Noticing structural thinking through the CRIG framework of mathematical structure

Mark Gronow
Central Queensland University
<m.gronow@cqu.edu.au>

Structural thinking skills should be developed as a prerequisite for a young person's future mathematical understanding and a teachers' understanding of mathematical structure is necessary to develop students' structural thinking skills. In this study, three secondary mathematics pre-service teachers (PSTs) learned to notice structural thinking through the CRIG framework of mathematical structure. The framework consists of Connections, Recognising patterns, Identifying similarities and difference, and Generalising and reasoning. I report here on how the CRIG framework helped the PSTs' notice structural thinking.

To develop an ability to notice structural thinking, teachers must first of all be aware of *mathematical structure*. Mason et al. (2009) defined mathematical structure as "the identification of general properties which are instantiated in a particular situation as relationships between elements or subsets or elements of a set" (p. 10). Stephens (2008) described structural thinking as an awareness of how mathematical properties develop into generalisations. Furthermore, Mason et al. (2009) promoted structural thinking as understanding the concepts and knowing procedures to use and when solving mathematical problems.

Varied theories exist about structure; as mathematical structure or structural thinking. Wertheimer (1945) proposed that mathematical structure is knowing how a formula is connected to a mathematical concept. Hiebert and Lefevre (1986) combined conceptual and procedural knowledge as 'proceptual' thinking across mathematical processes. Stephens (2008) defined 'structure' as synonymous with relational thinking (Skemp, 1976). Schwarz et al. (2009) proposed that structural thinking is knowing the relationships and connections between mathematical concepts.

The concept of structural thinking in mathematics is not clearly understood by many teachers of mathematics (Richland et al., 2012). Mason et al. (2009) stated that teachers' awareness of structural relationships transforms students' thinking and disposition to engage, they believe that teachers need to focus on structure so students can think structurally. Research in teachers' awareness of mathematical structure or structural thinking is limited. Gronow et al. (2020) explored secondary mathematics teachers' understanding and use of mathematical structure. Their study investigated how teachers used mathematical structure and encouraged structural thinking through components of mathematical structure: Connections, Recognising patterns, Identifying similarities and differences, and Generalising and reasoning. The four components, known as CRIG pedagogical framework of mathematical structure developed during Gronow et al.'s (2020) study found teachers' identified with structure but were not aware they were using it in their teaching. The CRIG framework, in this study, is introduced to PSTs as an effective mechanism for learning to notice structural thinking. The four components of the CRIG framework are detailed next.

Connections. Vale et al. (2011) recognised connections between mathematical representations as fundamental to structural thinking. Making connections between contexts or concepts allows learners to develop mathematical understanding. Mathematics teachers

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 211-218. Singapore: MERGA.

make connections between prior, present and future learning, and in real-world contexts of mathematical representations.

Recognising patterns. Patterns are essential in children's mathematical development which begin with their observations of the natural world. Children recognise, observe and generate patterns before reaching school and learn patterning in formalised learning situations that develop structural thinking processes that lead to a deeper understanding of abstract mathematical concepts. Mulligan and Mitchelmore (2009) found that children's structural thinking, identified in patterning awareness, is essential for mathematical concept formation in future learning.

Identifying similarities and differences. Learners develop structural thinking through noticing the differences in mathematical representations. Mason (1996) believed structural thinking is noticing similarities and differences in mathematical relationships. Mulligan and Mitchelmore (2009) discovered that children who found similarities and differences in patterns were involved in structural thinking.

Generalising and reasoning. Mason (2008) described this as an activity that develops a more in-depth experience of mathematics. Mathematical thinking that eventuates into a generalised fact is structural thinking, it connects mathematical relationships from concrete representations to abstract ideas. Mason et al. (2009) wrote that appreciation of structure involves the experience of generality. Stephens (2008) applied structural thinking to designing arithmetic questions. He asserted that children who could articulate a generalised principle underlying a whole problem were thinking structurally.

The framework of noticing also supports the process PSTs learning to notice structural thinking. Scheiner (2016) identified how noticing is not restricted to a single process. Mason (2002) asserted that "every act depends on noticing" (p. 7), he used the term "awareness" to characterise the ability to notice, referring to noticing as an awareness of what one is attending to. In this study, noticing structural thinking implies an awareness of understanding and use of mathematical structure.

By adopting Mason's (2002) approach to noticing, the development and use of mathematical structure has emerged as a form of directing PSTs' attention to their mathematical thinking. Mason studied what he noticed when doing mathematics and called what he noticed the structures of attention of how one thinks mathematically. The aim of this study is for the PSTs to notice structural thinking through learning the components of the CRIG framework of mathematical structure. The PSTs use of the CRIG framework provides an opportunity to detect their awareness of structure, thus answering the research question: *How does the CRIG framework help PSTs to notice structural thinking?*

Method

Context and participants

PSTs in their final year Bachelor of Education/Bachelor of Arts (secondary mathematics) degree at a Sydney university were invited to participate in this study. Three PSTs, referred to as Ms K, Ms M, and Mr T, volunteered to participate in the study during their professional experience placement. Each PST taught mathematics at a secondary school in metropolitan Sydney. Ms K taught an accelerated Year 9 class, Ms M taught a top streamed Year 8 class, and Mr T taught a mixed ability Year 7 class. The PSTs were familiar with the concept of mathematical structure through the content of courses studied in their undergraduate degree; however, they had no prior knowledge of the CRIG framework.

Study design, instruments, and data collection

The study design comprised of three cycles of: professional learning workshops (PLWs), which were audio recorded. Video recordings of PSTs' mathematics lessons and a noticing reflection audio recording of PSTs reviewing a recorded segment of their mathematics lessons.

Analysis

The audio recordings of the PLWs and noticing reflections were all transcribed to a word document and uploaded to NVivo (QSR International, 2017). The videos of the mathematics lessons were also uploaded to NVivo. NVivo was used to code the data from the PLWs, mathematics lessons and noticing reflections for PSTs' utterances and comments that identified a CRIG component. The data were analysed for evidence of the PSTs' noticing of structural thinking through the PSTs attending to the CRIG framework. The videos acted as the main source of evidence for identifying the PSTs noticing structural thinking through their use of the CRIG framework when teaching. The PLWs and the noticing reflections provide further evidence of the PSTs attention to the CRIG framework.

Results

This section presents a summary of the data collected for each PST from the three cycles of PLWs, mathematics lessons and noticing reflections. An outline of the results from the PLWs are given, followed by exemplars of each PSTs' utterances from the mathematics lessons and comments made during the noticing reflections in Tables 1, 2 and 3, coded to a CRIG component.

During the PLWs, the PSTs were taught to notice structural thinking through the CRIG components. The first PLW began with a presentation on the CRIG framework, followed by a viewing of a video titled Related Problems: Reasoning About Addition (Teaching Channel, 2017), where a teacher used the CRIG components to teach addition to a Year One primary class. Ms K *Recognised patterns* in the teacher's instructions to students. Ms M also *Recognised patterns* as a teaching strategy to engage the students. Mr T noticed that the students used *Similarities and differences* to make generalisations.

In PLW 2, the PSTs viewed a video recording of a child attempting several different arithmetic problems, they were asked to examine the child's mathematical thinking when solving the problems. Ms K noted the child relied on calculations and did not *Identify Similarities and Differences* between the numbers. Ms M noticed the child was using *Generalising and Reasoning* in her structural thinking when she recognised that the problem could be solved another way. Mr T stated the child "Got it after the CRIG prompt, meaning she has structural understanding."

In PLW 3, the PSTs considered how the CRIG framework could be applied to teaching the expansion of binomial products. Ms K made *Connections* to the distributive law and expanding the expression using the FOIL method. Ms M was *Identifying Similarities and Differences* when changing numbers, pronumerals, signs and coefficients in the binomial expression. Mr T stated that *Generalising and reasoning* was identified as a way to summarise the process of expansion and apply it in other mathematical contexts.

Table 1
Exemplars of Ms K using the CRIG Framework to Notice Structural Thinking

Cycle	Mathematics lesson	Noticing reflection
1	*Topic: Simultaneous equations* *Connections* to the relationship between the graphs' intersection points and solving the equations simultaneously. *Recognising Patterns* of the power of x to determine the curve's shape. *Identifying similarities and differences* "What is different about the line's shape?"	*Connections* between the equation and the graph. "I think to show how the y^2 and the x^2 is giving us part of the circle, that relationship." *Identifying similarities and differences* between graphs and equations: "So, they could see that all of them had a square except the last one."
2	*Topic: Angle sum of polygons* *Connections* to prior learning "How did we prove the angle sum of the quadrilateral?" Angle sum of a polygon formula: *Recognising patterns:* "Can you find the pattern of what is going on between the number sides and triangles?" *Generalising and reasoning:* "Calculate the interior angle sum of any polygon."	*Recognising Patterns* to develop the formula: "They understood it better with the pattern." *Identifying similarities and differences* different patterns helped students' thinking. "I had the triangles meeting at a point. I adjusted it as I saw the pattern they were working out."
3	*Topic: Quadratics* *Connections* "Quadratics and parabolas go hand-in-hand. The visual representation of a quadratic is a parabola." *Identifying Similarities and Differences* of the x^2 expression in an equation "This is not of degree two; it is a power of negative two. So, this is not a quadratic." *Generalising and Reasoning* relationships between the equation and the graph.	*Connections*: "I was connecting it to when we did the non-linear simultaneous equations." *Recognising Patterns*, "Rather than drawing random graphs, I'd link them to recognise any patterns from factorised quadratics." *Generalising and Reasoning* "Generalising the solutions of when crossing the x-axis."

Table 2
Exemplars of Ms M using the CRIG Framework to Notice Structural Thinking

Cycle	Mathematics lesson	Noticing reflection
1	*Topic: Circumference of a circle* *Connections* to a real life example of a pizza as a sector of a circle. *Recognising patterns* in the ratio of a circle's circumference and diameter. *Similarities and Differences* comparing the circle's radius and diameter.	*Generalising and Reasoning* through students' discussion when dividing the circumference by the diameter. "I'm looking at what they just did. I'm asking them to contribute what they found and see what they conclude from what they've done."

2	*Topic: Area of composite shape* *Identifying Similarities and differences* to explain the formula of the area of circles. "Area equals πr^2 which is the same as saying $\pi \times r \times r$." *Generalising and reasoning* "How come we have π for every circle? Because the circumference divided by the diameter was always equal to π."	*Recognising patterns* "asking them how to figure out the area. That could have been kind of recognising patterns." *Identifying Similarities and differences* "How to write something in exact form and not exact form *Generalising and reasoning* "Asking them questions they can conclude."
3	*Topic: Volume of a cylinder* *Connections* of a real-world problem: "This is a picture of the sinkhole. What shape does it look like?", *Generalising and reasoning* "What do we need to know to solve this problem? What are we trying to find in the end?"	*Connections* "How they could use previous things they've learnt." *Recognising patterns* "By helping them recognise patterns to work mathematically." *Generalising and reasoning* "Recognising the meaning and interpreting the information."

Table 3
Exemplars of Mr T using the CRIG Framework to Notice Structural Thinking

Cycle	Mathematics lesson	Noticing reflection
1	*Topic: Ordering fractions* *Connections* to a real example "What is one-third of my chocolate bar." *Identifying Similarities and Differences* in ordering fractions "When you look at this, which one's bigger? Or, which one's smaller?" *Generalising and Reasoning* defining a rule "The size of the parts needs to be the same."	*Connections* "I should have reworded the question because this was what we did last lesson." *Recognising Patterns "*What do you notice I'm doing with these numbers?" *Identifying similarities and differences* "Show the diagram of shaded fractions not symbolically."
2	*Topic: Adding and subtracting fractions* *Identifying Similarities and Differences* "What do you notice about the numerators?" *Generalising and Reasoning,* using a whole number method to add fractions. "So, if $1 + 1 = 2$, then, if I use the same thing, for a $\frac{1}{2} + \frac{1}{2}$, is $1 + 1 = 2$, and $2 + 2 = 4$, so it's over $\frac{1}{4}$. Right?"	*Recognising patterns*: "I tried to set up some patterns and then asked them to recognise the patterns." *Generalising and Reasoning* "I've tried to incorporate generalisation in terms of asking them, 'What do you think would be the next pattern?'"

3	*Topic: Stem and leaf plot graphs* *Similarities and Differences* between graphs and stem-and-leaf plots. "Now what were the things we compared. What's similar?" *Generalising and reasoning* to analyse stem-and-leaf plot data. "Take a look at your graph and talk to the other person and tell them what the graph tells you?"	*Recognising patterns*: "So I should have put one number on so the students to see a pattern." *Identifying Similarities and Differences* "I should have asked about the placement of these three numbers: "How are they different?"

Discussion

During this study, the PSTs' noticing of structural thinking developed through their learning of the CRIG pedagogical framework of mathematical structure. Noticing of structural thinking was evident in their references to the CRIG framework drawn from the statements made during the PLWs, utterances in their mathematics lessons, and noticing reflection comments. Exemplars given demonstrate the PSTs' noticing structural thinking through the CRIG components.

The PSTs use of the CRIG components were identified in varied pedagogical strategies. Ms K encouraged students to use a pattern to find the rule for the angle sum of a polygon, Ms M used real world examples for each of her lessons to connect students understanding to the mathematical concept and Mr T used the CRIG components in his questions.

The PSTs' teaching accommodated the CRIG framework and supported their understanding of the mathematical content. Ms K considered other patterning approaches to finding a rule for the angle sum of a polygon and Ms M noticed similarities and differences in binomial expansions. The PSTs' pedagogy focused on a structural thinking learning environment, Ms K promoted students' thinking by challenging them to connect the equation to a graph, Ms M connected mathematical concepts to real-world examples and Mr T asked questions so students would notice patterns, and similarities and difference. In their noticing reflections, the PSTs stated how the CRIG framework supported their teaching. Ms K, was thinking of her future teaching: "If I were to do this again, I'd teach the patterning way, and I would incorporate the CRIG more." Ms M stated CRIG helped her understand student thinking "They're trying to understand the difference between volume and capacity." Mr T reflected on how CRIG improved his explanations. "I should have made it more explicit, by connecting to their prior experience." The CRIG framework in these cases supported the PSTs' noticing of structural thinking.

Prescott and Cavanagh (2007) found that secondary mathematics PSTs tended toward a traditional teaching pedagogy. Awareness of the CRIG framework encouraged the PSTs in this study to move beyond traditional teaching pedagogy. The PSTs were more inclusive of student learning, as noted when asking CRIG component focused questions. Mr T's questions promoted students' structural thinking. He challenged students' thinking about why using a whole number method when adding fractions was incorrect. "So, if $1 + 1 = 2$, then, if I use the same thing, for a $\frac{1}{2} + \frac{1}{2}$, is $1 + 1 = 2$, and $2 + 2 = 4$, so it's over $\frac{1}{4}$. Right?" The PSTs diverse pedagogical strategies also saw them use the CRIG components when instructing or communicating with students. In her second mathematics lesson, Ms K used *Recognising patterns* to help students develop the angle sum of a polygon formula. As the students had discovered a different pattern, one that was not considered by Ms K, she acted in-the-moment and noticed the students' new approach, she encouraged her students to continue with their strategy and asked one student to explain it to the class. Ms M promoted student involvement in her lessons by arranging students in groups to complete activities, many of which had a real-world experience, such as, here final lesson of finding the volume of a cylinder as a sink hole.

The professional learning program to understand and use the CRIG framework helped the PSTs' to notice structural thinking. Ivars et al. (2018) identified the need for a specific framework for PSTs to have effective noticing. The CRIG framework provided this focus. The ability of the PSTs to understand the CRIG framework and to use it demonstrated its simplicity as a practical and useful tool for teachers of mathematics. The PSTs' content knowledge was established from their extensive mathematical background in their university studies. The CRIG framework, however, deepened the PSTs structural understandings of mathematical relationship, for example Ms K's students finding an alternative approach to finding the angle sum of a polygon.

The PSTs' lack of professional experience before this study could have influenced their fundamental understanding of the CRIG framework and their ability to notice structural thinking. However, having more teaching experience in the future will provide continual opportunities notice structural thinking through the CRIG framework when doing mathematics and when teaching. The PSTs' teaching experience was restricted to their university professional experience program. Researchers have identified how PSTs' limited experiences influence what they attend to when teaching. Star and Strickland (2008) found that secondary mathematics PSTs were not good at noticing mathematical content. Mason (2002) also asserted that PSTs lack experience in recognising and using classroom interactions effectively to promote mathematical understanding. Contrary to the results of these studies, the PSTs in this study produced mathematics lessons that engaged students with activities, instructions and questions that focused on developing students' structural thinking through using the components of the CRIG framework. The PSTs effectively demonstrated an ability to learn and apply the CRIG framework as a new pedagogical skill to mathematical content that they had not taught before. The introduction of structural thinking through the CRIG framework could be regarded as an extra burden for the PSTs to consider when teaching. Nevertheless, the evidence indicates that the PSTs were comfortable with identifying and including the components of the CRIG framework in their lessons and were able to notice structural thinking.

The PSTs were able to articulate the benefits of the CRIG framework to notice structural thinking they indicated that the CRIG framework had shaped their noticing structural thinking and had changed their teaching. Ms K stated that thinking structurally helped her make sense and explain mathematical concepts. In the final PLW, Ms K stated, "You structure your practice to facilitate deeper thought as to what and how things made sense."

Conclusions and Further Research

The CRIG framework proved to be useful for helping PSTs to notice structural thinking. The CRIG framework provided the PSTs with a foundation for teaching mathematics that helped them focus on developing their understanding of mathematical structure. Moreover, this provided PSTs opportunities to notice structural thinking.

Mason (2002) introduced the concept of noticing into the lexicon of mathematics education, and with his colleagues (Mason et al. 2009) the notion of teachers' noticing of structural thinking has emerged as a significant contribution to mathematics teaching. PSTs noticing of structural thinking as the focus of this study has demonstrated, as evident from the results, that there is potential to advance the discourse of mathematics teaching in this area.

The introduction of mathematical structure in the teaching and learning of mathematics and the noticing of structural thinking has implications for future research in mathematics teaching. Future research could consider how developing noticing structural thinking through the CRIG framework may benefit practicing teachers of mathematics (e.g., primary, secondary, pre-service, novice, experienced, and out-of-field teachers).

References

Gronow, M., Mulligan, J., & Cavanagh, M. (2020). Teachers' understanding and use of mathematical structure. *Mathematics Education Research Journal,* 1-26. https://doi.org/10.1007/s13394-020-00342-x

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hierbert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Erlbaum

Ivars, P., Fernández-Verdú, C., Llinares, S., & Choy, B.H. (2018). Enhancing noticing: Using a hypothetical learning trajectory to improve pre-service primary teachers' professional discourse. *Eurasia Journal of Mathematics, Science and Technology Education*, *14*(11). https://doi.org/10.29333/ejmste/93421

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.) *Approaches to Algebra: Perspectives for research and teaching* (pp. 65-86). Dordrecht: Kluwer Academic Publishers.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: RoutledgeFalmer.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 57–94). Mahwah, NJ: Lawrence Erlbaum.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, *21*(2), 10–32.

Mulligan, J., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal*, *21*(2), 33–49.

Prescott, A., & Cavanagh, M. (2007). Pre-service secondary mathematics teachers: A community of practice. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.) *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education.* (pp. 276–284). Seoul, Korea: Society of Educational Studies in Mathematics.

QSR International. (2017). NVivo 12 [Computer software]. http://www.qsrinternational.com.

Richland, L. E., Stigler, J. W., & Holyoak, K. J. (2012). Teaching the conceptual structure of mathematics. *Educational Psychologist*, *47*, 189–203.

Scheiner, T. (2016). Teacher noticing: Enlightening or blinding? *ZDM – Mathematics Education*, *48*, 227–238.

Schwarz, B. B., Dreyfus, T., & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In B. B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11–41). New York, NY: Routledge

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77,* 20–26.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education*, *11*(2), 107–125.

Stephens, M. (2008, September). *Designing questions to probe relational or structural thinking in arithmetic*. Paper presented at the 5th Annual Conference of the International Society for Design and Development in Education, Cairns.

Teaching Channel. (2017). *Related problems: Reasoning about addition*. Retrieved from https://www.teachingchannel.org/video/reasoning-about-addition-nsf

Vale, C., McAndrew, A., & Krishnan, S. (2011). Connecting with the horizon: Developing teachers' appreciation of mathematical structure. *Journal of Mathematics Teacher Education*, *14*(3), 193–212.

Wertheimer, M. (1945) *Productive thinking*. London: Tavistock.