"Because 7 and 8 are always in all of them": What do Students Write and Say to Demonstrate their Mathematical Fluency?

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Fluency is an important aspect of mathematics learning and plays a major role in developing proficiency as students are required to use skills and apply knowledge. This paper draws upon findings from a large project that aimed to explore primary students' mathematical fluency. Data from 160 primary students (K-6) from a NSW Departmental school were analysed as part of a process to test the efficacy of a Fluency Framework. Results showed that students displayed the proposed characteristics suggested in the framework and that similar characteristics were observed across all grades. These results indicate that the framework, once refined, will be an effective tool for teachers to use in identifying students' fluency.

Students' fluency with basic number facts and mathematical procedures has long been a focus in primary classrooms and related teaching resources. The term *procedural fluency* used by Kilpatrick, Swafford and Findell (2001) can however be problematic when interpreted as simply being able to follow a set formula or to compute mathematics quickly. Kilpatrick et al. (2001) recognised this issue affirming that "one of the most serious and persistent problems facing school mathematics in the United States is the tendency to concentrate on one strand of proficiency to the exclusion of the rest" (p. 11). When procedural fluency is focused on in isolation from the underlying conceptual understanding it can be detrimental to a student's skill and knowledge development in mathematics. The importance of the relationship between procedural fluency and other aspects of mathematical proficiency is also emphasised by Kilpatrick et al. (2001) who indicate that "as a child gains conceptual understanding, computational procedures are remembered better and used more flexibly to solve new problems" (p. 134). Foster (2017) stated that the "development of robust fluency with mathematical procedures" is a current focus of the UK national curriculum "developing procedural fluency and conceptual understanding" simultaneously (p. 122). When exploring the conceptualisation of procedural fluency, Graven and Stott (2012) found that "where flexibility and efficiency were high, conceptual understanding was progressively intertwined with procedural fluency and the distinction between these strands became increasingly murky" (p. 6).

A broader interpretation of fluency is important so teachers can build students' fluency not only in procedural knowledge, but also in understanding and in their use of appropriate strategies. *Mathematical fluency* involves carrying out procedures flexibly, accurately, efficiently and appropriately *as well as having* "factual knowledge and concepts that come to mind readily" (Watson & Sullivan, 2008, p. 112). For the remainder of this paper the term 'fluency' is in reference to the broader term, mathematical fluency, unless otherwise indicated. Watson and Sullivan's (2008) definition combines both the ability to perform the mechanics of mathematics (procedural) and the understanding of the mathematics being learned (conceptual). Even though fluency is placed at the centre of the Australian curriculum (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2015) as a necessary aspect of students' mathematical development, little research exists that specifically observes students' mathematical fluency beyond procedural knowledge. Previous research studies have focused on procedural fluency with number and assess

2019. In G. Hine, S. Blackley, & A. Cooke (Eds.). Mathematics Education Research: Impacting Practice *(Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia)* pp. 156-163. Perth: MERGA.

procedural fluency through the use of time-restricted testing (Miller & Heward, 1992; Stott, 2013). A shift is required to focus not only on mathematical content such as number facts, but also on the mathematical processes that aid fluency. This thinking is consistent with Star's (2005) comments surrounding procedural and conceptual knowledge:

> Methods for assessing students' procedural knowledge are somewhat impoverished at present, with procedural knowledge often measured simply by what a student can or cannot do. Research methods can instead focus on how students can and cannot do and on the character of the knowledge they have (including its depth), which supports their ability to perform procedures (p. 8).

Mathematical fluency involves students' abilities to *use* procedures *flexibly* and *appropriately* indicating a need for *decision making* and *choice*. These features are generally not analysed in existing research regarding fluency, where the attention is on the *accuracy* and *efficiency* of answers (Dole, Carmichael, Thiele, Simpson, & O'Toole, 2018; Gallagher, 2006; Mong & Mong, 2010). Further research is needed surrounding students' *choice* and *use* of their knowledge as an indicator of fluency. Hopkins and Bayliss' (2017) research discusses the importance of student choice of strategy to solve number tasks with confidence and accuracy. To observe mathematical fluency, extending *choice* to include decision making of not only the strategy but the numerical operation/s is required.

The aim of this paper is to provide evidence of the efficacy of the proposed fluency Framework. The research questions for this study are: Which of the proposed characteristics of mathematical fluency are observable in students' work samples (written and verbal)? and What, if any, additional characteristics of mathematical fluency are observable?

Conceptual Framework

The objective of the current study was to validate and refine the Characteristics of Fluency Framework (Table 1) to advance knowledge of mathematical fluency through the exemplification of each characteristic theorised. The Framework comprises the teacher-reported characteristics that emerged as themes from teachers' descriptions of what mathematical fluency looked like in their students (Cartwright, 2018). It is critical that the characteristics be observed in students to validate the Framework if it is to be utilised by teachers. The development of a stable, illustrated set of characteristics will be a significant contribution to current research in mathematics. The Framework will provide teachers with identified aspects of student fluency that may need strengthening or extending through appropriate teaching instruction.

Table 1
Characteristics of Fluency Framework (Cartwright, 2018)

Strategic competence	Conceptual understanding	Adaptive reasoning
Multiple strategies	Comprehension	Justifying strategy or method (the why)
Variety of strategies/ ways	Making connections between concepts (known to unknown)	Transfer to other contexts or problems (application into new situations)
Choice of/ identification of appropriate strategy	Flexible use of numbers and their relationships	Self-checking method (reasonableness)
Accurate process (articulation)	Explanation of method (the how)	Working through errors
(Ease of) mechanics- *automaticity*	Sharing strategies [with peers] (communicate)	
Fluidity (switch between strategies)		

The Study

The overarching purpose of the research project is to investigate the characteristics of mathematical fluency students display, exploring which characteristics teachers notice and what instructional decisions teachers make to further develop students' mathematical fluency. The study reported here took an explanatory approach gathering student data as illustrations of fluency characteristics to help validate the Framework proposed by Cartwright (2018) (Table 1). Johnson and Christensen (2008) highlighted that an explanatory approach aims to "empirically test a model to determine how well the model fits the data" (p. 384). The intention of applying an explanatory approach was to discover if the set of characteristics formulated were observable and how well, as a set of observable features, they fit within student data.

This paper reports on results from the school that participated as a pilot school to 'road test' the Characteristics of Fluency Framework. The primary school (approximately 300 students) was located in a medium socio-economic (school ICSEA 1022) metropolitan area of Sydney. Seven classes of students participated in the study: A Kindergarten class (n=16), two Year 1/2 classes (n=35), a Year 3 (n=27), a Year 3/4 (n=23), a Year 4/5 (n=27) and a Year 6 class (n=32) (N=160 students).

Instrument and Procedure

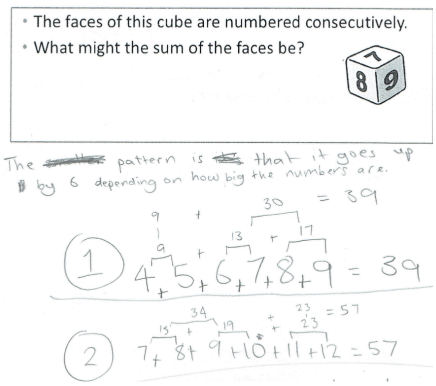
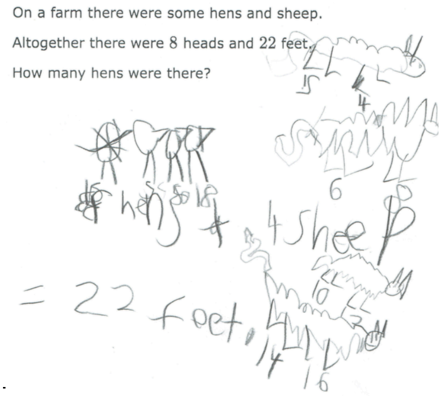
Student responses to problem solving tasks were the source of data. Problem solving tasks that have an element of challenge for students were utilised for the study. These types of tasks increase opportunities to observe mathematical fluency in action as they allow students to *choose* and *use* procedures and strategies. Russo and Hopkins (2017) made a similar point in their research where the tasks needed to be "engaging for students, have multiple solution pathways, involve multiple mathematical steps and take considerable time to solve" (p. 22). The tasks chosen related to number concepts as the majority of prior research on fluency focuses on number sense or the four operations. For example, "The faces of this cube are numbered consecutively, what might the sum of the faces be?" Students' oral (audio recordings), written and pictorial representations (work samples) were collected to support observation notes when analysing students' mathematical fluency. The tasks were conducted as part of whole-class mathematics lessons implemented by the researcher. The researcher taught the lessons for consistency of delivery as a number of the tasks were used repeatedly across classes or year groups. Individual work samples (N=160) were collected from all students and audio recordings were taken of a random selection of students per class explaining their solutions (n=57).

On a second visit to the school, one week later, the researcher taught another mathematics lesson in each class. During these follow-up lessons, students worked on a different task in small groups, recording their work on large sticky-note posters. The researcher used the "Explain Everything" iPad app to record randomly selected groups' work as the students explained their thinking and solutions to the researcher. Only the analysis of the individual student work samples, recordings and observational notes (Lesson 1) will be discussed and reported on within this paper.

Data analysis

A deductive approach to data analysis was employed, looking for evidence that the characteristics of mathematical fluency presented in the Framework were identifiable in the students' work sample data. All 160 student work samples (grouped and analysed by class) were indexed against the Framework using the pre-defined characteristics as a set of codes. Work samples were viewed numerous times and fluency characteristics that were visible were tallied and recorded against a copy of the Framework. For example, where there was evidence that the student had crossed-out work and recorded another response this was coded as 'self-checking method'. The Framework was also used when analysing the transcripts of the individual student audio recordings. A spreadsheet was used to organise the transcript data, comments were recorded and characteristics noted. Analysis of students' use of diagrams and representations to display fluency has also been included as some characteristics were identifiable through different domains such as; written, spoken and drawn representations. Two student work samples are presented in Table 2 with analysis notes listing the characteristics of fluency that can be observed in each sample.

Table 2
Examples of how Individual Student written work samples were Analysed

<table>
<tr><th>Students' work samples</th><th>Characteristics of fluency observable in the sample</th></tr>
<tr><td>
</td><td>Student uses an accurate process to find solutions, efficient strategies for addition are visible, multiple solutions are given, an explanation of the findings is written (including noticing of the pattern), diagrams are used to show partitioning and lines are used to organise different solutions, evidence of self-correction in the writing is visible.
Year 6 work sample</td></tr>
<tr><td>
</td><td>Student uses diagrams as part of working out, numerical and symbolic representations are used to label the working out of a solution, evidence of efficient strategy can be seen in the use of a count-by-twos method for counting the legs, evidence of self-correction (crossed-out hen) and words are used to write the solution.
Kindergarten work sample</td></tr>
</table>

Results

The results from the individual work samples are represented in Tables 3, 4 and 5. During coding it was necessary to modify the wording of a number of the characteristics for clarity. For example, 'Multiple strategies' was removed as it was similar to 'variety of strategies'. The term 'appropriate strategy' was interpreted as 'efficient strategy' - for the task. These characteristics were further amended post analysis for future use in the broader research study to clarify these terms: 'appropriate strategy' (for the problem) and 'efficient strategy' (for the student's stage of learning). 'Transferring to other contexts' was removed for the analysis as students were not provided with opportunities to solve further problems in *new* situations. Transference may be better observed over time by the classroom teacher.

Analysis of Students' Written Work Samples

Data analysis and coding were recorded according to year/ class level and similar patterns in the results appeared, it was therefore decided to present the data holistically (Table 3). Similar characteristics of fluency were identified across all year levels with the exception of ‘justifying strategy’ that was only present in Year 6 samples, and ‘making connections’ that was not identified until Year 3 samples.

Table 3
Percentage of Students’ Written Work Samples Displaying Characteristics (N=160)

Strategic competence	%	Conceptual understanding	%	Adaptive reasoning	%
Variety of strategies/ ways	25.6	Comprehension (understands the task)	91.8	Justifying strategy or method (the why)	1.2
Choice of efficient strategy	50.6	Making connections between concepts (known to unknown)	23.1	Self-checking method (reasonableness)	35.0
Accurate process (articulation)	60.0	[Flexible] use of numbers and their relationships	44.3	Working through errors	31.8
(Ease of) mechanics-*automaticity*		Explanation of method (the how)	31.2		
Fluidity (switch between strategies)		Sharing strategies [with peers] (communicate)			

The majority of students (91.8%) across all year levels were able to comprehend the problem and 60% of students were able to use an accurate process. Although, only 50.6% of students were able to show evidence that they chose an efficient strategy to use in solving the task. Students who simply wrote a correct solution were unable to be coded as ‘choosing an efficient strategy’ as it was not clear if they had, for example, used a count-by-ones strategy to solve a more complex task. Additional characteristics also emerged that were not captured in the original Framework and are presented in Table 4.

Table 4
Percentage of Students’ Written Work Samples Displaying Characteristics not Identified in the Original Framework (N=160)

Strategic competence	%	Conceptual understanding	%	Adaptive reasoning	%
Multiple solutions	40.6	Describes thinking/ findings*	61.4	Locates a pattern in solutions	11.8
				Makes generalisation/ justifies findings (why)	1.2

* The tasks implemented in K and Year 1/2 classrooms did not provide scope for students to describe their solutions therefore this data only relates to n=109

During the lessons, the students were specifically encouraged to provide a written explanation of their method (31.2% in Table 3). In general, students were reluctant to write full sentences to support how they worked out their solutions. It was necessary to further refine the coded data regarding ‘explanation of methods’ as many of the students (61.6% Table 4) described their thinking or findings either instead of, or in addition to, writing an explanation of their method. As noted in both Table 3 and Table 4, only 1.2% of students were able to write about ‘why they chose their strategy’ or to ‘justify their findings’, further extending prompts may be required to be used by the teacher to ascertain these characteristics of fluency.

Table 5
Percentage of Students' Written Work Samples Displaying Domains Through Which Fluency was Communicated (N=160)

Discourse/ language	%	Representation	%	Visualisation	%
Writing to support numerical/ visual work*	76.2	Numerical and/or symbolic	90.6	Use of colour to separate solutions	15.6
Use of high modality words e.g. always, must	9.3			Use of lines/ sectioning off areas of work space	21.8
				Drawing of own diagram**	53.3

* This number is inclusive of written words/ sentences that may have been unfinished or incorrect
** Note that for the Year 1/2 and Year 3/4 tasks a diagram was provided and all students utilised the diagram; therefore, these data only relate to n=75

From Table 5 it can be observed that students used a range of domains to show their fluency with the majority (90.6%) using numbers and/or symbols to represent their working and solutions. One-quarter of K-4 students drew pictures utilising more than one colour whereas the use of more mathematical diagrams, for example, tree diagrams, were more common in Years 4, 5 and 6 (40% of student samples). A number of characteristics could not be identified in the written work sample analysis as they need to be observed at a moment-in-time or communicated verbally, for example, automaticity, fluidity, and sharing strategies [with peers].

Analysis of Students' Verbal Descriptions

From the audio recordings students' automaticity with number knowledge was identifiable by the way they spoke about their strategies and solutions with accuracy and confidence, as were students' abilities to flexibly use numbers and number relationships. For example, "So 6 plus 7 is 13, plus 8 is 21, and then plus 9 is 30 then plus 10 is forty and then plus 11 is 51" (Tobi, Year 6); and "So, this equals 7, but I knew that … so 6 … maybe 5 and 1, that might equal another 6, and 6 plus 6 equals 12" (Alex, Year 1/2).

The analysis of the transcripts also found that high modality words were used more frequently verbally compared with students' written work samples:

Elle: Well would a pattern be the same, would it still be a pattern if I'm saying there's a 15 in every [line] of the numbers?
Researcher: Right, so why is there 15 in every single one of them?
Elle: Because 8 and 7 are always in all of them (Year 4/5 student)

Clio: I found out that if you add all the numbers together, then always, then you always get plus two (Year 4/5 student)

Molly: So, what I'm doing is I'm going to first count how many little squares are in here. And then if there's 16, but then I also know the answer that's in this one, so then I know the answer will be the same in all the other ones. (Year 3/4 student)

Tom: 39 plus 6 equals 45 … plus 6 to 45 is 51 … plus 6 to 57 is the next answer, so they are all plus 6
Researcher: So why is the difference 6 each time?
Tom: That's what we are going to work out! (Year 6 student)

Discussion and Conclusion

When comparing the coded data across year groups students in Kindergarten and Year 1/2 (27%) were less able to use an accurate process compared with 75% of Year 3 to 6. This could be due to the tasks relying on students *applying* knowledge to an unfamiliar task, or could reflect students' lack of exposure to tasks of this nature. These students may have shown *procedural* fluency if provided with straightforward number fact questions. However, if mathematical fluency involves understanding and knowing when and how to *use* mathematics appropriately, more open-ended task experiences are required. Despite the challenge of the tasks, students in all classes persevered and remained in-task for the entirety of the lesson, attempting to find multiple solutions. Anecdotally, students confident with the mathematics of the task wanted to work with their peers despite it being an individual task. Some students used an accurate process for the operation they chose, but inappropriate for the task. The accurate process indicates some level of fluency with numbers, however, knowing the mechanics without the understanding of when to use them is not fluency (Russell, 2000). The task design also impacted the students' abilities to demonstrate their fluency. The Year 1/2 task was conceptually too difficult and many students were unable to find a solution. The tasks used with Years 4-6 involved a lower level of mathematics and therefore provided more space for students to spend time exploring solution patterns.

The additional characteristic of 'the use of high modality words' that emerged during the analysis was of interest. Language development affects students' ability to communicate their mathematical fluency. There is a "close relation between students' reading skills and mathematical reasoning competence [where] specific reading comprehension strategies are also needed" for proficiency (Segerby & Chronaki, 2018, p. 4). Students observed as having mathematical fluency used high modality words, particularly when speaking and reasoning about their solutions. This aligns with Chapman's (1997) findings where "as they become more certain of and confident with their mathematical meanings, so they establish a higher modality" (p. 170). Modality and the role language plays in the acquisition of mathematical fluency should be further investigated.

From the analysis, in deciding if a student was 'fluent', they needed to be able to show evidence of the *use* and *choice* of an efficient strategy. Either by (a) showing numerically and/or symbolically how they came to their solution, the steps – a numerical answer alone was not enough; (b) using written words to explain how they worked out a solution; or (c) verbally explaining how they worked out their solution. Some characteristics were evident in verbal but not in written form, ideally audio recordings and written samples should be analysed together to gain a full picture of a student's fluency. Students regarded as 'fluent' were able to use a range of representations and/or articulate verbally their strategies and solutions. Day, Stephens and Horne (2017) make similar observations regarding reasoning where students have the ability to "move fluidly between multiple representations" and have the "language and discourse to reason mathematically" (p. 655). Students observed as 'fluent' demonstrated adaptive reasoning, in particular their ability to self-check and take a different tack by working through errors along the way.

The Framework will continue to be refined as the students' group task work samples and recordings are analysed. Future investigations will also test the relationships between the categories within the Framework, confirming or refining the interrelatedness of the three categories when and if mathematical fluency is present in students. For example, where evidence can be seen for a student displaying conceptual understanding and adaptive reasoning, does this equate to mathematical fluency? Do students need evidence in all three categories to be considered 'fluent'?

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