Spatial reasoning and the development of early fraction understanding

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Young children are capable of engaging with ratio, measurement and operator meanings of fractions earlier than many national curriculum standards indicate, yet current trends in children's understanding of fractions in Australia, remain weak. Research suggests that spatial reasoning can positively influence mathematical knowledge; however, the connection between spatial reasoning and fraction understanding remains under-researched. This paper will present qualitative data from a Design Based Research study that examined a spatialised approach for teaching fractions to 6-and 7-year-old children. Findings indicate that spatial reasoning played an important role in helping children develop early fraction knowledge.

Examining the various perspectives of early fraction development reveals spatial reasoning may play an important role in the construction of such ideas. For example, research relating to young children's proportional and fraction understandings suggests that children engage in spatial scaling when reasoning in such contexts, which requires mentally shrinking or expanding spatial information to determine the relationships between the relative magnitudes (see Huttenlocher et al., 1999; Möhring et al., 2015) This work aside, Bruce et. al. (2017) state there are many 'gaps' in relation to what is known about spatial reasoning and its impact on mathematics education, including how different aspects of spatial reasoning may support young children's engagement with, and understanding of, early fraction concepts. To explore this phenomenon, the following research question was examined in a Design-Based Research (DBR) intervention study: *In what ways does the inclusion of a spatial reasoning approach to fraction instruction in the early years of schooling influence children's understanding of key fraction concepts?*

Background

Fractions are an essential building block of mathematical knowledge yet are complex because they are represented in multiple interpetations, such as *fraction as a relation* (ratio/rate/ proportion); *fraction as operator*; and, *fraction as a measure* (see Confrey 2008; Orbersteiner et al., 2019). Partitioning as an experienced based activity, provides the foundation for the development of children's understanding of fractions (Lamon, 1996; Siemon, 2003) including the closely associated concepts of unitising and equivalence. These concepts should be explored through the three aforementioned fraction contexts to enable flexible and sophisticated understandings to develop (Confrey, 2008). However, current research indicates that the key difficulties young children exhibit in developing early fraction ideas are concerned with making the connections between the concepts of partitioning, unitising and equivalence and the various representations and interpretations in which they are explored (Bobis & Way, 2018; Way et al., 2015).

A growing body of research indicates that young children can engage with these concepts utilising spatial reasoning (Congdon et al., 2018; Möhring et al., 2015). This research demonstrates that young children can adequately problem solve in ratio and proportional contexts when presented with spatial, non-symbolic representations. These fraction ideas are typically not introduced into the curriculum until upper primary and middle school years.

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Whilst this body of research is limited, it does provide the warrant to explore the impact spatial reasoning may have on helping children understand the relationships between early fraction concepts, contexts and associated representations, to mitigate the persistent challenges children exhibit in this area of mathematical learning.

Theoretical Perspectives

Spatial reasoning is defined using the National Research Council's [NRC] (2006) framework, which describes spatial reasoning as a problem-solving activity, involving the coordinated use of space, representation, and reasoning. For the purposes of this paper, the spatial reasoning constructs of spatial visualisation, spatial structuring and gesture will be of focus.

Spatial Visualisation

Lowrie et al. (2018) define *spatial visualisation* as "the ability to mentally transform or manipulate the visuospatial properties of an object…for example, visualizing a cube from its net or predicting a pattern on a piece of paper that has been unfolded" (p. 3). This spatial skill is the multi-step manipulation of objects generated or retrieved in one's mind. Given this definition, this skill involves visualising how different objects and contexts may be manipulated mentally to help develop ideas of partitioning unitising and equivalence within the three different meanings of fractions.

Spatial Structuring

Spatial structuring can be defined as "the mental operation of constructing an organization or form for an object or set of objects"(Battista & Clements, 1996, p. 503). This focusses on identifying objects' spatial components and their composites, and establishing what relationships exist between these elements. Fraction understanding is founded upon partitioning, unitising, multiplicative thinking, and patterning which are also foundational to spatial structuring (Papic et al., 2011).

Representations

Internal and external *representations* are key components of the spatial reasoning framework. Goldin (1998) describes internal representations as systems of verbal/syntactical representations, which describe the way a learner processes imaginative or mental images that include visual and spatial cognitive configurations. These representations involve children mentally organising a problem and mapping the processes for problem solving. In the context of fractions, the *external representations* such as concrete materials, pictorial and graphical representations, and language are central to this component of spatial reasoning and mathematics education. Additionally, gesture is considered an external representation which mediates mathematical meaning, particularly in learning and communicating spatial information (see Alibali et al., 2014; Bobis & Way, 2018) and is an important theme in relation to the present study.

Gesture

Gestures are described as the movement of a part of the body (typically one's hands or head) that is used to covey an idea or meaning. It can be used to connect, illustrate and exemplify complex mathematical ideas so that children develop a deeper level of understanding and play a significant role in the cognitive processing of spatial information (Alibali et al., 2014). The visuospatial nature of gesture makes it suitable for capturing spatial information, in this case, information pertaining to early fraction ideas such as magnitude, as it brings the imagined or abstract spaces and objects into a more concrete form.

Research Design

This paper reports on a sub-set of data collected as part of a larger DBR study, that comprised of three cycles: a pilot cycle (Cycle 1) and two cycles of a teaching intervention which included an identical pre and post-assessment and unit of work that replaced the daily mathematics program for each class (Cycle 2 and 3), over a period of approximately three weeks (per cycle). This methodology was chosen based on the premise that the educational context is imperative for developing and extending theories for learning, and that "learning, cognition, knowing, and context are irreducibly co-constituted and cannot be treated as isolated entities or processes" (Barab & Squire, 2004, p. 1). Results presented for discussion in this paper are drawn from two tasks in Cycles 2 and 3: (i) a pre/post assessment item and (ii) a mapping-based task from the unit of work. Participating students did not receive any additional mathematics instruction during the intervention period. The participating classroom teachers also agreed not to teach their regularly planned fraction unit before their class participated in the intervention.

Participants

44 children aged 6-and 7-years participated in Cycles 2 and 3 of the intervention. The participating classes (Year 1-2 in Cycle 2; Year 2 in Cycle 3) were from separate, regional South Australian government primary schools. The teacher of each class did not teach any mathematics during this intervention; however, they acted as an additional researcher, by observing each lesson and recording their own reflections, interpretations and interactions with the children throughout each lesson.

Research Instruments

A 13-lesson unit of work was developed for this study. The unit of work was created and taught by the author of this paper. In Cycle 1, each lesson was piloted to determine its suitability for inclusion in the unit of work, and to determine the spatial skills and representations the children engaged with during each activity. Each of the lessons in the unit of work was approximately 50 minutes in length. An example of a task from this unit of work is based on a provocation developed from the picture book *Knock, Knock Dinosaur* by Caryl Hart: "The dinosaurs have escaped the house. They've decided to explore the neighbourhood. Help us find them!". Children were given clues and directions for where the dinosaurs had been 'seen' throughout the town. Using laminated maps and large carpet maps of fictional cities and towns, the children were asked to identify the locations of the dinosaurs, based on clues that contained fractional information (e.g., a quarter of the way along the train track; halfway along the bicycle path etc.). Many of the pathways chosen were not represented on the maps in a straight line, or were open to interpretation (e.g., negotiating which end of a path determined the 'start' of the measure). Thus, spatial reasoning was explicitly embedded into the anticipated problem solving strategies for this task.

An identical pre-and post-assessment was developed to assist in identifying the changes in understandings and strategies developed from the unit of work. The assessment was administered in a one-to-one task-based interview format with the researcher. Each interview consisted of 24 questions relating to the children's whole number knowledge, fraction knowledge and their spatial reasoning abilities. A rubric was developed to assess each item and to make comparisons between children's initial and final understanding. Children's work samples were collected for analysis in this study and a journal for observations, interactions and reflections was maintained throughout the project.

Data Collection

Each child completed the pre- and post-assessment tasks within two days immediately before and after the unit of work was taught. The assessment took approximately 25 minutes for each child to complete. Children's work samples were collected and their dialogue, gestures and use of materials was documented by the researcher during each item.

The unit of work consisted of a 50-minute lesson each day for 13 consecutive school days. During each lesson, the classroom teacher and researcher kept separate journals of observations and interactions throughout each lesson. At the conclusion of each lesson, the classroom teacher and researcher held a de-brief about the perspectives of the learning.

Data Analysis

All data was analysed using Hybrid Thematic Analysis (Swain, 2018). The method of analysis chosen for this study enabled key themes and relationships to become visible, which were important for developing an understanding of the possible connections between spatial reasoning and fraction knowledge.

Analysis from two tasks revealed the relationship between spatial visualisation, spatial scaling and gesture. The first task was taken from the identical pre-post assessment. It was designed to explore how children conceptualised unit fraction magnitude when asked the following question: *Which is bigger, a third or an eighth? How do you know?* Children were asked to explain their reasoning with access to a range of materials including counters, popsicle sticks, strips of paper, and drawing materials made available (but not compulsory) for use. The intention was for children to demonstrate how they visualised and represented their understanding of magnitude.

The second task, "*The dinosaurs have escaped the house!",* taken from the unit of work, indicated the influence spatial reasoning had on children's understanding of fraction as measure contexts. This task invited children to explore partitioning and unitising with an emphasis on spatial visualisation.

Whilst both tasks had an intentional focus on spatial visualisation, the findings suggested that spatial structuring and gesture were deeply embedded in the children's conceptualisation and representation of their knowledge.

Results and Discussion

In the assessment task, *which is bigger, a third or an eighth? How do you know?* every child from Cycles 2 and 3 (n=44) answered this question with "an eighth" in the pre-assessment phase. The most common explanation to the second part of this question, *how do you know?* was "eight is bigger than three" indicating a reference to whole number magnitude understanding. Additionally, no child chose to use any materials for their explanation, nor use any gesture other than a shrug of the shoulders to indicate they did not know the reason for their answer. Conversely, in the post-assessment, 34 of the 44 children assessed within Cycles 2 and 3 not only answered correctly, but provided rich descriptions supporting their answer that included gesture, evidence of spatial reasoning, and the use of materials to support their understanding of unit fraction magnitude. To exemplify, two responses from this question are presented (see Table 1) that are indicative of the interconnections between spatial and gestural elements evident in the majority of children's post-assessment responses to this item.

Table 1
Post-assessment interview (Pseudonyms assigned)

Speaker	Interview transcript
Adam:	It's a third, because look – if have this square paper *(A4 rectangular sheet)* and I imagine, like cutting in this way *(gesturing cutting the paper across two evenly spaced places, horizontally across the page)*, I get threes, each of these are a third. To get eight, you have to make more cuts and get more pieces, but the pieces get smaller and there's more of them, but they're heaps smaller – I can see them shrink. And it doesn't matter what size paper you use – a three is always bigger than an eighth.
Troy:	It's a third. When I see the parts in my head, I imagine a line and I can break it up evenly. Just…it's like… it's the more pieces or groups [of things] you need to make out of something, the smaller they get or less you have *(gesturing the forming of parts with hands, moving imagined objects to imagined groups in the air in an array like structure).*

Adam's response suggests some understanding of partitioning as he described how he was able to visualise the process, using gesture to communicate his claims. He demonstrated spatial visualisation through his description of visualising the units "shrinking" as he applied more partitions, which required holding multiple pieces of information in his mind's eye at once, whilst manipulating different components of the mental images (Lowrie et al., 2018). Adam's response indicates an understanding of quantitative equivalence in his discussion of relative magnitude, evidenced in his explanation of the relationship between the fractional units (i.e., a third is always bigger than an eighth regardless of the common whole) which demonstrates emergent multiplicative thinking. Adam's justification of this relationship suggests some abstraction about the essential foundations of fractional knowledge. These foundations include an appreciation for equal parts, and understanding that when the number of partitions increase, the size of the parts decrease (and vice versa) (Lamon, 1996; Siemon, 2003). Spatial visualisation, in addition to the use of gesture, appeared to assist Adam to communicate his understanding of fraction magnitude suggesting he is developing ideas of the relationship between partitioning (division) and multiplication. Additionally, Adam's explanation reveals there was an organisational structure to how he visualized the different partitions, by the way he gestured column and row structures when explaining how multiple unit fractions were created within the same whole. This suggests he was drawing on his internal representations of the patterning and the repeated units related to partitioning and unitising (Papic et al., 2011) which supported emergent multiplicative understandings and indicated an awareness of spatial structure.

Troy's response indicated a transfer of knowledge with reference to partitioning in continuous and discrete models. That is, Troy's response demonstrated an understanding of the measurement meaning of fractions by his description of a line that he mentally partitioned into thirds. Troy's response also exemplified the transfer of partitioning knowledge from continuous to discrete contexts, by visualising and gesturing the unit fractions of a set. The transfer from continuous to discrete contexts is an important landmark in early fraction understanding, as these ideas require different cognitive demands (Confrey & Maloney, 2010). The demands include recognising that a continuous model is the formation of multiple, contiguous parts; and the discrete model involves the need to perceive a set within one entity. In this case, gesture appeared to be closely associated with how Troy structured and visualised multiple partitions of either discrete or continuous contexts. Alibali et al. (2014) argued that gesture is a vehicle for communicating spatial information, which was evident in his gestures regarding the size and orientation of the unit fractions (i.e., an array like formation). Moreover, Troy's description and use of gesture throughout this task suggests that the spatial composition of the of the unit fractions and the relationship to the fraction construct of measure, was an essential part of his understanding and ability to transfer such ideas across continuous and discrete models.

The second task used for this analysis provides further evidence to address the research question by explicating a connection between spatial visualisation, spatial structuring, gesture and fraction as measure ideas. For example, to introduce the set of dinosaur tasks described above, the following question was posed: *A T-Rex was spotted halfway between the central fountain and the duck pond – where would she be?* From observational data and work sample analysis, most children recognised the fraction as measure context for this activity and engaged in a spatial strategy to solve the problem. This was indicated by drawing straight lines 'as the crow flies' on the map (some children gestured paths with their hands) to determine how the paths could be partitioned between the landmarks to represent where the dinosaur was located. Several children (n=8) interpreted this task as finding the halfway point of the path the dinosaur *may* have taken from the central fountain to the duck pond. That is, the children drew non-linear paths from one landmark to another and then identified the half-way point, as Shaun's work sample illustrates (see Figure 1).

Figure 1. Shaun's work sample

Shaun's path has been marked 'no' at one location and the path marked with an 'X' (digitally enhanced for ease of reading) at another point. When the researcher asked him what the "no" meant, Shaun explained that he initially copied the location his friend had marked for determining half of the path, but Shaun soon realised that his friend was indicating the halfway point of a different path to what he had drawn. Shaun stated that he had to "straighten out the line [drawn path] in my head" (whilst gesturing pulling his hands apart) and when he considered the first mark ("no"), he realised this was "more like a three-part of the way [a third] (using their hands to gesture the three parts of the path), than a two part [half]". Shaun then placed an 'X' on the path (above the yellow car) as the halfway mark instead. To paraphrase, Shaun stated that it did not matter how long the path was or in what shape/ orientation; to be half meant there were two equal parts of the concerning path. His recognition of the differing path lengths and its relationship to the target fraction demonstrates an emerging understanding of proportional thinking (fraction as relation). Although Shaun initially copied his friend's map, he recognised it could not be an accurate representation of the same fractional measure, as their paths were different lengths. Shaun stated he would have to mentally manipulate these paths (using spatial visualisation) to enable a comparison of measures. This type of thinking also suggests spatial structuring was an important component to his conceptualisation of the problem, particularly when combining components into spatial composites such as units of thirds and halves (Battista & Clements, 1996), to establish the relationships between these measures within his own representation and in comparison to his friend's path. Shaun demonstrated an understanding of relative magnitude, by explaining the differences of absolute measures through visualising and comparing the different paths and used gesture as a vehicle to demonstrate the iterative unit fractions of halves and thirds. Emerging proportional thinking as illustrated in this example was evident in 19 children's responses throughout the intervention, which highlights the abstraction and transfer of these concepts.

The relationship identified by this study between spatial visualisation, gesture and the concept of partitioning (in a fraction as measure context) extends Lamon's (1996) description of partitioning as being an 'experience-based activity'. The deep engagement between spatial visualisation with gesture forms an important part of this experience as it served as the vehicle for children articulating their experiences of partitioning. Moreover, spatial structuring was an important component in children's development of unitising and equivalence ideas that formed from their engagement with spatial visualisation and gesture, which in turn suggested it positively influenced the children ability to conceptualise the multiplicative nature of fractions in both discrete and continuous contexts. It is clear that the common multiplicative foundations spatial structuring and early fraction concepts share, influenced the way children visualised and used gesture when representing key fraction ideas.

Conclusion

The relationship between visualisation, gesture, spatial structure and fractions is an important finding and contribution to understanding how young children develop such ideas. Importantly, this study revealed that this relationship also contributed to children's abstraction and transfer of understanding of these concepts in both discrete and continuous models which is an essential component for developing conceptual understanding of fractions. Moreover, the results from this study go some way to addressing the persistent problems young children face in developing deep connections between the concepts and contexts in which fractions are explored and represented (Bobis & Way , 2018). These new understandings imply there are considerable benefits in adopting a spatialised approach to teaching fractions in the early years of primary school, because it can allow for a better exploration and understanding between the nature of young children's spatial reasoning, representations (internal and external) and the role these factors play in young children's development of fraction knowledge. The limitations of this study include the sample size of participants involved, and lack of video recordings for greater fidelity measures. However, future research directions could include a longitudinal study to provide greater insights into the connection between different aspects spatial reasoning and their impact on children's development of rational number knowledge more broadly.

References

Alibali, M. W., Nathan, M. J., Wolfgram, M. S., Church, R. B., Jacobs, S. A., Johnson Martinez, C., & Knuth, E. J. (2014). How teachers link ideas in mathematics instruction using speech and gesture: A corpus analysis. *Cognition and Instruction, 32*(1), 65-100.

Barab, S., & Squire, K. (2004). Design-based research: Putting a stake in the ground. *The journal of the learning sciences, 13*(1), 1-14.

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education, 27*, 258–292

Bobis, J., & Way, J. (2018). Building connections between children's representations and their conceptual development in mathematics. In V. Kinnear, M. Lai, & T. Muir (Eds.), *Forging Connections in Early Mathematics Teaching and Learning* (pp. 55-72). Springer, Singapore.

Bruce, C. D., Davis, B., Sinclair, N., McGarvey, L., Hallowell, D., Drefs, Francis, K., Hawes, Z., Moss, J., Mulligan, J., Okamoto, Y., Whiteley, W., & Woolcott, G. (2017). Understanding gaps in research networks: using "spatial reasoning" as a window into the importance of networked educational research. *Educational Studies in Mathematics, 95*(2), 143-161.

Confrey, J. (2008). *Learning Trajectories and Rational Number Reasoning* [PowerPoint slides]. Retrieved from https://www.human.cornell.edu/sites/default/files/HD/nsfalw/Confrey-NSF.pdf

Confrey, J., & Maloney, A. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. In *Proceedings of the 9th International Conference of the Learning Sciences, 1* (pp. 968-975). International Society of the Learning Sciences.

Congdon, E. L., Vasilyeva, M., Mix, K. S., & Levine, S. C. (2018). From intuitive spatial measurement to understanding of units. In *Visualizing Mathematics* (pp. 25-46). Springer, Cham.

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior, 17*(2), 137-165. https://doi.org/10.1016/S0364-0213(99)80056-1

Huttenlocher, J., Newcombe, N., & Vasilyeva, M. (1999). Spatial scaling in young children. *Psychological Science, 10(*5), 393-398.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170-193.

Lowrie, T., Logan, T., Harris, D., & Hegarty, M. (2018). The impact of an intervention program on students' spatial reasoning: student engagement through mathematics-enhanced learning activities. *Cognitive research: principles and implications, 3*(1), 50.

Möhring, W., Newcombe, N. S., & Frick, A. (2015). The relation between spatial thinking and proportional reasoning in pre-schoolers. *Journal of Experimental Child Psychology, 132,* 213-220. https://doi.org/10.1016/j.jecp.2015.01.005

National Research Council, (2006). Learning to think spatially. Washington D.C.: The National Academy Press. https://www.nap.edu/read/11019/chapter/1

Obersteiner, A., Dresler, T., Bieck, S. M., and Moeller, K. (2019). Understanding fractions: integrating results from mathematics education, cognitive psychology, and neuroscience. In A. Norton & M. W. Alibali (Eds.). *Constructing Number*, Cham: Springer International Publishing, 135–162. https://doi.org/10.1007/978-3-030-00491-0_7

Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of pre-schoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42(*3), 237-268.

Siemon, D. (2003). Partitioning: the missing link in building fraction knowledge and confidence. *Australian Mathematics Teacher, 59*(3), 22-24.

Swain, J. (2018). A hybrid approach to thematic analysis in qualitative research: Using a practical example. *Sage Research Methods*.

Way, J., Bobis, J., & Anderson, J. (2015). Teacher representations of fractions as a key to developing their conceptual understanding. In *Proceedings of 39th Psychology of Mathematics Education conference* (Vol. 4, pp. 281-288).