Teacher Noticing of Primary Students' Mathematical Reasoning in a Problem-solving Task

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Mathematical reasoning is a key proficiency in mathematics, however primary school teachers may have trouble noticing students' mathematical reasoning mid-lesson. In this study, students' mathematical reasoning has been analysed using video data collected as one grade of Year 5 and 6 students worked through the 'Painted Cube' task. Analysis reveals that students displayed a range of sophistication of mathematical reasoning. An ecological analysis of what student reasoning actions were visible to the teacher mid-lesson suggests that many of these actions were too subtle to be picked up by the teacher. Introduction

While mathematical reasoning is a key proficiency in the Australian Curriculum, research has shown that many Australian primary school teachers need support in enacting and assessing mathematical reasoning (Bragg, Herbert, Vale, Loong, & Widjaja, 2016; Loong, Vale, Bragg, & Herbert, 2013). Loong et al. (2017) reported that primary school teachers struggled to define, recognise and implement reasoning. Hiebert et al. (2015) stated, "it takes time for teachers to become aware of the nuances of mathematical reasoning and be able to express their understanding of it" (p. 35). This calls to attention the need to develop teachers' capacity to teach and assess this vital mathematical proficiency. Ellis, Özgür, and Reiten (2018) underscore the important role of a teacher in supporting student mathematical reasoning. They framed teacher pedagogical moves into four categories: eliciting, responding, facilitating and extending.

What teachers notice mid-lesson has been argued to affect teachers' pedagogical decisions as they teach (Jacobs, Lamb, Philipp, & Schappelle, 2011). Hence, if teachers are to teach and assess mathematical reasoning, they should be able to notice students' reasoning mid-lesson. This study seeks to investigate the degree to which a class of Year 5 and 6 students' mathematical reasoning was noticeable to the teacher as the students worked through a problem-solving task called the 'Painted Cube' (Driscoll, 1999). In analysing students' reasoning in this task, an ecological account of teacher noticing (Jazby, 2016) will be applied to data relating to teacher actions mid-lesson.

Theoretical Framework

Mathematical Reasoning in Primary School Classrooms

Earlier studies have highlighted the importance of making mathematical reasoning age-appropriate (Komatsu, 2010; Reid, 2002; Stacey, 2010; Stylianides, 2007). In primary schools, this often involves the use of concrete manipulatives, diagrams or other visual representations including gestures. This approach is often referred to as 'pre-formal proof' (Blum & Kirsch, 1991) or 'action proofs' (Stylianides, 2007).

Research underscores the need to unpack subtle differences in the way in which students reason mathematically and communicate their mathematical reasoning. However, there is still a lack of research addressing what counts as valid generalisation and justification, and what forms of reasoning and representations are within the conceptual reach of primary school students engaged in forming conjectures, justifying and generalising. Lin, Yang, and Cheng (2004) raised the importance of proof and disproof construction to be built around what and how students justified their conjectures. Jeannotte

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and Kieran (2017) underscored the importance of teachers' awareness of students' ways of communicating their reasoning.

> This [teachers' awareness of students' reasoning] requires a well-elaborated vision of MR [Mathematical Reasoning] where discourse is fundamental and that not only reflects the didactical discourse of the discipline but also serves as a conceptual tool for teachers (and researches) to analyze students' discursive activity (p. 3).

Teacher Noticing of Students' Mathematical Reasoning

If a teacher is to effectively develop students' capacity to reason mathematically, then they will theoretically need to have a clear conceptualisation of what mathematical reasoning is and have methods for highlighting reasoning in class (Bragg et al., 2016; Loong et al., 2013). In order to assess the mathematical reasoning proficiency strand in the Australian curriculum, they then will also need to be able to identify the reasoning that students are using. The construct of teacher-noticing is concerned with analysing what teachers notice mid-lesson, as what is noticed affects the pedagogical decisions that teachers take (Jacobs et al., 2011; Jazby, 2016). Most accounts of teacher-noticing, such as Jacobs et al.'s (2011), employ information processing models, where noticing is a passive, internal mental process of analysing what is perceived and deciding to respond (Sherin & Star, 2011). In these models, a teacher views a noteworthy event, makes sense of it, then decides how to respond.

Jazby's (2016) ecological model of teacher-noticing analyses the way in which a teacher interacts with the classroom environment so that the environment is more likely to produce noteworthy events. Teachers are urged to orchestrate events in their classrooms – by selecting the tasks and materials in a lesson, for example – and the way in which they orchestrate events may increase the likelihood that a noteworthy event will occur. Using head-mounted cameras and post-lesson interviews to track what teachers attended to mid-lesson, Jazby (2016) was able to identify mathematics teachers' perceptual routines. He argued that teachers deployed their attention in particular, purposeful ways as they taught, and that the ways in which attention was deployed then affected which parts of the classroom were perceptually accessible or occluded from the teacher. Through this ecological lens, noticing students' mathematical reasoning involves active behaviour on the teachers' part; behaviours which create the conditions for noteworthy events to occur when the teachers' attention is deployed towards them.

From an ecological point of view, environmental constraints could limit a teacher's capacity to notice students' mathematical reasoning. The nature of attention is selective (Kirlik, 2007); a teacher cannot attend to all of the events that occur in a classroom simultaneously when there are multiple student groups spread across a classroom space. Hence, even a teacher who had exemplary mathematical and pedagogical knowledge would need to split their attention between student groups. This study seeks to identify some of the environmental constraints that may affect a teacher's capacity to notice students' mathematical reason as a lesson unfolds.

This analysis is expected to provide insight into the different ways in which students in the same class engaged in the same task at a Year 5 and 6 level employed mathematical reasoning. In this paper, the following research questions will be addressed:

RQ 1. What types of mathematical reasoning are evident when Year 5 and 6 students are given in a particular problem-solving task?

RQ 2. To what extent is children's mathematical reasoning perceptible to a teacher mid-lesson?

Method

The research team worked with teachers from a Melbourne suburban school to develop a lesson that would require students to engage in mathematical reasoning. The Painted Cube task (described below) was selected by the research team as a task that would be challenging for students while also having opportunities for students to engage in mathematical reasoning. One class of Year 5 students and their teachers travelled to the data collection facility where students were presented with the task for the first time. For the purposes of this study, three small groups of students (2 - 3 students) were selected for preliminary analysis: Max's group worked through the problem quickly and were accurate; Carlo's group took an average amount of time (compared to their classmates) and were mainly accurate, Richard's group completed the task using all of the available time and were less accurate than other groups. As these three groups differed in terms of the time taken and level of accuracy, it was hoped that they would exhibit a range of mathematical reasoning that would represent the range of reasoning present in the wider class. By asking the children to work in groups, it was hoped that between-student discussion would be facilitated. In all of the groups analysed, an individual student took the lead during the task and each group is identified by this individual student.

Data collection took place at the International Centre for Classroom Research's (ICCR) data collection classroom at the University of Melbourne. The ICCR research classroom is a multi-camera data collection facility which can capture multiple channels of video and audio data as a lesson unfolds. Each student table had two small groups of students working at it and each table had a dedicated camera and two microphones recording what occurred during the lesson. Two additional cameras then tracked the teacher as she moved during the lesson, and the teacher wore a radio microphone to capture audio as she moved. Post-lesson, the teacher was interviewed in the ICCR studio about what she had noticed during the lesson.

The Painted Cube Task

The Painted Cube task is rich and complex, providing students with opportunities to explore a variety of patterns that can be described spatially, numerically and algebraically. The task was adapted from Driscoll (1999) to offer opportunities for primary school students to share and communicate their thinking and reasoning as they looked for patterns and formed generalisations. Students are provided with a worksheet – shown in Figure 1 – to help them work systematically through the task so that they can test ideas and to ask themselves questions about further cases. This task relates to the Year 6 content description of the Australian Curriculum Mathematics under Number and Algebra strand ACM133 elaboration "identifying and generalising number patterns" (www.acara.edu.au).

The teacher introduced the task to the Year 6 students by first showing a 3 x 3 x 3 unpainted Multi-Base Arithmetic (MAB) cube. Students were invited to brainstorm the features of a cube with each other such as the numbers of edges, corners or vertices, and sides of a cube. Students were asked to imagine that the cube was dipped in a paint can and then the "painted" cube was pulled out and separated into individual cubes, i.e., the 1 x 1 x 1 cubes. The task requires students to determine how many mini cubes are painted on 3 sides, 2 sides, 1 side and not painted at all, and record them in a table. Students were challenged to extend this to other sized cubes. The action of generalising was supported in this lesson when students were asked to express the relationships or patterns they observed in the table for columns A, B, C, D, and E shown in Figure 1.

CUBE SIZE	Total number of small cubes (A)	Number of cubes painted on 3 sides (B)	Number of cubes painted on 2 sides (C)	Number of cubes painted on 1 side (D)	Number of cubes not painted at all (E)
2 x 2 x 2	8	8	0	0	0
3 x 3 x 3	27	8	12	6	1
4 x 4 x 4	64	8	24	24	8
5 x 5 x 5	125	8	36	54	27
6 x 6 x 6	216	8	48	92	64

Figure 1. A completed table from the painted cube task.

Data Collection and Analysis

In order to address the first research question of the study, data had to be analysed in a way that would identify students' mathematical reasoning. It was hoped that students would discuss and justify their conjectures with their group members as they worked through the problem, as this is what had occurred when the same group of students had participated in a science-focused lesson at the same facility a week earlier. Hence, the first pass of the data analysis involved identifying instances when students had provided a justification or explanation of their thinking. Unfortunately, most groups did not make many utterances that justified their conjectures in this lesson. This meant that the research team had to employ abductive reasoning to infer the mathematical reasoning of students.

Photographs of student worksheets collected at the end of the lesson were used to identify whether students had been accurate. The table that students had been given contained 25 cells and this could be used as a unit of analysis if each cell was considered to be an instance of mathematical reasoning. When combined with video data, the order in which students had filled out the worksheet table could also be identified and timecoded (see Figure 1). This enabled analysis of how and when each group approached each step of the problem. Video data also showed the degree to which students used the physical cubes they were given as they worked through the problem, or whether they ignored the cubes and worked from a mental model.

For each calculation that was performed during the lesson, student utterances and gestures were also used to infer student reasoning. Six calculation strategies could be identified in the data. A *counting-based* strategy could be inferred when students picked up a cube and began to tap the smaller cubes while saying, "1, 2, 3…" and so one, to arrive at a total. *Partial counting/multiplicative* strategies could be identified when students used a counting-based strategy to count some of the required cubes, then multiplication was used to find the final total – count the cubes on one face, then multiple by six, for example. *Multiplicative* without counting could be identified when a student made a statement such as, "a 4 x 4 x 4 cube … that 4 x 4 is 16 … 16 x 4 is 64" with no evidence of counting gestures. A *layer model* could be identified when working on column A of the table. Carlo, for example, was trying to ascertain how many cubes are needed to make a 4 x 4 x 4 cube and stated, "there's 4 layers of 16" while making four slicing gestures over the cube. The use of *geometric properties* could be identified when students made statements about the properties of cubes. Max, for example, stated, “well there’s going to be 6 faces so times 6…”. These utterances suggested that some students considered the number of faces, ‘corners’ and edges each larger cube had. A final category of calculation strategy – *category unclear* – was used to code instances where there was not enough data to make an inference about student reasoning. Six times Max wrote an answer in the table without making any utterances or gestures, hence there is not enough data in these instances to make inferences about how he had arrived at a solution.

In order to answer the second research question of this study, student data needed to be contrasted with data relating to what was noticed by the teacher. In the post-lesson interview, the teacher was asked to identify which students she thought had displayed the best mathematical reasoning during lesson. The teacher’s movements, which were tracked by two cameras, were placed on an activity map of the classroom. Timecodes were noted when she shifted her position in the room. This provides data relating to what she could see at particular moments during the lesson as well as what was occluded from her view. Using an ecological lens, in order to notice students’ mathematical reasoning, the teacher would need to deploy her limited attentional resources in a way in which she can see or hear students recording, saying and doing things that showed their mathematical reasoning. Analysis of student data provides evidence of when and where the three groups analysed took actions that revealed their mathematical reasoning, and this can be cross-referenced with data relating to the teachers’ position and field of view to ascertain whether these events were noticeable to the teacher. This analysis does not provide evidence of what was attended to be the teacher, but merely tries to ascertain whether events that could reveal students’ mathematical reasoning were within her field of view or not.

Results

Student Reasoning in the Painted Cube Task

While each group’s accuracy in the task varied, both Richard’s and Carlo’s groups showed similar mathematical reasoning in the task, while Max’s group relied more on multiplicative and geometric reasoning. Carlo’s group made the most utterances (seven in total) which explained their reasoning, while Richard’s and Max’s groups rarely made such utterances (two and one utterance respectively). Table 1 shows a summary in the differences in mathematical reasoning between each group. Both Richard’s and Carlo’s group had similar use of the *counting* only strategy (39% and 38% respectively), however Carlo’s group was more accurate using this strategy. Max’s group, in contrast did not use the *counting* only strategy. All groups used the *counting and multiplying* strategy, and this was often paired with either a *layer model* or *geometric properties*. Both Richard’s and Carlo’s groups mainly used this strategy with the layer model, where one layer would be counted before being multiplied by the number of layers. Again, Carlo’s group was more accurate than Richard’s when these strategies were used. Both Richard’s and Carlo’s groups used *geometric properties* to recognise the number of cubes painted on three sides was synonymous with the number of corners. In contrast, Max’s group used *geometric properties* the most (63% of the time) and was able to see that the number of faces and edges of a cube could also be used to help solve how many cubes were painted on one and two sides.

Table 1
Students' mathematical reasoning in the painted cube task

	Richard (n=23)		Carlo (n=24)		Max (n=19)	
Strategy	Rate of usage	Accuracy	Rate of usage	Accuracy	Rate of usage	Accuracy
Counting	39%	56%	38%	78%	0%	
Counting and multiplying	35%	75%	25%	83%	47%	100%
Multiplying without counting	0%		0%		21%	100%
Layer model	26%	67%	17%	100%	5%	100%
Geometric property	27%	100%	21%	80%	63%	100%
Strategy unclear	13%	66%	25%	100%	5%	100%

Each group worked through the cells of the table shown in Figure 1 in a different order. Richard's and Carlo's groups tended to work from left to right across rows in the worksheet table. Max's group worked down each column of the table. Max recognised and utilised geometric properties as he worked through the task as working down each column required him to look across cases.

Teacher-Noticing of Student Reasoning Mid-lesson

When asked in the post-lesson interview to identify the student group that had displayed the 'best' mathematical reasoning, the teacher nominated Carlo's group. As students worked through the task, tracking of the teacher's movements showed that she moved between groups and stopped to interact with each group either once or twice during the between-desk segment of the lesson. As she faced a particular group of students and interacted with them, she turned her back on other students so that they were occluded from her view. Over the 17-minute period of between-desk instruction, Richard's group was within the teacher's field of view twice for a total of approximately three minutes, Carlo's group was within her field of view twice for a total of approximately 2 minutes 30 seconds, and Max's group was within her field of view twice for approximately 1minute 20 seconds. Students tended to stop working on the problem when the teacher was near and responded to her questions about their work. Hence, none of the subtle gestures, utterances or use of physical objects that researchers used to ascertain students' mathematical reasoning were visible to the teacher as she moved between desks. The teacher was able to hear three utterances related to explaining student reasoning – two from Carlo's group and one from Max's. Carlo's utterances were more detailed than Max's and he used a cube and gestures to explain to the teacher how the *layer model* worked in relation to working out the total number of cubes.

Discussion

Analysis of three groups of children working through the Painted Cube task reveals that each group used different types of mathematical reasoning. Max's group was the most accurate, finished the task quickly, worked across cases, and relied on multiplicative strategies and geometric properties the most of the three groups analysed. Of the three groups, this group used the most mathematically sophisticated reasoning. The other two groups analysed (Richard's and Carlo's groups) used a similar rate of counting-based strategies although Carlo's group was more accurate than Richard's. Hence, analysis of these three groups shows a range of mathematical reasoning is present as this Year 5 and 6 class works through the problem.

While careful analysis of video data by a research team post-lesson revealed that Max's group had more sophisticated mathematical reasoning, the teacher's perspective post-lesson was that Carlo's group demonstrated the best mathematical reasoning. An ecological analysis of what was within and occluded from the teacher's view mid-lesson reveals that most of the subtle gestures and utterances that students made, which the research team could review on video and code, were not visible to the teacher mid-lesson. Thus, if the teacher had not noticed that Max's group had the most sophisticated mathematical reasoning, it is because many of actions that make Max's reasoning visible are subtle and occur when the teacher's attention is directed towards other students. From an ecological point of view, this is not due to a deficit in teacher knowledge or skill; the classroom environment and task design have an impact on how visible students' reasoning actions are to the teacher who has a limited amount of attention that must be distributed amongst multiple student groups in a dynamic environment.

The teacher did seem to notice when groups made utterances explaining and justifying their thinking and her attention was directed to that group. Carlo's group made the most of these kinds of utterances and this was also the group that the teacher claimed had the best mathematical reasoning. Loong et al. (2013) claimed that teachers need to have a clear idea of what constitutes mathematical reasoning if they are to teach and assess it and, in this study, the teacher seems to equate mathematical reasoning with providing justification.

While it is not surprising that a research team who can review video data multiple times was able to identify differences in student reasoning that were not noticed by the teacher mid-lesson, the analysis provided in this report has important implications regarding how students' mathematical reasoning may be made more visible to teachers. Teachers do not have the affordances that a research team has in terms of being able to view, pause and review one student group at a time to identify subtle indicators of mathematical reasoning. From an ecological point of view, teacher-noticing is constrained by the classroom environment (Jazby, 2016) but manipulation of that environment can make some structures within an environment easier to notice. If we take the design of the task, for example, the worksheet students were provided with became filled with answers to calculations (see Figure 1). If teachers are more likely to notice mathematical reasoning when students engage in constructing justifications for their answers, then the task could be modified so that students must construct and record justifications. The scenario for the task could involve a worker who has to paint little cubes that then are put together into larger cubes of various sizes. When orders come in, they have to get the right number of not painted, painted on one side, painted on two sides and painted on three sides cubes for each order. The worker will not accept advice on how many cubes are needed unless you can explain to them how you worked out your answer. This kind of modification to the task would theoretically create more justification actions amongst students and increase the recording of justification of strategies. Thus, students' mathematical reasoning would theoretically become more noticeable via modification of the task rather than having to train the teacher how to notice.

The results of this study demonstrate just how difficult it is for a teacher to notice students' mathematical reasoning mid-lesson. Even the research team would not have been able to pick up the subtle student behaviours that were analysed from review of video data if they were also running the lesson in real time. Rather than focusing purely on developing teachers' capacity to recognise students' mathematical thinking (Jacobs et al., 2011), an ecological analysis suggests that mathematics education researchers could theoretically take a more human-centred approach to design. If the attentional limitations of people are kept in mind when designing mathematics tasks, the designer can make mathematical reasoning actions more likely to occur in ways that are easier to see for teachers that are having to monitor multiple student groups simultaneously.

References

Blum, W. & Kirsch, A. (1991). Preformal proving: Examples and reflections. *Educational Studies in Mathematics, 22*, 183-203.

Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers, grades 6-10*. Portsmouth, NH: Heinneman.

Bragg, L., Herbert, S., Vale, C., Loong, E. Y-K., & Widjaja, W. (2016). Primary teachers notice the impact of language on children's mathematical reasoning. *Mathematics Education Research Journal, 28(*x), 523-544.

Ellis, A., Özgür, Z., & Reiten, S. (2018). Teacher moves for supporting student reasoning. *Mathematics Education Research Journal.* https://doi-org.ezproxy-b.deakin.edu.au/10.1007/213394-018-0246-6.

Herbert, S., Vale, C., Bragg, L., Loong, E., & Widjaja, W. (2015). Developing a framework for primary teachers' perceptions of mathematical reasoning. *International Journal of Educational Research 74*, 26-37.

Jacobs, V. R., Lamb, L. L. C., Philipp, R. A., & Schappelle, B. P. (2011). Deciding How to Respond on the Basis of Children's Understandings. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics Teacher Noticing: Seeing Through the Eyes of the Teacher* (pp. 97-116). New York, NY: Routledge.

Jazby, D. (2016). An ecological analysis of mathematics teachers' noticing. In B. White, M. Chinnappan, & S. Trenholm (Eds.), *Opening up mathematics education research: Proceedings of the 39th conference of the Mathematics Education Research Group of Australasia* (pp. 358-365). Adelaide: MERGA.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*(1), 1-16.

Kirlik, A. (2007). Ecological resources for modeling interactive behavior and embedded cognition. In W. Gray (Ed.), *Integrated models of cognitive systems*. New York: Oxford University Press.

Komatsu, K. (2010). Counter-examples for refinement of conjectures and proofs in primary school mathematics. *The Journal of Mathematical Behavior, 29*(1), 1-10.

Lin, F.-L., Yang, K.-L., & Chen, C.-Y. (2004). The features and relationships of reasoning, proving and understanding proof in number patterns. *International Journal of Science and Mathematics Education, 2*, 227-256.

Loong, E. Y.-K., Vale, C., Bragg, L. A., & Herbert, S. (2013). Primary School Teachers' Perceptions of Mathematical Reasoning. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow: The proceedings of 36th annual conference of Mathematics Education Research Group of Australasia* (pp. 466-473). Melbourne: MERGA.

Loong, E., Vale, C., Herbert, S., Bragg, L.A., & Widjaja, W (2017). Tracking change in primary teachers' understanding of mathematical reasoning through demonstration lessons. *Mathematics Education Research Journal, 19*(1), 5-29.

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education*, *33*(1), 5-29.

Sherin, B., & Star, J. R. (2011). Reflections on the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 66-78). New York, NY: Routledge.

Stacey, K. (2010). *Mathematics teaching and learning to reach beyond the basics*. Paper presented at the ACER, Melbourne.

Stylanides, A. J. (2007). The notion of proof in the context of elementary school mathematics. *Educational Studies in Mathematics*, *65*(1), 1-20.