bigger', the properties of multiplication, and the development of written methods for calculating. The MTQ questions were based on these ideas. It became apparent that several sets of connecting ideas existed and a new model was constructed.

A model for multiplicative thinking

Four sets of connecting ideas were identified and have been termed Connections 1, 2, 3, and 4. The MTQ questions are clustered into four sections, each of which corresponds to one set of connecting ideas. Connections 1, based on the multiplicative situation, is shown in Figure 1.

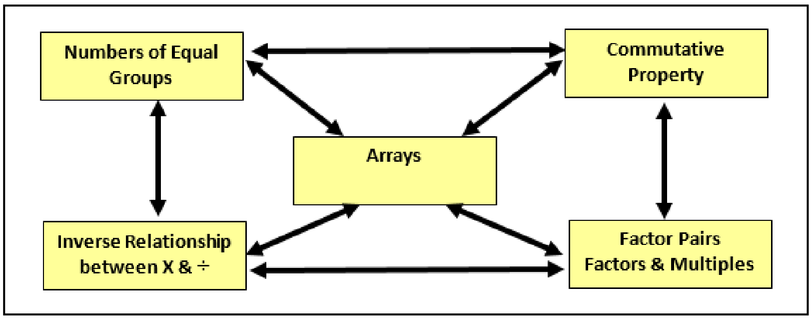

Figure 1: *Connections Set 1 of the model for multiplicative thinking*

Other sets of connecting ideas that comprise the model are based on the following ideas:

- Connections 2 – place value partitioning, and the distributive property of multiplication
- Connections 3 – the 'times bigger' notion, extended number facts, and movement of digits across places.
- Connections 4 – ratio and proportional relationships

Unfortunately, the scope of this paper does not enable the presentation of the complete model.

The Multiplicative Thinking Quiz (MTQ)

The MTQ contains 18 questions, each of which is designed to gather data about one set of Connections 1, 2, or 3. At the time of writing, quiz questions for Connections 4 had not been written. Typical questions from the MTQ based on Connections 1 include the following:

- What is the answer to 4 × 3? Using dots, crosses or something similar, draw a picture to show what 4 × 3 means.
- What is the answer to 8 × 7? What do the 8 and the 7 tell you? Write a story problem about 8 × 7.
- Write as many factors of 30 as you can? How do you know they are factors of 30?
- Which of the following will give you the same answer as 6 × 17? [16 × 7] [7 × 16] [17 × 6]. How do you know?

Parallel versions of the MTQ were developed to enable it to be used as both a pre-teaching and post-teaching assessment.

Targeted semi-structured interviews

The format for interviewing students has changed from a full interview covering the complete model for multiplicative thinking to shorter interviews that target a particular idea. Students' understanding of a specific aspect identified from the MTQ can then be probed more fully. One such idea targeted in this symposium is the use of materials to demonstrate or explain what is happening in multiplication. Students are asked to work out the answer for a one digit by two digit multiplication example, such as 7 × 15. They are then asked to show what is happening by using sets of bundling sticks, some of which are pre-bundled in tens. If children are unable to calculate say 7 × 15, they are offered an easier task, such as 4 × 9.

Teacher professional learning

The project has a professional learning model aimed at developing teachers as action researchers. Teachers are initially asked to analyse some student work samples taken from previously completed interviews and quizzes, to reflect on what the samples indicate about a student's mathematical understanding, and suggest what interventions are needed to help the student progress. This is followed by discussion and sharing of responses. The teachers are engaged in professional learning based on the multiplicative thinking model, trained to administer the MTQ, and to record the students' responses. Spreadsheets have been developed for this purpose and guidelines for recording data are also provided to ensure that the data recorded are reliable and consistent. For example, in relation to the above question about factors (Write as many factors of 30 as you can? How do you know they are factors of 30?), teachers are asked to record how the factors are written. That is, has a student written them in the format 1, 2, 3, 5, 6, 10, 15, 30, randomly as 5, 10, 2, 30, 1, 6, 15, 3, or in pairs (1, 30) (2, 15) (3, 10, (5, 6).

The model was initially implemented over a period of five months during 2017 with some 16 teachers from the Dunedin area of New Zealand. Teachers were supported throughout by regular visits from 'academic critical friends' who facilitated discussions and reflections in cluster meetings. Particular aspects of multiplicative thinking were identified from the initial data gathering and specific tasks were chosen from a bank of tasks created for this project. Teachers and their academic critical friends used student work samples gained from these tasks as the basis for discussion and further planning. Teachers compiled reflective notes throughout the five months and participated in a whole-day final debriefing professional learning session, one aim of which was to reflect on the data gathered from the parallel MTQ and to consider gains made by students.

Conclusion

The project has generated a lot of data that suggests very strongly that children have the capacity to think multiplicatively but that their knowledge is fragmented and unconnected. Askew et al. (1997) conducted some seminal research about the most effective teachers of numeracy having a 'connectionist' orientation. Results from this project suggest that this is the case and that teachers can be assisted to hold their content knowledge in a more connected way in order to be able to make connections explicit for their students. Early indications are that the professional learning model has changed teachers' practices about teaching multiplicative thinking and the data gathered by the teachers involved indicate that these changes in practice are translating into improved outcomes for students.

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Growth in Students' Multiplicative Thinking: Evidence from the Data

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This paper examines the learning by students who were participating in a project designed to promote multiplicative thinking. We used parallel pre and post quizzes to investigate their learning of connections within the multiplicative situation, place value partitioning, the distributive property, times bigger, extended number facts and movement of digits across places. Overall there was a highly significant increase in student attainment with a very large effect size. Items in the quizzes that related to students' explanations of the mathematics showed particularly large gains in attainment.

Multiplicative thinking is much more than just repeated addition and involves understanding of the multiplicative situation, including relationships between multiplication, division, fractions, factors, multiples and products. It also involves understanding of the commutative, associative and distributive properties, the ability to work with the relative magnitudes of large and small numbers, and use of appropriate language (Hurst, 2017; Siemon, Bleckley, & Neal, 2012). This view of multiplicative thinking places great importance on the connections between related ideas and conceptual understanding rather than just procedural knowledge. Multiplicative thinking is a fundamental component of proportional reasoning, fractions, decimals, ratio, statistical sampling and algebraic reasoning (Siemon, Breed, Virgona, Dole, & Izzard, 2006), and is therefore extremely important for learning mathematics. Crooks, Smith, and Flockton, (2009) found that only 56% of New Zealand Year 8 students could correctly calculate 39×6 and only 26% of Year 4 students could correctly calculate 19×4. For the Year 4 students who showed their working, half used an additive rather than a multiplicative strategy. In this part of the study the research question addressed is, "What was the impact on student achievement of engagement in tasks designed to promote multiplicative thinking?"

Methods

Twenty-three teachers from schools in the Dunedin area of New Zealand originally applied to participate in the project, but only 16 provided full sets of student data. Two hundred and forty-two students from these sixteen classes consented to take part in the study and completed both pre and post assessments. The students were predominantly from Years 5 and 6 and nine of the classes were composite Year 5 and 6 classes (ages 9 to 11). However, nine percent of the students were from Years 3, 4, 7 or 8 because seven of the classes were multi-level. The students were taught by their teachers, who were participating in the *Multiplicative Thinking Project* during Terms 2 and 3 of 2017.

Diagnostic assessment data, in the form of a written quiz, were gathered at the start of the study. The quiz assessed not only students' knowledge, but also their language and their explanations of the mathematics. There were three sections to the quiz: Connections 1 - the multiplicative situation; Connections 2 – place value partitioning and the distributive

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property of multiplication; Connections 3 – times bigger, extended number facts and movement of digits across places.

During Term 2 and early Term 3 the students engaged in tasks related to multiplicative thinking. All teachers used at least twelve of the provided tasks with their students. Most teachers used more than one lesson for some of the tasks and some teachers used more than twelve tasks to meet the learning needs of their students. All teachers used some of the *Bags of Tiles* tasks discussed in the results below. At the end of Term 2 or early in Term 3 a parallel written quiz was administered to provide summative assessment data.

Diagnostic assessment

There was a huge range of achievement in the pre quiz, with students attaining total scores between 0 and 49 out of a maximum of 60 correct or acceptable answers (M=24.5%, SD=19.3%). There was also considerable variability between items, with the number of students answering the item correctly ranging between 6 students (2.5%) and 189 students (78.4%).

There was little difference between students' percentages of correct or acceptable responses in the three different sections of the pre-quiz (Table 1).

Table 1
Students' scores on pre-quiz sets of connections

Quiz section	Description	Mean %	S D %
Connections 1	The multiplicative situation	21.0	14.4
Connections 2	Place value partitioning and the distributive property of multiplication	24.4	21.0
Connections 3	Times bigger, extended number facts and movement of digits across places	20.7	22.6

The items with the highest frequencies of correct responses were related to knowledge of multiplication facts and performing calculations, e.g., 8×7 (67.8%), 6×17 (42.3%) and 16×10 (59.5%). The items with the lowest frequencies of correct or acceptable responses were related to students' explanations of the mathematics: e.g., description of 8×7=56 in terms of factors and/or multiples (9.1%); explanation of an alternative strategy to place value partitioning for 6×17 (6.2%); and explanation of 16×10=160 in terms of powers of 10 and/or digit movement (7.9%).

The diagnostic assessment information informed the teachers' choices of tasks and approaches to teaching. Each teacher chose different tasks to address the learning needs of their students but all teachers were focused on developing students' understanding of multiplication rather than learning facts and procedures.

Tasks used and links to the sets of connections

Within the limited scope of a symposium paper it is not possible to describe all the tasks used by the teachers. However, four of the original twelve tasks provided were in the *Bags of Tiles* series and were based on students using 2cm×2cm square plastic tiles to construct arrays and addressed objectives from Connections 1 and 2. In each task students were provided with a given number of tiles, e.g., 24, and asked to make an array with no tiles left over. The different arrays were compared and teachers modelled the descriptions using correct mathematical language of factors, multiples and products. Each *Bags of Tiles* task was focused on a different issue, including the commutative property of multiplication and division, relationships between fractions and division, square numbers, prime numbers, and the distributive property of multiplication and division. The overarching goal of these tasks was to provide students with an understanding of the rich connections within the multiplicative situation. The use of this meaningful concrete representation was intended to move students from a perception of multiplication and division as separate facts and procedures to an appreciation of the multiplicative situation as a coherent whole

Overview of students' results

There was a highly significant difference in student attainment on the sixty parallel items in the pre-quiz (M=14.7, SD=11.6) and post-quiz (M=30.1, SD=13.1), using a paired t-test t(241)=23.7, p=0.000. The effect size of this comparison was extremely large (Cohen's d=1.52) and it can be concluded that participation in the project had a major impact on students' multiplicative thinking.

This improvement in attainment was observed in all three sets of connections (Table 2). Paired t-tests revealed highly significant differences in students' scores on all three sets of connections and Cohen's d demonstrated large or very large effect sizes.

Table 2
Pre-/post comparisons of sets of connections

Quiz section	Mean (pre)	S D (pre)	Mean (post)	S D (post)	Paired t(241)	Cohen's d
Connections 1	4.8	3.3	10.1	3.9	20.5*	1.31
Connections 2	2.7	2.3	4.7	2.5	14.2*	0.92
Connections 3	6.6	7.2	14.4	8.2	18.6*	1.20

* p=0.000

Main themes of improvement

A more informative picture of the improvement of students' multiplicative thinking was obtained by examining the items on which there were the largest increases in the number of students answering correctly.

Table 3
Frequency of correct responses

Quiz section	Item(post-quiz in parentheses)	Pre-quiz	Post-quiz	Increase
Connections 1	*In the number sentence 7×5=35 (9×7=63), identifies which numbers are factors	41	161	120
Connections 1	Represents the number fact 4×3 (5×4) as a multiplicative array	54	158	104
Connections 3	*Identifies 400 (700) as 10 times bigger than 40 (70)	75	177	102

Connections 1	Writes an appropriate story for 8×7 (7×6)	87	189	102
Connections 3	*Writes "÷10" or "×0.1" to transform 30 into 3 (60 into 6)	89	183	94
Connections 1	Identifies 144÷6 (135÷5) and 144÷24 (135÷27) as inverses of 24×6=144 (27×5=135)	103	182	79
Connections 2	Calculates 6×17 (9×14) using mental calculation	102	165	63

* Similar items with similar increases have been omitted for brevity

It can be seen from Table 3 that there were substantial increases in the number of students giving correct responses to questions requiring appropriate use of mathematical language, use of arrays to represent multiplication, interpretation of times bigger, inverse operations, and use of the distributive property. During Workshop 1 all of these categories had been identified by the teachers as aspects of multiplicative thinking that they wished to address. It is also interesting to note that students' knowledge of the basic fact 8×7 (7×6) increased from 164 students to 213 students, even though the teachers had not been focusing on basic facts.

Conclusions

There were substantial gains in achievement on all assessment items, clearly demonstrating that students' knowledge and skills related to multiplication and division increased through engagement in the tasks. There were particularly large gains in achievement on items related to students' explanations of the mathematics, rather than on items related to facts and procedures. The tasks used in the project provided opportunities for students to make connections, use mathematical language, use concrete materials as tools, and engage in worthwhile tasks. Anthony and Walshaw's (2009) summary of effective pedagogical practices argues that providing the opportunities described above are components of good practice and so it is perhaps not too surprising that the students made such substantial progress in their multiplicative thinking.

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Using materials to support multiplicative thinking

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This paper reports on a small aspect of a large multiplicative thinking project, which first started in 2014 in Western Australia. The research question 'How does the use of materials impact on multiplicative thinking?' is a focus of this iteration, which took place in sixteen classes in New Zealand schools. This paper evidences the benefits of using materials when solving and communicating problems, and describes some issues that arise through unintended consequences of using materials.

Using materials in New Zealand classrooms

Materials, which encompass concrete materials, manipulatives, equipment and fingers, have been widely used in New Zealand classrooms for some time. The introduction of The Numeracy Development Project (NDP, 2001) was an impetus to a dramatic change in teachers' mathematics practice in New Zealand schools. Part of the NDP was to promote quality teaching with tools and materials and this was pivotal to the success of the students' achievements. Materials and professional development were provided to help teachers understand the conceptual development in students' thinking and to offer them an effective model for teaching strategic thinking in number. Today's classrooms are very different from the latter half of the 20th century with rooms now reflecting the value of mathematics in the environment, with students discussing mathematics and using materials (personal communication, Peter Hughes, February, 1999). The picture looks great but a question does arise. Do the teachers and students have the same understandings about the purpose of the materials or are they working at a tangent with their differing thoughts?

Solving problems with materials is nothing new; indeed materials have been used for centuries and many cultures had some form of counting using materials. The benefits of materials for learning mathematics, over many decades, are lauded in researched or synthesised papers written with the focus of materials in mind (e.g., Kinzer and Stanford, 2014; Higgins, 2005) and more with the use of materials to develop multiplicative thinking as the focus (Boaler, 2017; Jacob & Mulligan, 2014). Anthony and Walshaw (2007) support the use of tools and representations as one of the ten principles of effective pedagogy in mathematics (p. 23). Black (2013) backs up that idea as she espouses the use of materials as "a powerful tool to support sense making, mathematical thinking and reasoning when they are used as tools to support these processes rather than adjuncts to blindly following a taught procedure to arrive at an answer" (p. 5).

There is some critique around the use of materials. Black (2013) questioned whether the materials are used in a way where they are "perceived as being central to the early development of mathematical ideas especially for children aged under 11" (p. 3). This aligns with Piaget's well-known concrete operational stage. When researching the use of materials in some schools in the NDP in NZ, Higgins (2005) questioned the use of materials used in a procedural way and suggested the schools followed an algorithmic way of learning, in contrast to the consistent messages in the NDP. In her paper Higgins, whilst

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positive about the use of materials, queried the "purpose of the equipment designer and the teachers' purpose in using the equipment" (p. 95). Both do not necessarily coalesce with the best outcomes. Furthermore using 'hands-on' materials does not necessarily mean students have their 'minds on' developing the mathematical concepts the materials are designed to engender.

Using materials to support multiplicative thinking

Materials are particularly useful to support children's multiplicative thinking. NCTM (2017) provided a comprehensive list of how materials can be used in mathematics classrooms. The following eight resonate particularly with aspects of multiplicative thinking:

- distinguishing patterns—the foundation for making mathematical generalisations;
- understanding the base-ten system of numbers;
- comprehending mathematical operations such as addition, subtraction, multiplication, division;
- recognising relationships among mathematical operations;
- engaging in problem-solving;
- representing mathematical ideas in a variety of ways;
- connecting different concepts in mathematics; and,
- communicating mathematical ideas effectively.

Methodology

To answer the question 'How does the use of materials impact on multiplicative thinking?', we used a design research methodology (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006) because we were engineering and systematically studying the students' learning related to multiplicative thinking after furnishing their teachers with professional development on a multiplicative model and a set of related tasks. The set of tasks were structured to include instructions and suggestions for pedagogy, including the use of materials (see Offen & Ingram, this issue for further information). A model for multiplicative thinking was used as a signpost to ascertain the multiplicative understandings they have in regard to four sets of connections (See Hurst, this issue for detail). Data was drawn from sixteen teachers and their 242 (Y3-Y8) consenting students in schools in urban Dunedin, New Zealand. Data was collected through assessments of children's multiplicative thinking, teacher feedback and teacher progressive feedback when using a set of multiplicative tasks provided. In class, when the students worked on the tasks, the teachers made a wide variety of materials available to the students. A further source of data was that 43 students from Years 3 – 8 were interviewed early in the study to elicit student thinking, particularly multiplicative reasoning, to gauge the connections between their calculations and materials. Each teacher was asked to nominate a high, middle and low achieving student from their class. During the interviews the students were asked to represent problems using bundles of sticks. The length of the interview depended on the level of questions the student answered. Once they showed evidence of lack of understanding the interview was immediately stopped. The data relating to materials was qualitatively analysed to explore students' representation of multiplicative thinking with materials.

Results

First, students' use of materials during the interview is related to their multiplicative thinking. Then, how materials were used during tasks to develop multiplicative thinking is explored through the teachers' perceptions.

Using bundles of sticks to represent 23×4

Twenty of the 43 students interviewed reached the stage when they were asked to solve 23 x 4. All but one student answered correctly. Most students used place value partitioning mentally or thinking through jottings. One student used compensation (25 x 4) – (2 x 4) successfully. Two used an algorithm. If the 19 students had sat a pen and paper test they would have been marked as correct. However, there were differing responses when asked to demonstrate a representation for 23 x 4 using bundles of sticks and only six were successful. The two students who used an algorithm were not successful.

Most students seemed unfamiliar with modelling mathematics using bundles of sticks. This inability to demonstrate representations was alarming as classrooms have many materials to support student thinking since the NDP and there is an expectation students should be very familiar with representing mathematics through materials. The students sampled indicated that, generally, there is a mismatch between the purpose of the use of materials and students' representation of multiplicative thinking. This reinforces the idea of Higgins (2009) who questioned whether the purpose of the materials was the same as the purpose teachers had in mind.

All students who successfully solved 23 x 4 were asked to represent the problem using bundles of sticks. There was a range of ways the student cohort failed to model the problem when using materials. For example, Student A simply modelled the answer of 92 with 9 tens and 2 ones rather than using the materials to support multiplicative reasoning during the process of problem solving. Student B incorrectly modelled the answer of 92 using 9 ones and 2 ones rather than 9 tens and 2 ones. Student C laid out 4 ones separate from 2 tens and 3 ones, and then laid out 9 ones and 2 tens to represent the answer. Student D did not see the purpose of materials and said "I could do it easily in my head but I thought you wanted to know all that", even though he correctly modelled the problem. Interestingly all of the six students who modelled 4 groups of 2 tens and three ones were also able to link the notion of times bigger to 400 x 23**.** A typical answer was "23 x 4 is 92 so 400 x 23 is 9200 because 400 is 100 times bigger than 4".

Using materials with the tasks

During the study the teachers saw the benefits of using materials to explore and demonstrate multiplicative ideas. Although there was some confusion with which materials to use for a particular situation, evidence from the teachers' feedback showed the use of materials benefitted the students' multiplicative thinking as they presented the tasks with more of an onus on discovery and investigation. This was particularly evident when the teachers provided children with square plastic tiles to create arrays.

> Using arrays is an extremely powerful way to show relationships. (Carol)
>
> Tiles were particularly helpful for visualising number sentences and all their related facts. (Diane)

In final feedback teachers realised previously there was a lack of using materials in developing multiplicative thinking in their classrooms and resolved to make better use of the tools.

> Realised I wasn't using them often enough. (Ginny)
>
> Will use more materials in my teaching so students can discover the answers for themselves. (Diane)

A change in the teachers' attitudes towards using materials in those ways was a pleasing result. However, one teacher clung to the idea "it was great to use materials to *show* the children" (Lisa), confirming Higgin's (2005) concern about using materials in a procedural way.

Students' multiplicative thinking improved dramatically in this project (see Linsell, this issue for details). There is evidence that the use of materials supported students' understanding of the multiplicative aspects particularly the aspects of place value partitioning, times bigger, the relationship between multiplication and division, communicating mathematical ideas effectively, and connecting different concepts. The more the children worked with materials, the more their confidence grew and the more risks they were likely to take. Unfortunately, the length of this paper limits reporting further results (see Hurst & Linsell for further results, sent for review).

Conclusion

Materials are a useful tool to support children's mathematical learning, and particularly pertinent to this project, their multiplicative learning. Through their involvement in this project, the teachers saw the benefit of using materials in the classroom and using appropriate materials with relevant tasks. The tasks provided by this study were designed so that, when used with materials, mathematical connections were woven in a sense making tapestry, confirming the tasks are so much richer when used with materials to support children's multiplicative thinking. If materials are in constant use in a classroom it is hoped that children will become self-motivated to choose the appropriate materials that will support their thinking. There has to be a consistent understanding of what the use of materials stand for and the teachers' flexibility and knowledge of how to use them. These elements have to work in harmony for effective practice and student growth in developing robust multiplicative thinking.

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Teachers' perspectives of tasks designed to promote multiplicative thinking

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In this paper, teachers' perspectives of their involvement in a multiplicative thinking project are explored in terms of their growth and their use of a set of specifically-designed tasks with their classes. Teachers reported growth in their own content and pedagogical knowledge. Furthermore, the use of the tasks gave them more confidence in teaching multiplicative thinking strategies and demonstrated the connections between aspects of multiplicative thinking. Teachers also identified a growing awareness of the use of explicit language and the importance of using materials from engaging with the tasks in this project.

Being multiplicative is a vital component of being able to think mathematically (Siemon, Breed, Virgona, Dole, & Izzard, 2006), yet students, and teachers, find working multiplicatively can be very challenging (Young-Loveridge, 2007). To best support children's multiplicative thinking, teachers need strong content and pedagogical knowledge (Ball, Hill & Bass, 2005), i.e., a connected understanding of the multiplicative situation, and effective pedagogy such as questioning, demonstration, discussion, reasoning, investigation, and interaction (Askew, 2016). Teachers need to source tasks that support children to grow their conceptual understanding (Sullivan, Clarke & Clarke, 2013) through deep thinking about the mathematical ideas (Anthony & Walshaw, 2009).

In earlier research Hurst (2017), working with various others, developed a model for multiplicative thinking based on four sets of connecting ideas (see Hurst, this publication). In this iteration, we sought to grow teachers' understanding of this model through professional development and the provision of a set of tasks related to the multiplicative situation to use in their classrooms. This paper reports on the teachers' perspectives of their own mathematical and pedagogical growth through the use of these tasks.

Method

This research can be considered 'design research' because, by giving suggestions for the content and structure of the tasks, a particular form of learning was being engineered and systematically studied (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006). This research was iterative in that it drew on learning from previous projects with Australian and English schools (e.g., Hurst, 2017).

Sixteen New Zealand teachers of Year 5-8 students attended two professional development days, where teachers learnt about the connections between aspects of multiplicative thinking and the importance of making these connections explicit with their students. Further to those days, teachers were supported by regular cluster meetings with the researchers; 'academic critical friends' who facilitated discussions and reflections.

The teachers chose tasks from a set of suggested tasks to use with their Year 5-8 students. The tasks were specifically designed to connect students' understanding across specific aspects of the multiplicative model. These tasks, which were variously sourced, were re-structured to include suggestions of: how to administer the task; what to look for in students' actions and responses; how to phrase questions and how to develop the task,

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depending on children's responses. Further to the task instructions, each task also contained teacher notes that explained the mathematical concepts, links to specific aspects of the multiplicative model and the mathematical language targeted by the task. A feature of each task was students were required to explain their thinking using concrete materials.

Data were drawn from the progressive feedback teachers gave during professional development days, field notes taken during the cluster meetings, a teacher questionnaire at the beginning of the project and further reflections on these questionnaires at the end of the project. Analysis of the qualitative data was based on grounded theory (Strauss & Corbin, 1998), where the data were coded into categories of themes that developed, which were related to the teachers' growth in knowledge, and how their pedagogy changed, namely the connections they made between the different aspects of multiplicative thinking, the use of materials, and finally, the mathematical language they used when teaching multiplicative strategies. The results, presented in the next section, are evidenced by representative quotes from the teachers. Pseudonyms have been used to protect the identity of the teachers.

Results

We know that students learnt more about the multiplicative situation (see Linsell, this publication). There was also qualitative evidence that both the teachers' mathematical content knowledge and pedagogical content knowledge related to multiplicative thinking were enhanced through this research, and change in their teaching practice.

Growth in knowledge

Within the data, 86% of teachers referred to an increase in their knowledge related to multiplicative teaching as a result of their participation in the project. The emphasis some teachers placed on the connectivity between the multiplicative aspects demonstrated that teachers understood the big idea of the multiplicative model.

> [My] own knowledge of multiplicative thinking has improved. (Danielle)
>
> I have learnt about strong relationships between all principles in multiplicative thinking. (David)

This growth in content knowledge had an flow-on on teachers' increased confidence in their teaching of the multiplicative situation and the use of the tasks.

> [Being in the project] helped me with my own understanding – what is important to teach re: multiplicative thinking. (Rose)
>
> [Being in the project] I have deepened my knowledge of how to teach mult/div thru [sic] using different materials and the tasks provided. (Stacey)
>
> The [tasks provided] scaffolding for teachers. (Lucy)

Indeed, the teachers suggested the structure of the tasks, the accompanying instructions and the teacher notes gave teachers confidence in teaching multiplicative thinking strategies with their students. Importantly, the teachers reported they felt more confident when teaching multiplicative strategies when using these tasks.

Furthermore, the cluster meetings and discussions with teachers from other schools meant teachers could reflect and grow their understanding of the mathematical content they were teaching. The teachers had the opportunity to discuss how they were using the tasks and ask questions. This meant that teachers had support in interpreting the tasks and their purpose, as well as getting ideas from each other.

Change in teaching practice

The professional development and the structured nature of the suggested tasks also changed the teachers' practice, especially in the areas of making connections, using materials, and using language to the multiplicative situation. It should be noted first, however, that some teachers continued to rely heavily on the task instruction, even when tasks were adapted for a different purpose. For example, tasks that were intended for smaller numbers were also used for problems with larger numbers. This caused children spending an inordinate amount of time organising the tiles individually and not enough tiles available for children to manipulate. This meant, in some cases, teachers reverted to abstract thinking or procedural steps before children had mastered the conceptual understanding. In contrast, teachers who were focused on the intent of the task, rather than the task itself, either used a different task, or adapted the materials accordingly. For example, when the problems used larger numbers, teachers used grid paper or dotty arrays.

Making connections. The connections the teachers made between aspects of the multiplicative situation changed their teaching practice.

> I'm making connections and not teaching [multiplicative strategies] in isolation. (Danielle)

> I've learnt to make connections between fractions, decimals, ratios, percentages and mult/div wherever possible. (Joanne)

The tasks helped teachers make links within and between the four connecting ideas of the multiplicative thinking model and other mathematical concepts. For example, some teachers noted the tasks fed naturally into learning about measurement, and especially area. For others, the natural link was with other aspects of multiplicative thinking. It was also noted in feedback, that as well the tasks lending themselves to natural connections in multiplicative thinking, they also discouraged multiplicative concepts being taught in isolation. Furthermore, teachers who planned to use the tasks for a three-week block as part of the project continued to use them for a term or more.

Using materials. The sense of making connections was enhanced by the visual nature of the tasks, and the use of materials.

> It all makes sense. I was going to teach fractions later in the year, but I could actually see how the tasks and the materials would just naturally flow into this. (Joanne)

The teachers used a variety of concrete materials to support their children's multiplicative thinking (see Holmes, this publication). The tasks highlighted for teachers that "even older kids benefit from using materials" (Robert). It was a surprise to the majority of teachers that initially, students who could accurately solve problems procedurally were unable to offer either a conceptual explanation or demonstrate the nature of the task using materials.

> I knew they could answer the questions correctly, but when I asked them to explain to other students, they stuck to step-by-step procedures and couldn't explain why those procedures were in place or show how they worked using the tiles. (Joanne)

After teaching using the tasks and the materials, Joanne noted that children were able to "demonstrate what was happening in their heads". Having children demonstrate using materials showed what they didn't know, as much as it showed what they did know, which was helpful for the teachers identifying misconceptions and determining next steps. Another key element of children being able to demonstrate what was 'happening in their heads' was their use of specific mathematics language to explain their ideas.

Using the language of mathematics. Teachers commented on the growth of the specific language of multiplicative thinking by students as a direct result of teachers using the tasks. Each task explicitly emphasised multiplicative language specific to the task (for example; factor, multiple and product). Feedback from teachers showed explicit use and teaching of the specific language enabled children to more clearly describe their thinking.

> I always thought that if they could do the maths, the language wasn't too important, but I could see kids had more confidence and clarity explaining the concepts when they knew the language to use. (Jane)

> I could tell the kids were really excited to be using real 'mathsy' words, and they used them whenever they could. (Lucy)

From a pedagogical viewpoint, teachers found the structure of the tasks with explicitly targeted language reinforced or alerted them to how important the language of mathematics is for children to explain their thinking.

Conclusion

The teachers responded positively to their participation in the multiplicative thinking project. Their content and pedagogoical content knowledge grew, and this had a positive affect on their teaching practice, particularly in the areas of making connections, using materials and using specific language. The professional development in this project not only met these criteria but also provided tools and a structure to implement the professional development. Teachers' overall conclusion of the professional development component of the project was they felt empowered and confident when teaching multiplicative strategies using the tasks. Mathematical language was enhanced and teachers noted students were more confident when explaining their thinking when they knew the correct language to use. Furthermore, the tasks enabled teachers to make connections of how multiplicative thinking linked naturally to other areas of mathematics.

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