RESEARCH PAPER

Are Learners Referring to the General or the Particular? Discursive Markers of Generic Versus Empirical Example-use

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Supporting students of all levels to move beyond empirical arguments, which employ example-based reasoning to endorse universal truths and are thus mathematically invalid, remains a challenging goal in mathematics education. Arguments that make use of generic examples are both mathematically valid and accessible for even young learners. However, discerning whether students are viewing or using an example as a specific case, or a general case, is difficult. In this paper, we open the space between empirical and generic use of examples and establish categories of example-use regarding odd and even numbers. We reveal discursive markers pointing towards whether a learner is referring to particularity or generality in their example-based reasoning.

Background

A wealth of research corroborates what Stylianides and Stylianides (2017) described as "key and persistent problems" (p. 124)—students' reliance on *empirical arguments* to endorse universal statements. Empirical arguments are example-based arguments that provide inconclusive evidence for mathematical generalisations; by verifying the truth of a universal statement on only a subset of all possible cases, they fail to eliminate the possibility of the existence of a counterexample. Therefore, mathematically speaking, these arguments are invalid. In comparison, *deductive arguments* use definitions and theorems to produce "logically rigorous deductions of conclusions from hypotheses" (NCTM, 2000, p. 56) and are considered mathematically valid (*proofs*).

While formal, deductive proofs might be out of reach for young learners, the usefulness of *generic examples* have been widely acknowledged (e.g., Hanna, 2000; Mason & Pimm, 1984; Reid & Vallejo Vargas, 2018; Stylianides, 2007). The idea of a generic example can be traced back to Mason and Pimm (1984), who defined it as "an actual example, but one expressed in a way as to bring out its intended role as the carrier of the general" (p. 287). Hence, unlike empirical example-use, a generic example removes the need to produce endless specific examples by showing general (rather than particular) properties of the cases it exemplifies. To illustrate, Stylianides (2007) provided an excerpt of an 8-year-old student using a generic example for "odd + odd = even" where the student drew two sets of seven lines and proceeded to circle them in twos, saying, "[All] odd numbers if you circle them by twos, there's one left over. So, if you ... plus one, um, or if you plus another odd number, then the two ones left over will group together" (p. 7). Generic arguments, such as this, are more accessible to young learners and they have explanatory potential. They can help support students "not only to see that it [a theorem] is true but also why it is true" (Hanna, 2000, p. 8).

The issue remains though, "How is it possible to determine whether students are using or viewing an example generically?" For instance, in the above example, it is quite possible that some may view the same two sets of seven sticks used by the student in Stylianides (2007) illustration as a specific example rather than a general one. Determining whether learners are using examples generically and are therefore seeing the general rather than the particular in examples, is neither obvious nor straightforward (Mason & Pimm, 1984; Reid & Vallejo Vargas, 2018; Yopp & Ely, 2015). This is particularly difficult where students are not aware of what constitutes a valid mathematical argument or where the production of a written argument is not appropriate (e.g., with primary school students). In such situations, it might be

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), pp. 330–337. Launceston: MERGA.

necessary to look for more subtle signs that point to students implicitly recognising the genericity of an example and reasoning deductively. Accordingly, in this paper we explore the empirical-deductive space and use the commognitive framework (Sfard, 2008) to examine the ways in which students use examples in their reasoning.

The Commognitive Framework

Sfard (2008) defined mathematical *discourse* as a special form of communication, made distinguishable via four interrelated characteristics: keywords (e.g., number-words like "three", "fourteen"); visual mediators (e.g., numerals, symbols, diagrams, pictures); narratives (e.g., definitions, proofs); and routines (repetitive ways of performing mathematical tasks). Learning is seen as a lasting transformation in a learner's discourse, which is identifiable by changes in one or more of these four characteristics. According to Sfard (2008), learning occurs both at the *object-level* and the *meta-level*. Object-level learning is signalled by an expansion in the routines and endorsed narratives within one's discourse. For example, when an individual who endorses even numbers as "numbers in the sequence 0, 2, 4, 6, 8 ..." learns that these are also "numbers ending in 0, 2, 4, 6, 8" or "divisible by two". Meta-level learning occurs when learners transition into a discourse that is different from their familiar one, requiring a change in endorsing "propositions about the discourse rather than about its objects" (Sfard, 2007, p. 573).

Transitioning from endorsing empirical arguments for universal statements to endorsing deductive arguments requires a meta-level shift in learning. Whereas in empirical discourses learners converse about specific objects, in deductive discourses learners are required to converse about abstract entities. In commognitive terms, learners performing an empirical substantiation routine will use numeric keywords (specific numbers) or visual mediators signifying specific numbers to model the resulting sums they make, and they rely on the sums of such numeric examples (such as, 3 + 5 = 8) to substantiate a universal narrative (e.g., odd + odd = even). In contrast, a deductive routine for substantiating a universal narrative (e.g., odd + odd = even) relies on a series of propositions supported by definitions (e.g., odd is even + 1; a multiple of 2 plus 1; or 2n + 1), theorems or axioms, whereby each proposition is logically deduced from the previous one in an organised way (e.g., (2n + 1) + (2m + 1) = 2n + 2m + 2).

Although examples are not necessarily part of a deductive routine, there is space for them to be so if they are used generically. Commognitively speaking, determining whether an example is being used empirically or generically should be visible via some change in a substantiation routine (i.e., changes in keywords, visual mediators or narratives). Hence, in this paper, we examine students' verbal responses and their accompanying actions using the commognitive framework to characterise primary school students' use of examples. We look for discursive indicators as subtle signs that examples are being used more empirically or generically. Specifically, we ask:

(i) *How can learners' use of examples in their reasoning be categorised?*

(ii) *What commognitive indicators are present in learners' reasoning that suggests that they are using examples of odd and even generically rather than empirically?*

Conduct of the Study

Data were collected from 28 Year 4 students (aged 8- and 9-years-old) from two New Zealand schools. As the unit of investigation in this study was discourse, teachers selected students to work in groups of four according to whom they considered would be willing and able to engage in a mathematical dialogue. The students were shown a cartoon dilemma which featured three characters, each with a speech bubble containing differing narratives on the sums of odds and evens (e.g., odd + odd = even; odd + odd = odd; odd + odd = sometimes even and sometimes odd) for students to reject or endorse and then substantiate their choices. The students had pens and paper, counters and Numicon tiles[1] available to work with. Each student group session was video-recorded and transcribed in its entirety, and the students' written work was added to the data corpus.

We used fine-grained discourse analysis that utilised Sfard's (2008) commognitive framework to examine episodes of students' dialogue for distinguishing features (words and their use; visual mediators and their use; narratives and routines) that marked their reasoning and example-use as being more specific or more general.

Findings

From the analysis of our data, we were able to classify all comprehensible instances of students' example-use into four categories: (1) Inductive use of numeric examples; (2) Inductive use of numeric-generic examples; (3) Deductive use of numeric-generic examples; and (4) Deductive use of nonspecific-generic examples. We observed a number of discursive markers within these categories that pointed towards generic, versus empiric, example-use when endorsing universal narratives about the sums of odds and evens:

(i) A change from numeric to nonspecific keywords and visual mediators;
(ii) A switch in the mathematical object of focus in substantiating narratives—from the number generated in sums to the structure of the addends and the sum;
(iii) A shift from present to future tense;
(iv) Changes in the use of determiners—from the use of specific nouns, definite articles and demonstrative pronouns to the use of indefinite articles and quantifiers that refer to the whole extent of the particular group or situation in focus; and
(v) The use of illustrative expressions such as "like" or "for example" to indicate the purpose of the example is to illustrate the general in the particular.

Inductive Use of Numeric Examples

Episode 1
Toby and Erin's Inductive Use of Numeric Examples to Endorse "odd + odd = even"

Speaker	What was said	What was done
Toby:	Yeah, [odd + odd = even] because five plus five equals ten. Seven plus seven equals fourteen …	
Teacher:	... Has anyone found an odd plus odd equals odd?	
Erin:	Yes! Two four... Oh yes, yes.	Takes 5 and 9 and Numicon tiles and forms a 7 x 2 array.  Starts counting her 7 x 2 array in twos.
Toby:	No. Because two, four, six, eight, ten, twelve. Fourteen.	Reaches over to count Erin's 9 + 5 array: counting in twos.
Erin:	So, it's even.	
Toby:	Yeah	

[1] Numicon tiles are tangible mediators of numbers 1-10 as dots within a frameless 2 x 5 rectangle.

In Episode 1, Toby initially recalled the sums of pairs of specific odd numbers [5 + 5; 7 + 7] to substantiate endorsing “odd + odd = even” and Erin visually mediated a specific instance of “odd + odd = even” [5 + 9]. The two students’ keywords were entirely numeric (e.g., “five”; “ten”; “six”) and they used the resulting number of the specific sums they have selected as the object in their substantiations for “odd + odd = even.” Furthermore, when the students physically formed symmetrical structures (a paired array) with their two odd (asymmetrical) tiles, it is the number rather than the symmetrical/asymmetrical shape that signified evenness. The students’ disregard of the symmetrical structure of the pairs of odd Numicon tiles they used was most evident when Erin briefly endorsed her selection of Numicon 9 and 5 tiles arranged in a *2 x 7* array as an example of “odd + odd = odd.” Erin and Toby then justified rejecting it as an instance of “odd + odd = odd” on the basis of confirming the *count* is “fourteen,” and therefore even (rather than referring to the symmetrical *shape* of the array). These discursive features suggest that these students are apprehending the particular, not the general, in their example-use.

Inductive Use of Numeric-generic Examples

Episode 2
Sadie’s Inductive Use of Numeric-generic Examples to Endorse “even + odd = odd”

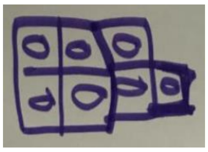
What Sadie said	What Sadie did
[Speaking to herself] Ok. It equals odd. Because this one here is hanging off.	Takes Numicon 9 and 2 and places them together. Points to the extra one from the two. 
So that one’s even. Ok. … because this bit here is hanging off.	Holding up Numicon 2. Places it with Numicon 9 (as before). Points to the extra one on top of the arrangement.
But can I have a experiment and see if Ruby’s [“even + odd = sometimes even and sometimes odd”] right?	Selects Numicon 10 and 3 tiles. Rotates Numicon 3 around to see if it will “fit” with Numicon 10.
Yeah, I think it’s Jed [“even + odd = even”]. This one’s not working. It’s still odd. Ok so I think it might be Jed that’s right. Yeah. Yeah.	

In Episode 2, even though Sadie selected pairs of Numicon tiles, she did not tend to refer to these addends with number-names (only once did she refer to the Numicon 2 tile as “my *two*”) nor did she use number-names for the sums. Instead, when Sadie selected Numicon tiles 2 and 9, she used the words, “one ... hanging off”, to substantiate the sum’s oddness, pointing to the “one” as she does so. Her use of these keywords and visual mediators indicate Sadie realised the generic mathematical (asymmetrical) structure of oddness. However, Sadie only used talk of mathematical structure to substantiate the odd *outcome* (“one ... hanging off”) for examples of “even + odd”; there were no dialogue substantiating why “even (with no “ones ... hanging off”) + odd (with “one ... hanging off”)” addends *together* make a shape with “one ... hanging off,” let alone why this would always be the case.

Sadie exhibited an ad hoc way of connecting the two Numicon pieces: she tried different rotations of Numicon 3 to see if there was a way to connect it with Numicon 10 to make a symmetrical (and therefore even) shape and commented, “it’s not working”, when she was unsuccessful. Accordingly, Sadie did not appear to be using the generic structure of the odd (asymmetrical) and even (symmetrical) addends to account for the resulting asymmetrical shape. Instead, her use of examples to endorse “even + odd = even” were: (i) related to more trial and check—she needed to “experiment” with more than one pair of “even + odd” to check for oddness; (ii) absent of deductive narratives and actions; and (iii) based on empirical observations of the two, confirming asymmetrical (odd) outcomes.

Deductive Use of Numeric-generic Examples

Episode 3
Jane and Zara’s Deductive Use of Numeric-generic Examples to Endorse “even + odd = odd”

Speaker	What was said	What was done
Jane:	[even + odd = odd] because there’s four and there’s gonna be three. It’s gonna be odd because there’s one more left. If that [extra one] wasn’t there, it’d be even because it would be three and three and that’s six. But if that’s there it’s seven so it’s not even.	Draws a Numicon-like 4 and 3 shapes, connected.  Pointing to the extra one on the bottom- right. Covers up the bottom- right square and then uncovers it again.
Zara:	Yeah, because if you have like two plus one, it will equal three. And there’s an even and there’s a one- and there’s an odd and it would still equal an odd. Cos there’d be that one extra.	Picks up Numicon tiles 2 and 1.
Jane:	For example, if there was six and one then we will have one still sticking out. Whatever odd number it is we’ll still have one more sticking out.	Takes Numicon tiles 6 and 1 and connects them. Changes Numicon 1 for Numicon 3 and connects it to Numicon 6. Then picks up Numicon 10 and 7.

At first glance, these utterances suggested that Zara and Jane were substantiating their endorsement of “even + odd = odd” with numeric examples and, as such, their example-use could be confused with that described in the first category (numeric talk with inductive substantiations). The two students used numeric keywords (e.g., three, six, seven), visually mediated numeric examples (e.g., Jane’s drawing of four plus three; Jane and Zara’s use of Numicon tiles) and used narratives that made use of those numeric examples—all of which were features consistent with the empirical example-use illustrated in the first category. However, there was several features that point towards generic example-use. First, in addition to using numeric keywords, both Jane and Zara also used generic keywords related to the mathematical structure of odd: “one more left”, “extra one”, and “one still/more sticking out”. Second, Jane visually mediated the resulting shape from adding different pairs of even and odd Numicon tiles and specifically pointed to the “extra one” by covering it (to indicate evenness) and uncovering it (to indicate oddness). Third, their narratives and actions indicated that they were using these examples for illustrative purposes: their narratives included the words “like” (Zara: “if you have *like* two plus one”), and “for example” (Jane: “*For example,* if there was six and one”), and Jane proceeded to interchange Numicon tiles to illustrate her substantiating narrative, “one sticking out”, holds for other examples.

Further, and even subtler, discursive markers, which also suggested that Zara and Jane were using numeric examples generically (rather than empirically), come from: (i) their use of determiners and (ii) a shift in tense. Note that in the previous category, Sadie used demonstrative pronouns "it" and "this" and she spoke only in the present tense (e.g., "*It* equal*s* odd. Because *this* one here *is* hanging off."). In contrast, Zara switched from using numeric words ("two," "one," "three") to using indefinite articles ("*an* odd", "*an* even"). Had Zara used more demonstrative determiners such as "this," "that," or the specific pronoun "it"—for example, "*this* odd," or "*that* even," or "*it* is even" and "*it* is odd"—her example usage would have suggested that the identity of the even and odd tiles she was referring to was known and they were specifically the ones (i.e., "two" and "one") she had chosen. Instead, Zara's use of the indefinite article "an" implies that the identity of the even and odd addends is neither known nor obvious and serves to make these nouns more general. In a similar way, Jane switched to using the determiner "whatever" when describing an odd addend with its generic "one sticking out", which implies her example holds for *any* odd number.

Furthermore, in this episode, both girls' shift in tense—from present to future—strengthens the sense one gets of them moving from the specific to the general in their example-use. Zara began in the present tense—"there'*s* an even and there'*s* an odd"—but then switched to future tense—"it *would* still equal an odd. Cos there*'d be* that one extra." This shift suggests that Zara may have used this particular numeric example to signify what *would* happen with *any* combination of "odd + even." Similarly, Jane switched to the future tense, when she presented the example of "six and one" and said, "... we *will have* one sticking out."

The object that both students used in their substantiations to endorse "even + odd = odd" was the structure of the sum (not the numeric result) and they showed how the generic structure of the even or odd addends combined to make the odd or even sum. Zara and Jane's use of "if ... then … because" narratives provide evidence of logical and conditional substantiations in hypothetical situations. For example, regarding her use of Numicon tiles for "four" and "three," Jane says: "*If* that [extra one] wasn't there [then] it'd be even *because* it would be 'three and three' and that's six. But *if* that's [the extra one] there [then] it's seven *so* it's [the sum] not even." In short, Zara and Jane's substantiating narratives made use of generic structural features to account for any hypothetical or potential case of "even + odd", which allow suggesting that Jane and Zara were indeed seeing the general in the particular examples they used.

Deductive Use of Nonspecific-generic Examples

Episode 4

Zara's Deductive Use of Nonspecific-generic Examples to Endorse "odd + odd = even"

Speaker	What was said	What was done
Zara:	Yes, so if you have something like a square. If you have something like this.	Draws an oblong rectangle.
Jane:	A rectangle.	
Zara:	Yes, it's an oblong. So, if you have like two circles on each, it will be even. And just keep on going down. But if you added on an extra one here, then it wouldn't be even. So, if you put like another one ["one" is taken here to mean another "odd"] there [referring to her drawing] then it would be even.	Draws two circles in the rectangle. Draws two lines going down from each of the circles. Draws the extra circle (bottom-right). 

In this episode, Zara drew a rectangle that signified her realisation of symmetry in a nonspecific even and her use of the words "square" and "oblong" referred to the generic symmetrical structure of "even" evident in her drawing. These keywords and visual mediators are consistent with generic example-use. Zara then drew a pair of dots inside the rectangle and lines from each of the dots that "just keep on going down." While her drawing resembled the Numicon tiles (which signify specific numbers) that she had worked with previously, it also implied that the even number continues indefinitely, signifies *any* even number, and highlights the "multiple of two" property in *any* even number. Zara then added "an extra one", which made her drawing asymmetrical and therefore prompted a realisation of "*not* even" (odd). This nonspecific odd embodied the same general properties as her former visually mediated nonspecific even had done; it implied that the odd number continues indefinitely, signifies *any* odd, and highlights the "multiple of two plus one" property in *any* odd number.

To endorse "odd + odd = even", Zara deductively used the generic structures of the two odd addends combined to make an even outcome. Note, as with the previous category, her use of "if ... then ... so" conditional statements and her use of "would" implies an imaginary or hypothetical situation. Note, also, Zara's use of the generic determiner "*another* one" implies the identity of the second odd addend (just like the first odd addend) is unknown—it is a generic odd. The combination of all these discursive markers indicate that Zara was seeing the general in nonspecific, and more abstract, examples.

Discussion

Supporting learners at all levels of mathematics education to develop valid arguments has been recognised as an obstacle in moving from inductive to deductive reasoning (e.g., Stylianides & Stylianides, 2017). Supporting learners to use generality in a particular example is part of the issue. Previous research pointed to the difficulty of knowing whether a student was aware of the general intent in an example (e.g., Reid & Vallejo Varga, 2018; Yopp & Ely, 2015). However, our study uncovered four different categories of example-use and several subtle discursive markers that *implicitly* pointed towards generic, versus empiric, example-use when endorsing universal narratives about the sums of odds and evens (Table 1).

Table 1
Categories of Example-use and Changes in Discursive Markers from Particularity to Generality

Categories of example-use	Inductive use of numeric examples	Inductive use of numeric-generic examples	Deductive use of numeric-generic examples	Deductive use of nonspecific-generic examples
Keywords and visual mediators	Numeric	↔		Generic
Object	Number	↔	Structure of the sum ↔	Structure of addends within the sum
Tense	Present	↔		Future
Determiners	Demonstrative pronouns	↔		Quantifiers expressing entirety
Articles	Definite articles	↔		Indefinite articles
Illustrative expressions	Absence of illustrative expressions	↔		Use of illustrative expressions

While example-use has previously been dichotomised as either empiric or generic, our findings show the occurrence of example-use to be more nuanced and multi-layered. The second category fell somewhere in between what has previously been considered *either* empirical *or* generic example-use. In this category, learners were better placed than those in the first category to potentially see the general in the particular because they recognised the generic (symmetric or asymmetric) structure of even and odd. Furthermore, where generic example-use has been viewed as a single category, our findings showed two distinct categories of generic example-use distinguishable by the level of abstraction: "numeric-generic example-use" and "nonspecific-generic example-use."

As is suggested by the overlapping of categories in Table 1, it is quite possible that students' routine ways of using examples may fall somewhere between two categories or be in flux between categories. It is also important to note that we are not claiming that all the discursive features described within these categories will necessarily be, or need to be, present in a learner's narrative to categorise the learner's example-use as fitting within one of the categories. Nor is it likely that one discursive feature alone will be sufficient to suggest that a learner is using examples generically. Hence, the intention behind observing discursive markers is not so that they may be used as an exclusive "must-have" tick list for distinguishing genericity from particularity in example-use, but for them to be used discerningly to strengthen researchers and teachers' conviction that a learner may be signalling generality in their example-use.

The present study is limited in its mathematical focus. The ways in which students use examples in this study might be different in other contexts. Equally, the four categories that emerged in this study are not necessarily transferrable to other tasks. However, by considering learning through a discursive lens, the study has brought otherwise unmentioned features of students' example use to the fore. Further work is required, and future studies would need to explore the way these discursive markers are present in learners' reasoning in different contexts and how they point to learners reasoning about and demonstrating generality.

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