Undergraduates' Uses of Examples in Introductory Topology: The Structural Example

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Example use in undergraduate mathematics has been extensively studied. However, little is known about undergraduates' example use in the context of introductory point-set topology. We present a case study of one undergraduate and describe her example use when constructing proofs in the context of introductory topology. We describe her use of a type of example previously undiscussed in the research literature, which we call structural examples. We conclude by drawing distinctions between structural examples and other, more specific forms of example.

The successful application of mathematics requires much more than the robotic computation and symbol manipulation many people think of as mathematics. On the contrary, mathematics is a highly creative field, often requiring a significant amount of ingenuity and the ability to think about concepts in novel ways. Expert mathematicians often use a sort of standard set of informal techniques to gain insights into new concepts; among these techniques are the production of visual representations and example generation (Carlson & Bloom, 2005). The generation and examination of examples can lead to deeper understanding of concepts, possibly by revealing a means of producing more examples or by leading to a clearer knowledge of essential behaviors and properties. Such deeper understandings can subsequently lead to the formalised algorithms and definitions we find in our textbooks. For professional research mathematicians, the identification of new algorithms, definitions, and theorems is often the ultimate goal of their work. Students, however, often have different goals in mind than the advancement of mathematics, such as simply getting enough of an understanding to pass their courses. Previous work has shown that students also reason using visual representations and examples to gain an informal understanding of mathematical concepts (Weber, 2004; Zazkis, Dubinsky, & Dautermann, 1996). However, such studies have not focused on the context of undergraduate topology courses. In this abstract field of study, examples, nonexamples, and counterexamples can be critical to understanding, yet we know little about how students in such courses generate or use examples. This study aims to add to our knowledge of how students use examples in the context of topology.

Watson and Mason (2005) described the *example space*, referring separately to the conventional example space and the personal example spaces of individuals. When trying to call students' attention to a pattern or to a particular property, teachers and textbooks often exhibit several examples in the hope that learners will abstract the general concept from the specific instances in which it is exhibited (Sinclair et al., 2011; Watson & Mason, 2005; Zodik & Zaslavsky, 2008). These examples make up the *conventional example space*, those examples which are presented to learners with the intention of those learners abstracting or generalizing a particular notion. Unfortunately, teaching and learning are not always so clear-cut; though the teacher intends for students to notice a particular common feature of a collection of examples, the students may interpret those examples differently in practice (Goldenberg & Mason, 2008; Mason & Pimm, 1984; Sinclair et al., 2011; Watson & Mason, 2005). A student's *personal example space* (PES) is composed of those examples which the

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student actually perceives to be representative of a more general idea along with any available methods for generating additional examples.

One's PES is not a haphazard collection of examples and methods; on the contrary, each individual's PES has structure, and the nature of this structure can be influenced by the level of experience and the values of the individual (Sinclair et al., 2011; Zazkis & Leikin, 2007; Zazkis & Leikin, 2008). According to Sinclair et al. (2011), one's personal example space may be structured along four dimensions: population, generativity, connectedness, and generality. These dimensions relate to the number and nature of examples to which the individual has access, as well as the connections among examples and the individual's ability to generate new examples. Thus, to understand a student's personal example space, it is not sufficient to know merely how many examples the student has access to; one must also understand the ease of access to examples and the connections between examples, as well as the *purpose* particular examples serve in the student's mind. Though students are often presented several examples during instruction, they may have difficulty generating additional appropriate examples due to a lack of experience with identifying dimensions of possible variation within a class of examples (Marton & Tsui, 2004).

At times, an individual may be reluctant to discuss a specific example at all, even when directly asked to do so. This reluctance may result from the fact that the individual possesses a method for producing infinitely many examples, or perhaps all examples, and thus does not feel the need to conjure a specific example (Sinclair et al., 2011). Similarly, it is not uncommon to consider only one example. In such cases, it may be that the individual considers this particular example to be paradigmatic; that is, one example may be intended to stand in for *all* examples in a given class, if all examples are perceived to be isomorphic in some way (e.g., all 'absolute value' functions may be considered isomorphic up to translation, stretching, and scaling). Such an example is known as a *generic example*: "[a] generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general" (Mason & Pimm, 1984, p. 287). On the surface, a generic example may appear no different than any other, specific example; it is the user's *intention* (or the observer's perception) that it should represent an entire class of examples which makes the example generic.

Of note about the dichotomy of the 'general' example strategies discussed in the preceding paragraph is that both strategies rely on the ability of the student to produce at least one *concrete* example, either using an example generation strategy or from memory. But what happens when students are unable to produce even one specific, concrete example of a concept or property? In particular, topology is a branch of mathematics in which abstraction is highly valued. Paradoxically, because of this abstraction, the value of concrete examples, non-examples, and counterexamples skyrockets, as these become more difficult to produce on the spot. The difficulty in producing concrete examples often presents a significant stumbling block for students in introductory topology courses.

In this paper, we describe instances of students reasoning about definitions in introductory topology using what we call *structural examples*. A structural example is an example that possesses only the essential properties of a definition but lacks any additional properties that might be present in a concrete example.

Theoretical Perspective

Piaget's (1985) notion of *constructivism* hinges on the idea that learners build knowledge for themselves and that this knowledge is built on prior understanding. In the context of solving problems (either computations or proofs), this notion is essential. If learners lack the ability to construct knowledge, then how can they be expected to make the logical connections necessary to advance through the stages of a solution or proof? Research mathematicians must make observations and be able to encapsulate those observations into new concepts as definitions or theorems, often with no idea what kinds of concepts might be necessary to solve a particular problem. For students, even if they already have some baseline knowledge of the concepts essential to solving a given problem, – perhaps from class or from a textbook – they must still be able to forge the connections among those concepts in the proper sequential order to construct a coherent argument. In particular, when students' example spaces are not populated by examples from lecture notes or from a textbook, students have no other option but to construct their own examples based on the understandings they bring with them to a problem.

Mason and Watson (2008) highlight two essential features in the generation of examples: *dimensions of possible variation* and *range of permissible change*. Understanding what features of an example can be changed and to what extent they can be changed without robbing an example of its 'examplehood' are essential skills in the successful production of examples. The real benefit of developing the recognition of these properties, however, lies in the consequent recognition of those properties which must be *invariant in the midst of change* (Mason & Watson, 2008, p. 195). These properties are common to all examples; they represent the bare minimum which must be incorporated for an example to 'count' as an example.

Investigating the ways in which expert mathematicians solve problems, Carlson and Bloom (2005) developed the Multidimensional Problem-Solving Framework. The complete framework describes four dimensions of mathematicians' attributes (namely, resources, heuristics, affect, and monitoring) that they leverage while solving problems, as well as a problem-solving cycle indicating the typical pattern of behaviors mathematicians exhibit from start to finish while solving a problem. This Problem-Solving Cycle consists of four main phases, labeled *Orienting, Planning, Executing,* and *Checking*. In this paper, we will make focus on the Orienting phase, as this is the phase during which problem-solvers are most likely to make use of examples and visual representations.

Methods

Data for this study were collected and analysed using a descriptive case study methodology (Yin, 2003). The primary goal of this study was to determine how students use examples when writing proofs in an introductory topology course. Descriptive case study was chosen as a methodology due to our desire to describe behaviour, rather than to quantify the frequency of such behaviours. Further, our data set is small and therefore lacks the size necessary for statistical generalisation. As such, we aim to describe the phenomena in this study as completely as possible with the goal of analytical generalisation.

Student volunteers were recruited from a semester-long, introductory, undergraduate-level course in point-set topology at a large research university in the United States. Volunteers were asked to attend weekly, one-hour group study sessions in which they were presented with one, two, or three proving tasks (the number of tasks in a given session varied depending on the speed with which the students completed the tasks). Tasks were designated as one of three types: Prove, Disprove, or Prove or Disprove. In Prove tasks, students were told explicitly that the conjecture given in the problem is true and asked to prove it. In Disprove tasks, students were told that the given conjecture is false and asked to disprove it. In Prove or Disprove tasks, the truth value of the given conjecture was left ambiguous, and the students were asked to determine its truth value and prove or disprove the conjecture according to their assessment. All group study sessions were video recorded, and these videos were transcribed. The lead author attended all regular class meetings (except for tests and exams), and the content of the proving tasks was chosen each week based on the content recently discussed in class. As compensation, the lead author offered additional office hours to discuss topics from class or difficulties with homework assignments.

In total, four students participated in at least one group study session throughout the semester: three undergraduates and one graduate student. In the interest of maximising the amount of data collected, weekly attendance at group study sessions was not a mandatory requirement of participation: students were permitted to attend sessions inconsistently from week to week based on their availability. Only one student, Stacey (all names used in this paper are pseudonyms), attended all group study sessions. When students attended their first session, they were asked to complete a survey indicating their academic major and minor fields of study as well as a comprehensive list of all mathematics courses they had taken at the university level.

The intent of the group study sessions was to replicate, as accurately as possible, the atmosphere of students studying mathematics in a natural setting, such as when working on homework problems or studying for a test. To this end, students were encouraged to collaborate and discuss ideas, and they were permitted to reference their textbooks. They were permitted to ask such questions of the lead author, who was present for all sessions, as they might ask an instructor at a review session. Whenever possible, the author attempted to redirect such questions toward other students in the group. The author intervened at times to clarify terminology or notation (to ensure that students were proving the intended statement rather than some different statement resulting from confusion due to notation or unfamiliar terminology), to ask probing questions, or to offer guidance when the students' work came to a complete halt.

This study investigated the ways in which students reason about and leverage examples when completing proving tasks in elementary topology. However, the group study design was intended also to benefit the student participants in the study; our hope was that a structured, active engagement with the content would strengthen the students' proving abilities and foster the development of rich concept images of topological concepts. We were not particularly interested in studying the students' success in producing correct proofs, so much as the approaches the students took to proving. Consequently, the author occasionally engaged in discussion with the students about correct notation or precision of word choice in proof writing, once it became clear what the ideas were that the students were trying to communicate. This was especially relevant during sessions in which Stacey was the only participant, as her direct communications with the facilitator were much more frequent.

Data

The transcripts of the group study sessions were initially coded for instances of example generation by the students. Codes for three different types of examples were generated: specific examples (used for isolated examples produced by students with no evidence to suggest that they were intended to represent any generality), generic examples (as in Mason & Pimm, 1984), and algorithmically-generated examples. These codes were applied to the transcripts of the group study sessions; however, examples remained which did not fit into any of these categories, and which had a distinctly more abstract nature than these three kinds of examples. The three existing codes were then grouped under the heading *concrete examples*. To categorise the other examples, we defined another code called *structural examples*. A structural example is an example which possesses all the essential properties of a given concept but no other properties which may be possessed by concrete examples. We describe two instances of structural example production by Stacey to clarify the nature of such examples. Throughout the nine sessions Stacey attended, eight out of twenty of the examples she produced under the *Prove* condition were coded as structural examples.

The first such instance we discuss of Stacey producing a structural example took place during Session 5 in response to the prompt to prove the true statement "A subset A of a topological space (X, T) is said to be dense in X if $cl(A) = X$. Prove that if for each open set $O \in T$ we have $A \cap O \neq \emptyset$, then A is dense in X." Note that prior to her work on this problem, Stacey had never encountered the concept of a dense subset.

Stacey began her work on this problem by drawing a diagram (Figure 1a), which we regard as a structural example.

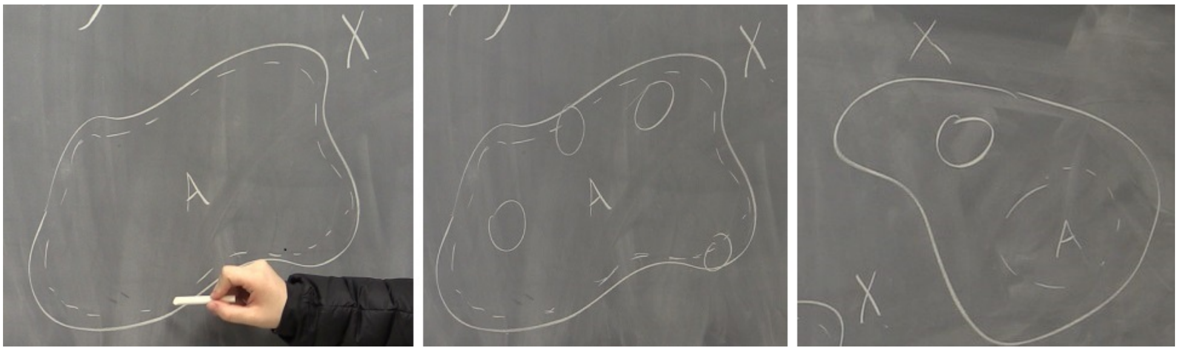

Figure 1(a-c). Stacey's structural examples in Session 5.

She explained:

> I can't really show it with a picture because I can't draw, like, a dashed line over a... solid line, but we have X on the outside, and then we have the set A, which is represented by the dashed, which I wish I could get closer to this [*pointing to the border of X*], but I can't. So if we had the closure of A, then it would just be the same as that solid line. So then if you take any open set anywhere [*drawing circles, Figure 1b*], there has to be some kind of intersection with A. So if it wasn't, like if you take... if the intersection could be... the empty set... [*drawing Figure 1c*] You've got X here... and A here, and you could have an open set here, and their intersection would be the empty set. But then this closure wouldn't be equal to X. I get it conceptually I think, but I'm not sure how to prove it.

The most salient feature of Stacey's diagram was that it represented her conception a set with a dense subset, but it did not represent a *specific* case of this idea, such as the rational numbers as a subset of the real numbers. It possessed all of the essential properties of the definition (as she visualises them) and the conjecture, but it lacked any additional properties, such as specific identities for the sets or a specific number of elements in either set, hence its classification as a structural example.

Recall that this was Stacey's first exposure to the concept of a set with a dense subset. Consequently, her example space for this concept was completely empty; any example she might have produced would be one of her own construction. This example served as a sort of template or skeleton for a specific example, as any specific example would necessarily have had these properties, but there was no possibility that Stacey was attending to any non-essential features, as this example did not have any. Furthermore, she was able to conduct thought experiments on this example, just as she might on a specific example. In Figure 1c, Stacey attempted to ascertain information about what might break down if the hypothesis of the conjecture were not true: if there existed an open set whose intersection with A is empty, then the closure of A could not have been equal to the entire space, X.

Stacey exhibited similar behaviour in Session 7 when prompted to prove the following statement: "Let (X, T) be a topological space. A separation of X is a pair U, V of disjoint open subsets of X whose union is X. X is connected if no separation of X exists. If the sets C, D form a separation of X, and if Y is a connected subspace of X, then either $Y \subseteq C$ or Y

⊆ D.” Again, Stacey’s first action was to draw a diagram (Figure 2a), which was coded as another structural example.

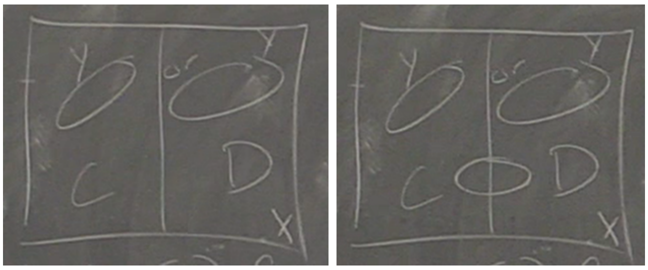

Figure 2(a-b). Stacey's structural examples in Session 7.

After completing the diagram, Stacey explained:

> If you have X, which is also the ambient space here, and then you have the sets C and D, they form a separation. That means that they’re disjoint, so they don’t have any of the same elements, and that their union is X, so that is satisfied for this. And then if Y is connected, which means it’s not in these sets that are disjoint whose union is Y, it’s just one cohesive set, then it has to be either in C or in D. It can’t be in both… because if it was like that [*draws Figure 2b*], it would be disjoint [*misspeaking: disconnected*].

As in Session 5, Stacey was interacting with a new concept. Stacey had experience with connectedness from class, but this was the first time she had been exposed to the definition of a *separation* of a topological space. She had no specific examples of separations in her example space, so she was forced to construct an example about which to reason in order to begin her argument. It was not strictly necessary for Stacey to construct a specific example, however, to complete this proof. Rather, she produced a nearly-complete proof by reasoning only about this object with minimal properties.

Discussion

Structural examples represent an as-yet undiscussed class of examples in the research literature. To more fully discuss the appearance and the role of such examples in proving, we describe these examples in comparison with more conventional types of examples.

Existing literature describes at least three kinds of *concrete examples*: specific examples, generic examples, and algorithmically-generated examples. A common feature among all types of concrete examples is their specificity. A concrete example is an example which uses specific numbers, sets, or relations and as such, exemplifies not only the relevant features intended to be manifested by its user, but potentially others as well. Consider, for instance, the set $\mathbb{Q}$ of rational numbers as an example of a dense subset of the set of real numbers $\mathbb{R}$. Certainly, the set $\mathbb{Q}$ possesses all the necessary properties of such an example; it is, of course, a set, and its topological closure is equal to the set $\mathbb{R}$. However, much more information is available about this example, even if it is temporarily ignored. For instance, $\mathbb{Q}$ is countably infinite, and it forms an abelian group under addition. Though these features were not relevant for the proof of the conjecture posed in Session 5, they are certainly features which can be identified about this concrete example.

Compare this concrete example with the structural example of a dense subset produced by Stacey during Session 5. Stacey’s example also possesses the necessary properties of being a set whose topological closure is equal to the entire space in which it is contained. In contrast with the concrete example of $\mathbb{Q} \subset \mathbb{R}$, however, no other intrinsic details can be recognised in this example. The cardinality of the subset is unspecified, no isomorphism with another set can be identified, the set has not been endowed with the properties of a group, ring, or field – the only information we have is that A is a dense subset of X. This abstractness – this distinct *lack* of specificity – is what makes Stacey's construction into a structural example.

One question naturally arises from this discussion: why did Stacey produce such an abstract entity about which to reason when concrete objects are readily available? The initial answer to this question is simple: *the concrete example was not readily accessible for her*. Based on the list of courses Stacey reported taking, she has been exposed to the rational numbers as a subset of the reals but may not have been explicitly asked to consider this property of the rational numbers. But this raises a further question: why didn't she just *construct* a concrete example?

To consider this question, let us return to the notions of dimensions of possible variation and range of permissible change (Mason & Watson, 2008). The individual must be able to identify the properties of an example that can be changed while maintaining its 'examplehood', as well as the extent to which those properties can be changed. When describing a specific (individual) example, learners show no evidence of attending to these features at all. For instance, consider the student who produces the example $\sqrt{2}$ when prompted for an example of an irrational number. Though this is a correct response, it is likely not the result of any consideration of a process for producing irrational numbers, nor is there probably much consideration of the properties of irrational numbers.

In contrast, consider another student who, in response to the same prompt, produces the example $\sqrt{2}$, but then proceeds to explain that the essential properties of $\sqrt{2}$ that make it irrational, such as its inability to be expressed as a ratio of two integers or its representation as a non-repeating, non-terminating decimal, are typical of *all* irrational numbers. The student may even have some sense that 'taking square roots' is somehow key to the irrationality of some numbers, but they may not feel it is necessary to generate further examples: since $\sqrt{2}$ is a generic example for this student, there is no additional information to be gained by producing a second or third example. Such a student attends to the *dimensions of possible variation*, but not to the range of permissible change. They are aware that changing the 2 in the example is one possible way to produce further examples, but not just any number will do.

Finally, think about the student who initially produces the example $\sqrt{2}$, but furthermore understands that replacing the 2 with any *prime* number will always result in another example of an irrational number. This student shows understanding of both the dimensions of possible variation *and* the range of permissible change by recognising not only what is allowed to change, but also what the allowable options are for that change. Of course, this procedure will not generate all irrational numbers, but is evidence that the student possesses an *algorithm* for generating infinitely many examples of irrational numbers.

Let us apply this rationale to Stacey's production of the structural example in Session 5. Because this is a structural example, it possesses only the essential components of the definition it is intended to exemplify. Like a specific example, no attention is paid to the dimensions of possible variation or to the range of permissible change – because, to change any of the essential properties would rob the example of its examplehood. What distinguishes this kind of example from a specific example, however, is the explicit acknowledgement of the features which must be *invariant in the midst of change*. While several concrete examples are typically considered in order for students to abstract these essential features, Stacey seems to start at the end of this process, recognising the more abstract representation of the concept without any exposure to concrete examples. In fact, Stacey may still face great difficulty generating concrete examples, as she has never been asked to do so for the concepts presented in Sessions 5 and 7 (Marton & Tsui, 2004).

Conclusion

Knowledge of the examples students use in different contexts is invaluable for teachers as this gives insight into students' intuitive understandings of abstract concepts. This information can be leveraged to help students work more efficiently toward a deeper understanding of the relevant mathematical concepts. Structural examples provide a way for students to demonstrate that they have some understanding of abstract concepts even if they are unable to produce concrete examples. With an awareness of structural examples, teachers can design more efficient instructional materials to utilise these examples and foster deeper student understanding.

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