By teaching we learn: Comprehension and transformation in the teaching of long division

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Despite recent calls to adopt practice-embedded approaches to teacher professional learning, how teachers learn from their practice is not clear. What really matters is not the type of professional learning activities, but how teachers engage with them. In this paper, we position learning from teaching as a dialogic process involving teachers' pedagogical reasoning and actions. In particular, we present a case of an experienced teacher, Mr. Robert, who was part of a primary school's mathematics professional learning team (PLT) to describe how he learned to teach differently, and how he taught differently to learn for a series of lessons on division. The findings reiterate the complexity of teacher learning and suggest possible implications for mathematics teacher professional development.

There have been recent calls to incorporate collaborative inquiry-based approaches embedded in teachers' practices to improve the teaching of mathematics. This has led to the adoption of collaborative professional learning activities such as video clubs (van Es & Sherin, 2002), Lesson Studies (Clea Fernandez & Yoshida, 2004), and collaborative lesson research (Takahashi & McDougal, 2016). However, it would be "wishful thinking" to expect that teachers would learn just because they gather "to talk about practice" (Bryk, 2009, p. 599). In Singapore, while there is extensive support for teachers to engage in learning communities for the purpose of working collaboratively to learn and improve their teaching, it is unclear whether and how teachers learn from these activities (Hairon & Dimmock, 2012). What really matters, therefore, is not the kind of professional development activities, but rather how teachers engage with these activities (Choy & Dindyal, 2019; Fernandez, et al., 2003). As claimed by Sherin (2002), learning from teaching occurs when teachers have opportunities to *negotiate* among three aspects of their teacher knowledge: understanding of mathematics, curriculum materials, and knowledge of how students learn. In this paper, we refer to Sherin's (2002) metaphor of teaching as learning to examine how a primary mathematics teacher, Mr. Robert, learned from his teaching through a dialogic process involving pedagogical reasoning and action (Shulman, 1987) as he worked with his colleagues on a series of lessons to teach division for Primary Three pupils (aged 9). The paper is framed by the following question: How does a primary mathematics teacher learn from his own teaching via his participation in a professional learning team?

Theoretical Considerations

Following Shulman (1987), we see that teaching "begins with an act of reason" and "continues with a process of reasoning" to culminate in a series of pedagogical actions, and "is then thought about some more until the process can begin again" (p. 13). In other words, with the aim of improving teaching, teachers need to learn to use their knowledge base for teaching to provide justifications for their instructional decisions through a process of

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pedagogical reasoning. This process involves taking what one understands about content and "making it ready for effective instruction" (Shulman, 1987, p. 14), through a cycle of activities involving comprehension, transformation, instruction, evaluation, and reflection leading to new comprehension. According to Shulman (1987), comprehension refers to how teaching first involves understanding the content and purpose. When possible, teachers should *comprehend* what they teach in different ways and relate these ideas to other ideas within and beyond the subject. The key distinctive of a teacher's work lies in how a teacher *transforms* his or her content knowledge into "forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students" (Shulman, 1987, p. 15). Transforming this knowledge involves preparation, representation, instructional selections, adaptations of these representations and tailoring the representations to specific students' profiles. Although comprehension and transformation can occur at any time during teaching, Shulman (1987, p. 18) sees these two processes as "prospective", occurring before *instruction*, an "enactive" performance in the classrooms. Moving on to a more retrospective process, Shulman highlights *evaluation* as the means to assess students' understanding and to provide feedback. But it is through *reflection*, by which a teacher looks back at the instructional processes and experiences, that a teacher learns from his or her experiences. This learning is encapsulated in the process of *new comprehension* where teachers have a better understanding of teaching and learning.

Shulman highlighted that *new comprehension does not necessarily follow through from reflection*. This explains that some teachers learn from their teaching experiences, while others do not. Hence, we argue that new comprehension of content, student learning, and teaching actions occurs when a teacher has a shift of attention, gaining awareness of new possibilities in teaching and learning (Mason, 2002), or simply when a teacher *notice* critical aspects of teaching and learning. These new insights expand the teacher's current cluster of resources, orientations, and goals (Schoenfeld, 2011), which in turn becomes the base from which the teacher make sense of instruction. Moreover, as Choy (2016) has highlighted, productive noticing can take place during planning, instruction, and reviewing of lessons. Consequently, new comprehension can occur during any of the activities of Shulman's model of pedagogical reasoning and action.

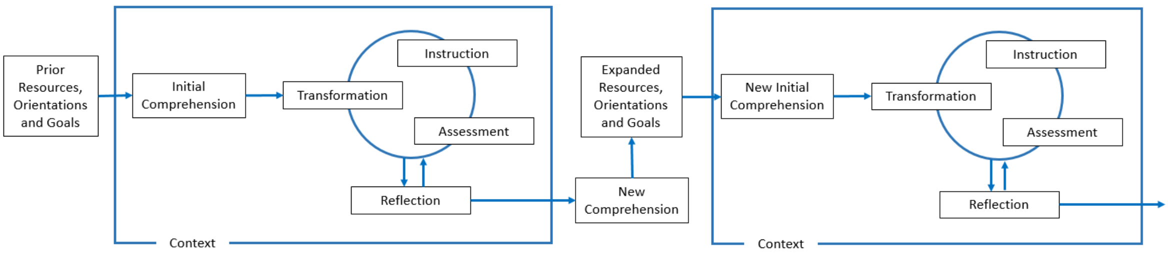

Figure 1. Adapted Model of Pedagogical Reasoning and Action.

Building on ideas from both Shulman (1987) and Schoenfeld (2011), we developed an adapted model of pedagogical reasoning and action to highlight the dialogic processes involved when learning from teaching. The strength of Schoenfeld's ideas lie in the fact that teaching is goal-directed, rests on a set of resources, and driven by a teacher's orientations. The orientations aspect is quite important as it explains why some teachers loop back to do happily what they have been used to doing and in doing so, submit to the exigencies of the context. Thus, in the model above, we show that teaching starts with some prior resources, orientations and goals (ROG) and some initial comprehension. The teacher then transforms the initial comprehended ideas into a form suitable for teaching the students. The iterative and cyclical processes of transformation, actual instruction and assessment of learning feed forward to the reflection of the teacher (to different extents for different teachers). This process leads to some new comprehension, which may or may not lead to a new expanded set of ROGs and the cycle repeats. What this adapted model affords us is the opportunity to capture the complexity of the dialogic processes involved when teachers learn from their practice. On one hand, teachers comprehend new ideas about content and teaching to apply them in their instruction. On the other hand, they learn new ideas as they apply their new comprehension in their instruction. We shall now illustrate the dialogic nature of a teacher's learning from teaching through the example of Mr. Robert, who learned and applied new ideas about division as part a professional learning team.

Methods

The data presented in this paper were collected as part of a larger project which aims to develop the proof of concept for a new professional learning model for mathematics teachers. Drawing on current theoretical perspectives of teacher noticing (Dindyal, et al., 2021; Fernandez & Choy, 2019), we conceptualized professional learning sessions where teachers would have opportunities, in the context of a *community of inquiry* (Jaworski, 2006), to work and co-learn with us by:

1. Focusing on unpacking the mathematics in the curriculum documents;
2. Investigating how a topic may be unpacked in terms of a sequence of lessons, and a lesson as a sequence of tasks;
3. Teaching a sequence of lessons as part of a unit;
4. Observing and reflecting upon a sequence of lessons;
5. Articulating their learning from the observations; and
6. Suggesting possible changes to the sequence of lessons and tasks based on their learning.

As highlighted by Jaworski (2006), sustainability is often an issue with communities of practice and learning. To ensure sustainability and feasibility, we co-designed protocols to guide each professional learning session as teachers worked together to plan and teach a unit of work. As each session lasted about an hour and so, it was crucial that we built in specific focus for each session to facilitate more productive discussions. We also provided teachers access to relevant research and practice-based articles when requested, as well as templates to facilitate teachers' inquiry processes. Data collected include voice and video recordings of the discussion during the sessions, photographs of lesson artifacts such as lesson plans, discussion notes, and when available, samples of students' work.

In this paper, we report how Mr. Robert, an experienced primary mathematics teacher from Eunoia Primary School (pseudonyms), perceived and harnessed affordances as he worked with a team of nine other teachers to discuss the teaching of long division to Primary Three pupils (aged 9). The sessions were facilitated by a Lead Teacher, Ms. Mandy, who had extensive experience teaching in the primary school. We were present at the sessions as knowledgeable others to share new ideas for teaching. We did not insist that the teachers adopt any particular idea that we had shared. Instead, we left all the instructional decisions to them because we wanted to investigate their decision-making processes. The vignettes described here were developed from data collected from four discussion sessions and a video recording of a short 20-minute segment of Mr. Robert's teaching. The voice recordings of the discussion sessions were parsed for segments related to discussions on the teaching and learning of long division. Notable episodes involving mathematically significant moments were marked for further analysis. Irrelevant incidents such as logistics and administrative matters were discarded. The marked segments were reviewed, and initially coded for processes related to our adapted model of pedagogical reasoning and action (See Figure 1). The reviewed segments were then transcribed before they were coded using a "thematic approach" (Bryman, 2012, p. 578) to highlight aspects of how Mr. Robert learned from his practice. We acknowledge that it is difficult to distinguish Mr. Robert's learning from the learning achieved by other teachers. Here, we assume that Mr. Robert, as an individual, can learn from his own teaching experiences, the ideas and experiences shared by his colleagues, as well as ideas we, as the research team, had shared with him. This corresponds to what Mason (2002) terms as the three worlds of experiences.

By Teaching We Learn: A Dialogic Process

Findings developed from our data suggest a dialogic process by which Mr. Robert had learned from his practice. First, we claim that he learned some new ideas about teaching division during the PLT discussions that offer opportunities to teach differently. Second, we propose that he taught differently by trying out some of the ideas learned, which in turn give rise to new comprehension. We will now describe vignettes of teachers' learning, focusing on Mr. Robert to highlight the dialogic process of learning from teaching.

Learning to Teach Differently

For the first two sessions, we worked with the teachers to unpack mathematical ideas related to division using the components of school mathematics as proposed by Backhouse et al. (1992), namely concepts, conventions, results, techniques, and processes. All the teachers were cognisant of the quotative and partitive notions of division and were fluent in performing the long division algorithm. They were also familiar with the key terms such as *quotient*, *remainder*, and *divisor* but not the term *dividend*. More specifically, they seemed to see quotient and remainder as part the answer to a division problem. For example, they would write $82 \div 4 = 20R2$, seeing 20 as the quotient and 2 as the remainder being the answer to $82 \div 4$. They did not think of other expressions that give the "same answer" as problematic. For instance, when we highlighted that $62 \div 3 = 20R2$, the teachers did not notice any issues with the notation. The usual way of writing the answer as "20R2" suggests that $82 \div 4$ is equal to $62 \div 3$. It appeared that the teachers did not notice this until we pointed out the issue to them. To highlight that the relationship between dividend, quotient, divisor, and remainder, we introduced the following "new equation":

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

For the teachers, this was something new and so we highlighted the relationship between division and multiplication, e.g., $20 \div 4 = 5$ is related to $20 = 4 \times 5$. More importantly, the equation involving dividend, quotient, divisor, and remainder was linked to how division can be demonstrated through manipulative, "splitting" the number into two or more components, and the long division algorithm. As an example, we showed how $82 \div 4$ can be visualised as distributing 80 items into 4 equal groups, with 20 items in each group; or seen as $80 + 2$, which can be rewritten as $4 \times 20 + 2$; and the long division which gives the quotient

20 and a remainder of 2 when 82 is divided by 4 (See Figure 2). The sharing of these new ideas provided opportunities for teachers to engage in comprehending the content and transforming their new-found knowledge to usable forms.

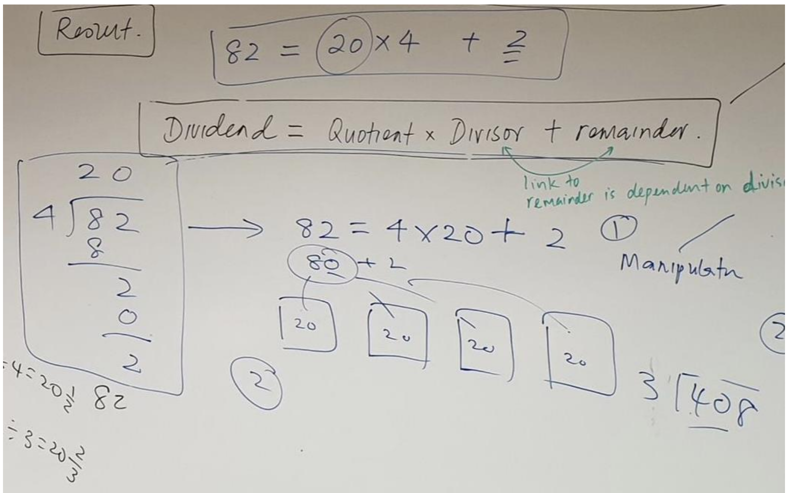

Figure 2. Snapshot of our sharing as documented on the whiteboard.

Teaching Differently to Learn

This “new” equation which highlighted the relationships between dividend, quotient, divisor, and remainder was taken up by Mr. Robert who tried to use this idea for his own teaching (Turn 15):

15.	Mr. Robert	I tried in my class, in fact I introduce in my class last week the quotient … like something like 9 = 4 + remainder something, you know the remainder thing? For the equation thing we did last week.
16.	Ms. Mandy	Dividend = Quotient × divisor + remainder.
17.	Teachers	[inaudible] remainder theorem.
18.	Mr. Robert	We did that last week. We could get the simple ones. But how you translate this to the long division working, it’s still a disconnect.
19.	Researcher	Yea. So, they could get this, they can understand this kind of thing …
20.	Teachers	Small numbers [inaudible]
21.	Mr. Robert	2 digits they can get, 3 digits they are gone.
22.	Researcher	Ok, so they could get 2 digits but not 3 digits.
23.	Mr. Robert	Maybe at the start we just started with 2-digit number. In fact, once it goes beyond 20, they are a bit lost already.

Mr Robert’s use of the “new equation” highlights how new ideas shared or discussed during PLTs can open up new opportunities to teach differently. As Mr. Robert *comprehended* these ideas for himself and *transformed* them into a sequence of examples involving 9, some 2-digit numbers, and even 3-digit numbers for his *instruction* (Turns 15, 21, and 23), he also began to be more aware of his students’ thinking (Turns 18 and 21). He was able to *assess* that his students may be confused when the numbers went beyond 20. However, it was his *reflection* about the possible disconnect between this “new equation” and the long division algorithm that opened up new threads of discussion and possibly opportunities to acquire new comprehension during the PLT.

Cycles of Learning to Teach Differently and Teaching Differently to Learn

Here, we begin to see how Mr. Robert's pedagogical reasoning and action had afforded opportunities for him to learn to teach differently. In the discussion that followed, we explored with teachers how students could make sense of division problems using different methods. For example, for 48 ÷ 3, students can do repeated addition: 3 + 3 + 3 + … = 48; or they can do repeated subtraction: 48 – 3 – 3 – 3 – … = 0. Students can also do skip counting: 3, 6, 9, …, 48; or reverse skip counting: 48, 45, 42, …, 0, amongst others. We also introduced the different chunking strategies (Putten et al., 2005), or what others refer to as partial quotients (Takker & Subramaniam, 2018), before we linked these informal strategies to the long division algorithm. For example, for 78 divided by 3, students may think of 3 × 10 = 30 and they will subtract 30 from 78 to give 48. Then they may subtract another 30 from 48 to give 18, and 18 divided by 3 is 6. Therefore, the answer is 10 + 10 + 6 = 26. This can be presented in this manner:

$$\begin{array}{r} 6 \\ 10 \\ \underline{10} \\ 3\,)\,78 \\ \underline{-\,30} \\ 48 \\ \underline{-\,30} \\ 18 \\ \underline{-\,18} \\ \underline{\;\;0} \end{array} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \quad 10 + 10 + 6 = 36$$

Mr. Robert then explored and used these ideas in his own teaching. As seen from the snapshots taken from the video snippet of his lesson (see Figure 3), we see how he had tailored some of the ideas for his students. Although Mr. Robert decided not to write the "new equation" explicitly, he used the ideas to go through some of the informal division strategies with his students. Mr. Robert's decision to use the "7R1" notation could be in part due to how all the approved textbooks present the answers.

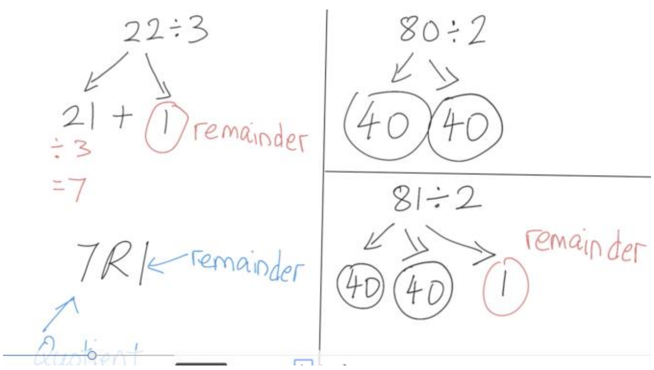

Figure 3. Snapshot of Mr. Robert's lesson to demonstrate informal strategies.

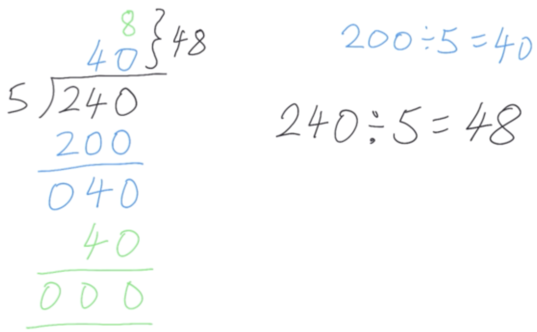

Figure 4. Snapshot of Mr. Robert's lesson to demonstrate the chunking strategy.

In another snapshot (see Figure 4), we see Mr. Robert demonstrating the chunking strategy (Putten et al., 2005) for his students. As seen from Figure 4, he used different colours to denote the different place values to make it clearer for his students. This use of colours was inspired by one of his colleagues in the same PLT who shared how the use of colours helped his students to grasp the importance of place value to understand long division. Here, Mr. Robert demonstrated the importance of learning new ideas from his colleagues and trying these ideas to see if they work. As we examine Mr. Robert's teaching and learning, we begin to gain insights into how he had learned from unpacking the mathematics, his colleagues, and knowledgeable others to be aware of different possibilities for teaching. But we also see how he had actually tried to teach differently in order to learn from his own teaching by assessing his students' understanding and reflecting upon the lesson.

Discussion

It was clear to us that the teachers in the PLT, including Mr. Robert, struggled with these ideas initially. However, it was also clear to us that teachers began to scrutinise these new mathematical ideas about division and explored the possibility of incorporating these ideas for their teaching. In other words, we argue that professional discussions involving experiences from different people, which focused on making connections between mathematics and pedagogy, have the potential for teachers to learn to teach differently. Nevertheless, for teachers' practices to change, it is necessary for them to try out these new ideas, as Mr. Robert had done, and reflect on their teaching to gain new insights. That is, for teachers to learn from their practice, it is necessary for them to learn about new ideas to teach differently and teach differently to learn these new ideas.

What Shulman (1987) implied in his model of pedagogical reasoning and action is that teachers can learn from their own teaching, or the idea of *docendo discimus*—by teaching, we learn. This idea aligns with the current notions of professional learning, which involve some form of job-embedded teaching inquiry activities, such as Lesson Study. However, implementing such teaching inquiry activities may be challenging due to time and resource constraints. There is a place and time for more elaborate teaching inquiry as part of a teacher's professional learning. But, what about the possibility of a teacher learning from his or her own teaching on a *day-to-day* basis? If we were to examine the processes of pedagogical reasoning and action, it became apparent that the model revolves around a teacher's day-to-day teaching activities. Hence, we propose two fundamental shifts in our thinking about professional learning. First, we see *every* teaching moment as an opportunity for professional learning. Second, we see pedagogical reasoning as the primary mechanism to effect changes in pedagogical actions, and eventually changes in one's system of resources, orientations, and goals. As exemplified by Mr. Robert, every moment in teaching can provide affordances for teachers to learn from their own practice.

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