Learning through critiquing: Investigating students' responses to others' graphs of a real-life functional situation

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Argumentation activities, such as constructing and communicating claims, critiquing others' ideas, and evaluating the strengths and weaknesses of claims, can play a foundational role in students' educational development. Yet there is more to understand about utilising these activities for effective mathematics learning. This paper discusses an exploration of 12 Australian students' responses to a purposefully designed argumentation task on graphing functions to provide insight into their attention to particular functions concepts. Issues related to argumentation task design and functions teaching and learning are discussed.

In recent years, there has been a growing appreciation of the importance of incorporating argumentation into the mathematics classroom. Firstly, argumentation is a valued mathematics practice, in that mathematicians socially construct knowledge through generating and evaluating alternative arguments. Secondly, research suggests that participation in argumentation activities that require the student to explore, confront, and evaluate alternative positions, voice support or objections, and justify different ideas and hypotheses, promotes meaningful understanding and deep thinking (Weber, Maher, Powell, & Lee, 2008). Implementation of explicit argumentation activities in the mathematics classroom is not yet common. Findings from studies have highlighted that such activities in classrooms are demanding and require from teachers and students specific intellectual and social skills (e.g., Ayalon & Even, 2016; Yackel, 2002). Mathematics teachers have demonstrated difficulties in improvising argumentative activities in the classroom context spontaneously; precise task design and subtle, adaptive management seem necessary. Such activities are also often emotionally challenging, and there remains the issue of how to engage students in productive argumentation that is neither teacher-centred nor devoid of dialectic (Schwarz & Baker, 2017). Further research on learning and teaching mathematics with purposefully designed tasks for argumentation is needed. In this paper we discuss data collected on Australian students as part of a larger project on Years 7 to 12 students' functions concept development (e.g., Ayalon, Watson, & Lerman, 2016). It addresses the following research questions: *1) What types of warrants do students use in constructing, critiquing, and revising graphs to match a real-life functional situation? 2) How might students' attention to particular warrants relate to elements of task design and prior learning experience?*

Background and Context

Two theoretical perspectives informed the design and implementation of this study: students' involvement in argumentation activity, and approaches to developing students' understanding of functions. These are briefly overviewed in turn and then information on the students' curriculum context is presented.

There are diverse definitions of argumentation in the education literature (Schwarz & Baker, 2017). A widely accepted definition is that argumentation is "a verbal, social, and rational activity aimed at convincing a reasonable critic of the acceptability of a standpoint by putting forward a constellation of propositions justifying or refuting the proposition expressed in the standpoint" (van Eemeren & Grootendorst, 2004, p. 1). Argumentation can be considered as having two important meanings – *structural* and *dialogic* (McNeill, Katsh-

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Singer, González-Howard, & Loper, 2016). The structural meaning of argumentation focuses on the aspect of discourse in which a claim is supported by an appropriate justification. We adapt Toulmin's (1958) model of argumentation for this structural definition (e.g., Yackel, 2002). The structure is conceptualised as consisting of three essential components: *claim*, *data*, and *warrant*. The claim (C) is the conclusion that answers the question or problem. The data (D) are the foundations on which the argument is based, the relevant evidence for the claim. The warrant (W) or justification connects the data and claim by, for example, appealing to a mathematical rule or definition or concept. In addition to a structural focus, there is also a dialogic meaning for argumentation, which focuses on the interactions among individuals in generating and critiquing each other's ideas.

The second area of the literature relevant to the study is the learning and teaching of functions. A key aspect of school algebra is the study of relationships among quantities and of variables as quantities that have variability (Usiskin, 1988). The notion of variables is fundamental to understanding graphical representations of functional relationships and is a prerequisite for making sense of a covariation view of function: how variations in one quantity relate to variations in others. A variable can be contextual or abstract, numeric or non-numeric, discrete or continuous, and might also be compound units or rates. Contextualised variables are most often continuous in nature, and many tasks involve time or a time-dependent variable. Students' ability to conceive of time as a quantity can affect their conceptions of continuous variation (Thompson & Carlson, 2017).

Tasks involving graphical representations can be classified into two, non-exclusive, categories of action: *interpretation* and *construction* (Leinhardt, Zaslavsky, & Stein, 1990). Interpretation is the action by which a student gains meaning from a provided graph. Construction involves generating new objects that are not given, such as building a graph, and can be local (e.g., plotting points) or global (e.g., sketching a graph). Students' everyday knowledge of realistic contexts can serve as a basis for learning how to interpret the graph of a function by drawing on their common sense and reality-checking strategies (Goldenberg & Kliman, 1988) but *constructing* graphs to match realistic situations has been shown to raise difficulties, particularly when the variables involved are unusual for the learner (Leinhardt et al., 1990). In a previous study of Israeli and English secondary students' choice of variables and graph selection (from given examples) to match a realistic situation, Ayalon et al. (2016) found that for the audience clapping situation (used for the task in this paper), the main difficulty for students across year levels was attending to its contextual features (the initial audience silence and reaching saturation). Our study provided the opportunity to investigate students constructing, justifying, and then critiquing others' graphical interpretations of this particular real-life situation.

In the Australian curriculum context (Australian Curriculum Assessment and Reporting Authority [ACARA], 2018), the content related to functions learning includes a hybrid of both traditional (equations-based) and reform (functional) approaches (Sutherland, 2002). The national curriculum prescribes informal graphical exploration of functional relationships and covariation concepts (rate of change, gradient) at Years 7 and 8, for example, locally through plotting points on the Cartesian plane, and globally through interpreting simple real-life linear and non-linear graphs with time as the independent variable (such as triathletes' performances, daily temperature variations, companies' sales figures, and filling glasses with juice). (In these lower secondary years, the more formal focus is on manipulating algebraic expressions and solving linear equations, the traditional approach to school algebra.) In Years 9 and 10, the curriculum prescribes that students are introduced more formally to different types of non-linear functions, particularly quadratics, circles, and exponentials. Students who opt to study the more rigorous mathematics units at Years 11 and 12 (Mathematical Methods) will be introduced to the term 'function' and the formal set-theoretic definition of a function, investigate a variety of different functions including trigonometric functions, and also study Calculus.

Research Design

The functions task given to the Years 7-to-12 students in this study was adapted from Swan (1980) and requires attention to different variables and contextual features present in a real-life situation. It involves both graph construction and interpretation and is relevant to Australian curriculum content from the lower secondary levels (ACARA, 2017). The two meanings of argumentation (Asterhan & Schwarz, 2016) were used in developing the task: the structural components of the Toulminian framework for constructing and critiquing arguments (claim, data/evidence, warrant/justification) and the dialogic process (although partial) for evaluating others' ideas. We developed a sequence of activities to involve students in making a *claim* (about the best way to represent the real-life situation graphically), using *data* (contextual features from the real-life situation) to justify their graph's features (*warrant*), critiquing other claims (three provided fictitious students' graphs) by identifying their strengths and weaknesses, and then revisiting their own claim.

We first gave students the following worded real-life situation, titled 'After the performance':

> After the concert there was a stunned silence. Then one person in the audience began to clap. Gradually, those around her joined in and soon everyone was applauding and cheering.

We asked them to construct a graph that they think best matches the situation (their claim), and explicitly requested that they choose and label their chosen variables on the provided axes, as a way to scaffold their attention to possible variables (e.g., time, volume, number of people) and their behaviour (Leinhardt et al., 1990). We then asked students to critique three fictitious students' graphs (in Figure 1) before deciding whether or not to revise their own constructed graph, and to provide justification for their decision.

Alex Ben Cath

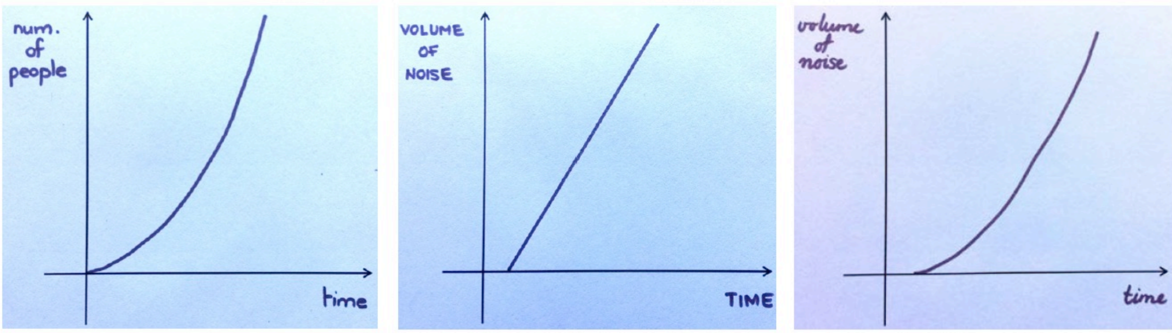

Figure 1. The three fictitious graphs students critiqued after constructing their own graph

We designed the three examples to each incorporate a combination of different weaknesses and strengths in terms of: the choice of variables, the relationship between them, and contextual features of the real-life situation, as shown in Table 1. We did not intend for the students' critique to be a 'multiple-choice task' for finding the correct graph, although we did not inform them that the graphs were all incorrect. We sought to investigate students' attention to these types of strengths and weaknesses in their critiques of the three examples and their subsequent revisions of their own graphs. We chose to leave out the contextual feature of saturation in all of the examples to see if students might attend to it and keep it in their claim, despite not being given any example.

Table 1
Design of the fictitious claims, incorporating correct and incorrect aspects

Types of warrants represented graphically	Alex's graph	Ben's graph	Cath's graph
Choice of variables	√	√	√
Relationship between variables	Should be discrete points	Should be non-linear	√
Contextual features of the situation			
- Initial silence of audience	×	√	√
- Saturation: reaching max. num. people/vol.	×	×	×

Twelve high-achieving secondary students, (boy and girl from each of Years 7 to 12) from a large independent school in Melbourne, each participated in a one-on-one videotaped task-based interview for an hour with the first author. The choice of a relatively small number of students addressed the need to have manageable data for analyzing individual responses in depth while also taking a wide range of student approaches into account. We asked the school's Head of mathematics to recruit only those students with high achievement results in mathematics as we reasoned that any confusion they might evidence with the tasks would more likely be widespread and reflect areas of genuine difficulty (Goldenberg & Kliman, 1988). Of course, our ultimate intent with these types of tasks is for students of all levels of understanding to benefit from participation in such argumentation activity in small-group and classroom settings. The students were interviewed in their school setting during the school day. They were encouraged to 'think aloud' throughout the interview so as to elicit insights into their self-explanations in solving a series of functions tasks, one of which is reported in this paper. Their verbalisations were transcribed and analysed along with their gestures on video, such as when they physically pointed to particular aspects of their own graph/s or the fictitious example graphs. We were interested in seeing students' responses across the range of secondary levels, in attending to functional concepts and also in choosing different types of warrants. Samples of the participant students' initial and final graphs (claims) for the argumentation task in this paper will be shared at the conference.

Findings and Discussion

The following discussion focuses on the three types of warrants to which students attended (or not) in their initial graph construction, critique of the three fictitious example graphs, and subsequent revision of their own graph. Table 2 presents the numbers of students who chose, justified, and changed particular aspects. In terms of *Choice of variables*, a majority of the students (10 out of 12) chose the continuous independent variable 'time' for the *x*-axis and the discrete dependent variable 'amount (typically chosen by the younger students) or number (older students) of people clapping/applauding' for the *y*-axis. This suggests a familiarity with 'time' as a variable in real-life functional situations and the mathematical convention of assigning it to the horizontal axis. The Year 10 and 12 girls, however, chose 'number of people clapping' as the independent variable on the *x*-axis and 'noise emitted' or 'volume/intensity' on the *y*-axis. Both subsequently noticed the use of 'time' when critiquing the examples and reasoned aloud about how best to match it with a suitable dependent variable: either 'number of people clapping' or 'volume'. The Year 10 girl initially reasoned that 'both are acceptable' then later decided that number of people was 'more concrete' and subsequently revised her graph to use time and 'number of people'. The Year 12 girl decided to keep volume on the *y*-axis and changed her *x*-axis to time.

Table 2

Students' attention to warrants in constructing, critiquing, and revising graphs (n = 12)

Types of warrants represented graphically	Initial graph (claim)	Critique of examples	Subsequent changes made to initial graph
Choice of variables (independent vs dependent)			
Time & Amount/number of people clapping/applauding	10	9 (from the original 10) + 2	1 (from num. people & vol.)
Time & Volume of noise	-	9 + 2	1 (from num. people & vol.)
Number of people clapping & Volume/noise	2	-	-
Relationship between variables			
Exponential continuous	8	12	2 (from linear, from exp. with vert. asymp.)
Exponential cont. with horizontal tapering	2	-	-
Exponential cont. with vertical asymptote	1	-	-
Linear continuous	1	12	-
Negative parabolic continuous	-	-	1 (from exp. cont.)
Contextual features of the situation			
Initial silence of audience	2	12	5
Saturation: reaching max. num. people/vol.	3	2	2 (1 added, 1 removed)

During their critique of the three example graphs, 9 of the 10 students (who had chosen time and number of people clapping as their variables) subsequently noticed the alternative dependent variable 'volume of noise'. (The other student (Yr 8 girl) did not evidence attending to the different variables in her critique.) Yet none of these 9 students opted to revise their initial choice. Five students reasoned that either number of people clapping or volume was fine:

> I want amount of clapping, amount of people clapping, I still want to keep it like that, because it's generally the same thing if you had both of them. (Yr 7 boy)

The other four students argued that the number of people clapping was better:

> The volume of noise, it can be measured but, in this circumstance, it wouldn't be great because one person could be clapping as loud as 5 people clapping. (Yr 9 boy)

Their noticeable puzzling over this issue might be related to their unfamiliarity with the measurement of volume of sound compared to the counting of discrete objects – 'people' in this situation. Noise levels and their unit of measurement (decibels) are not typically explored in school mathematics. It could also relate to difficulties imagining volume of noise realistically when individuals in the audience might clap at different speeds and volumes, as suggested by the Year 9 boy's response. A different wording for the situation (e.g., 'the people all clapped as loud and as fast as they could') might change the way students reason in the task about the volume of noise as a variable.

In terms of the *Relationship between variables*, all of the students chose and kept a continuous representation as opposed to discrete data points, even though a majority used a discrete variable (number of people clapping). The literature highlights developmental progression from discrete to continuous graphs (Nathan & Kim, 2007), yet it was not clear if these students had 'moved on' from discrete representations and now drew continuous graphs by default, or if they had not encountered the need to differentiate between discrete and continuous data types. One student (Yr 7 girl) initially constructed a *linear* graph, by drawing scales on the axes (with time in seconds) and plotting individual points (a local approach rather than sketching globally). The literature highlights younger students' propensity towards constructing linear representations in realistic contexts (e.g., Lehrer & Schauble, 2001), and there is a noticeable focus on plotting linear graphs in the lower secondary curriculum content (ACARA, 2017). However, she changed to a non-linear curve after critiquing the example graphs (and without plotting points this time). She justified her revision:

> I like the thought that it'd take longer at the start and then when more people join in, people get more confident and it suddenly goes like that. (Yr 7 girl)

Even though the students were found to construct continuous exponential graphs (and also evidenced a mixture of local and global approaches) across the year levels, the younger students tended to begin their curve at the origin (0, 0) whereas the older students began at (0, 1). This is suggestive of prior learning experience with exponential graphs, e.g., $y = 2^x$, which has a y-intercept of (0, 1). These are prescribed at Year 9 in the national curriculum. A Year 8 student commented that he hadn't yet been taught non-linear graphs but nonetheless drew an exponential curve based on his local reasoning (with 100 people and 50 seconds). There was also a noticeable pattern of development, particularly from Year 9 onwards, in students verbalising more formal covariation terms when justifying their choice of the graph, for example:

> Everybody starts clapping right now, as in – I'm not really sure how to put it, it's just – it's gradually, those who are around her, so it's just an immediate clap, clap, more, more, more. (Yr 7 boy – *informal language*)

> This [pointing to y-axis] is the dependent variable and this [time on the x-axis] is the independent variable… I believe that it will increase exponentially. (Yr 12 boy – *formal language*)

This suggests that covariational reasoning about familiar real-life functional situations, particularly with time as a variable, can be intuitively justified by younger students – in this example, by using words like 'immediate clap, clap', and 'more, more, more'. This reasoning can then be expressed in more formal mathematical language from the stage it is introduced to students – in the above example, using 'dependent' and 'independent variable', and 'increase exponentially.'

In terms of *Contextual features of the real-life situation*, the initial silence of the audience – only the two Year 8 students attended to it in their initial graph construction. And although all of the students did then discuss this feature in their critique of the example graphs (Ben and Cath both drew gaps for silence), surprisingly only five (out of 10) chose to revise their graphs to include it, suggesting the need for further research into types of warrants that are salient enough to students to prompt revision of their own claims.

In terms of the other key contextual feature of the situation – a maximum being reached in the situation once everyone is clapping – only three students (Yr 9 boy, Yr 10 girl, and Yr 12 boy) initially attended to it, two with a horizontal leveling-out of the curve (Yr 9 boy; the Yr 12 boy also added a horizontal asymptote). Interestingly, the previously mentioned Year 10 girl (who didn't use time as a variable initially) *had* attended to saturation in her initial graph by drawing a vertical asymptote to the right of her (exponential continuous) graph and explained that it represented reaching 'the maximum amount of people in the audience'. Surprisingly, she left it out of her revised graph (with time on the x-axis). Vice versa, the Year 12 girl (who also did not use time on her original graph) did not initially attend to saturation in her first graph, but it seemed that her decision to change to 'time' actually stimulated her reasoning about how the revised graph would now need represent saturation. She opted for a sketching a negative parabola:

> The volume of noise wouldn't be constantly increasing forever. It would eventually have to come – which I guess is the issue for my graph as well – that it has to come back down again. (Yr 12 girl)

None of the example graphs had highlighted this saturation feature so it is not surprising that it was not mentioned by most of the students. The Year 12 boy, who had attended to saturation in his initial graph, actually noticed its absence in the three fictitious examples and commented:

> I stand by my justification of it asymptoting at a certain level due to in my perspective there being a maximum number of people within the audience. (Yr 12 boy)

This suggests that this student was not fazed by the three examples all sharing the same weakness and maintained his position.

Conclusion

This exploratory study found that the inclusion of critiquing fictitious student responses in a graphing task seemed to stimulate students' attention to their warrants in their own graph, particularly in helping them notice alternative choices of variable, the nature of the relationship between the variables, and representing the contextual feature of the audience initially sitting in stunned silence. Initially most, and subsequently all, of the students chose time as the independent variable, and placed it on the *x*-axis of their graphs. A majority of students had chosen the discrete variable 'number of people clapping' but then noticed 'volume of noise' in the fictitious examples. Their puzzling over the differences between them suggests that loudness may be a less familiar variable in terms of how to measure and represent it than 'number of people'. Also, initially only two and then subsequently all students noticed, and correctly interpreted, the way two of the example graphs represented the initial stunned silence of the audience. Very few students attended to a contextual feature which was deliberately excluded from the examples (reaching a maximum). Additionally, the discrete variable (number of people clapping) was not represented correctly in the examples, and none of the students attended to this aspect correctly either. This suggests that task design – the choice of correct and incorrect features in the fictitious graph examples – played a role in influencing the students' learning by drawing their attention to or away from those different features.

The influence of prior learning was noticeable in a younger student attending to inappropriate linearity once she had critiqued the example graphs, and in the increasing sophistication of the language used by the students in their arguments. Although the older students used more formal language than the younger students, they all demonstrated an intuitive capacity to reason about covariation concepts. This resonates with the literature that time-dependent situations are a familiar setting for students to explore covariation (Goldenberg & Kliman, 1988; Leinhardt et al., 1990). The students also evidenced being comfortable with the explicit use of argumentation terms (claim, data, warrant) and definitions from the Toulminian framework, in the task handouts and in the interviewer's questions. We surmise that the use of fictitious task response examples has potential for developing students' analytical skills, for drawing their attention to specific mathematical concepts, and also for providing an emotionally safe context to learn to critique others' ideas. An implication for mathematics teachers more generally might be to include opportunities for their students to respond to a task themselves, then analyse and discuss the strengths and weaknesses of some fictitious task responses or solutions (made up by the teacher or from anonymous students) that draw attention to a particular mathematical concept/misconception or problem-solving issue.

In our context of graphing real-life situations, further research on the number and choice of fictitious example graphs is planned to explore these issues. We also intend to research the dialogic processes of argumentation through sequences of collaborative activities in the social context of the classroom. There is more to research on students learning to critique each other's mathematical ideas appropriately and effectively.

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