Key Shifts in Thinking in the Development of Mathematical Reasoning

Dianne Siemon (Chair)
RMIT University
dianne.siemon@rmit.edu.au

Rosemary Callingham
University of Tasmania
rosemary.callingham@utas.edu.au

Lorraine Day
University of Notre Dame Australia
lorraine.day@nd.edu.au

Marj Horne
RMIT University
marj.horne@rmit.edu.au

Greg Oates
University of Tasmania
greg.oates@utas.edu.au

Rebecca Seah
RMIT University
rebecca.seah@rmit.edu.au

Max Stephens
The University of Melbourne
m.stephens@unimelb.edu.au

Jane Watson
University of Tasmania
jane.watson@utas.edu.au

Jill Fielding (Discussant)
University of New England
jill.fielding@une.edu.au

This symposium will draw on the evidenced-based learning progressions for multiplicative thinking, algebraic reasoning, geometrical reasoning, and statistical reasoning presented at previous MERGA conferences (see references by symposium authors in the papers that follow). The four papers will consider key shifts in thinking identified within each progression, without which students' progress may be seriously constrained.

Paper 1: *A Disposition to Attend to Relationships: A Key Shift in the Development of Multiplicative Thinking*
[Dianne Siemon]

This paper draws on multiple data sources to better understand the shift from additive to multiplicative thinking, which is crucial to all further participation in school mathematics.

Paper 2: *Key Shifts in Students' Capacity to Generalise: A Fundamental Aspect of Algebraic Reasoning*
[Max Stephens, Lorraine Day, & Marj Horne]

This paper will elaborate five levels of algebraic generalisation and two key understandings based on an analysis of students' responses to RMFII algebraic reasoning tasks.

Paper 3: *Cognitive Flexibility and the Coordination of Multiple Information in Geometry and Measurement*
[Rebecca Seah & Marj Horne]

This paper analyses students' solutions to problems in geometry and measurement situations in order to identify key components needed to nurture reasoning.

Paper 4: *Facilitating the Shift to Higher-order Thinking in Statistics and Probability*
[Rosemary Callingham, Jane Watson, & Greg Oates]

Students have difficulty moving from concrete representations and procedural mathematical statistics to context-based appreciation of data. This paper examines the barriers to this shift to higher-order thinking based on the Statistical Reasoning Learning Progression.

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), p. 41–57. Launceston: MERGA.

A Disposition to Attend to Relationships: A Key Shift in the Development of Multiplicative Thinking

Dianne Siemon
RMIT University
dianne.siemon@rmit.edu.au

This paper draws on numerous data sources to better understand the shift from additive to multiplicative thinking in years 4 to 9. Research studies that have used the Scaffolding Numeracy in the Middle Years assessment tasks have found that while students can be supported to move through the early and upper zones of the Learning and Assessment Framework for multiplicative thinking, it has been difficult to move students through Zone 4 at the same rate. A closer examination of item responses at this level reveal that a disposition to notice and work with relationships between quantities may explain this phenomenon.

Access to multiplicative thinking has long been recognised as critical to success in school mathematics in the middle years and beyond (e.g., Harel & Confrey, 1994; Hilton et al., 2016; Lamon, 1993; Siemon et al., 2006). However, many students at this level do not have access to this critical capacity (Brown et al., 2010; Siemon, 2019) suggesting that the transition from additive to multiplicative thinking is more complex than previously recognised (e.g., Clark & Kamii, 1996; Van Dooren et al., 2010; Vergnaud, 1983).

Research studies that have used the *Scaffolding Numeracy in the Middle Years* (SNMY) assessment tasks have found that while students can be supported to move through the early and upper zones of the Learning and Assessment Framework (LAF) for multiplicative thinking (Siemon, 2016, 2019), this appears not to be the case for Zone 4, which is where students are starting to use multiplicative thinking on a more consistent basis (see Figure 1 for examples). This and the fact that the proportion of students in Zone 4 is typically higher than in any other zone confirms the difficulty of acquiring multiplicative thinking, but it also prompts the question, "What can be learnt about the barriers to multiplicative thinking from a closer analysis of student responses to tasks that span Zone 4?"

Solves more familiar multiplication and division problems involving two-digit numbers (e.g., *Butterfly House c* and *d*, *Packing Pots c*, *Speedy Snail a*). Tend to rely on additive thinking, drawings and/or informal strategies to tackle problems involving larger numbers and/or decimals and less familiar situations (e.g., *Packing Pots d*, *Filling the Buses a* and *b*, *Tables & Chairs g* and *h*, *Butterfly House h* and *g*, *Speedy Snail c*, *Computer Game a*, *Stained Glass Windows a* and *b*). Tends not to explain their thinking or indicate working. Able to partition given number or quantity into equal parts and describe part formally (e.g., *Pizza Party a* and *b*), and locate familiar fractions (e.g., *Missing Numbers a*). Beginning to work with simple proportion, for example make a start, represent problem, but unable to complete successfully or justify their thinking (e.g., *How Far a*, *School Fair a* and *b*).

Figure 1. Rich text description of Zone 4 (Siemon et al., 2006).

Approach

The *Stained Glass Windows* task (Figure 2) was selected for analysis as the item difficulties ranged from Zone 3 to Zone 7 and the setting, while accessible, did not conform with the more familiar multiplicative models implicit in problems such as *Packing Pots* (i.e., equal groups or arrays). It was also selected because the context invited additive thinking, which tested the extent to which students could see past that to the underlying multiplicative structure (e.g., Vergnaud, 1983), which was hinted at in the task stem. These same criteria were met by another

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), pp. 42–45. Launceston: MERGA.

task, *Canteen Capers*, which involved lunch order options given two choices of rolls, four choices of filling, and three choices of drink. The first item required students to identify the number of options for a roll with a specified filling and a drink item (2 x 3). The second item required them to determine if everyone in a class of 26 children could have a different lunch order made up of a roll, filling, and drink. In both cases students were asked to explain their reasoning using as much mathematics as they could.

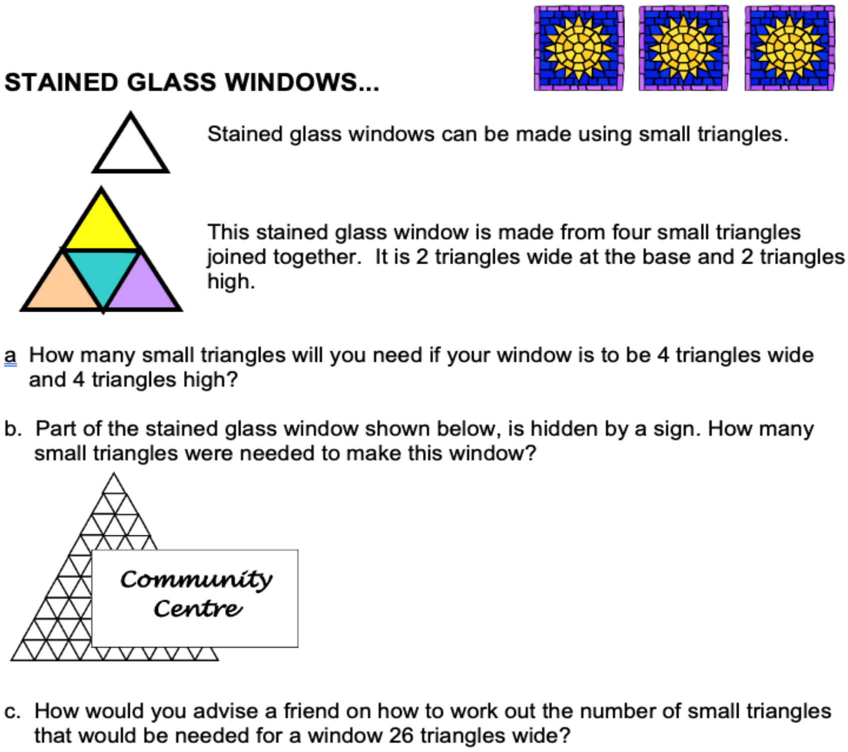


Figure 2. Stained Glass Windows task from SNMY Assessment Option 1 (Siemon et al., 2006).

Data sets from four different projects are used in the analysis reported here. That is, the SNMY project (Siemon et al., 2006a), the *Reframing Mathematical Futures Priority* project (Siemon, 2016), the *Reframing Mathematical Futures II* project (Siemon et al., 2018), and the *Growing Mathematically—Multiplicative Thinking* project (Callingham & Siemon, 2021). The student populations across the four projects ranged from Year 4 to Year 9 of whom approximately 65% were from low socio-economic backgrounds.

A total of 11,775 students (67% in Years 7 or 8) responded to the Stained-Glass Windows task and 4985 students (83% in Years 7 or 8) to the Canteen Capers task. Student responses were marked by project schoolteachers using partial credit scoring rubrics and entered into a deidentified spreadsheet which was forwarded to the research team for analysis.

Analysis and Discussion

Table 1 shows the proportion of students scoring a 1, 2, or 3 on items a, b, and c of the two tasks with the last entry for each item indicating the proportion of students providing a multiplicative response. The very low proportion of students evidencing either an additive or a multiplicative response to both problems is at odds with the suggestion that strategy usage is impacted by the numbers involved or the extent of the challenge (Downton & Sullivan, 2017; Larsson et al., 2017). It is undoubtedly the case that "some students use strategies that are only as complex as they need" (Downton & Sullivan, 2017, p. 303). However, the proportion of students providing a correct answer supported by additive reasoning (i.e., a score of 2 on items a and b of Stained Glass Windows and item a of Canteen Capers) is surprisingly low, given that the majority of the students were from Years 7 or 8.

Table 1
Proportion of Students Scoring a 1, 2, or 3 on Each Item of Each Task

	Stained Glass Windows (*n* = 11,775)			Canteen Capers (*n* = 4985)	
Score	A	b	c	a	B
1	22.9%	29.5%	11.3%	22.2%	29.8%
2	28.5%	13.8%	22.4%	22.6%	24.9%
3	13.7%	18.5%		17.8%	

An insight into why this might be the case is afforded by the item difficulties shown in Table 2 for the Stained Glass Windows task. On the ordered list of item difficulties produced by the Rasch analysis a score of 3 on item a (sgwa3) was located towards the top of the scale in Zone 7. However, the item difficulties associated with recognising and using the same relationship in items b and c (i.e., sgwb3 and sgwc2) were located in Zone 6, which suggests that noticing the rule is harder than applying the rule despite the strong suggestion of the rule in the stem (2 x 2) and the likelihood that 4 and 16 would be recognised as square numbers.

Table 2
Scoring Rubrics for Stained Glass Windows by Item Difficulty (LAF location)

Item	Rubric (item difficulty code)	Score	Zone
A	Incorrect based on inaccurate drawing and/or counting of triangles, or correct with little/no explanation (sgwa1)	1	3
	Correct (16 triangles), with evidence of additive reasoning based on drawing and counting (sgwa2)	2	4
	Correct (16 triangles), with evidence of multiplicative reasoning based on 4 x 4 (sgwa3)	3	7
B	Incorrect based on inaccurate drawing and/or counting of triangles, or correct (81 triangles) with little/no explanation (sgwb1)	1	3
	Correct (81 triangles), with evidence of additive reasoning based on drawing and counting, or inappropriate use of area formula (e.g., L x W) (sgwb2)	2	4
	Correct (81 triangles), with evidence of multiplicative reasoning based on pattern (e.g., 9 by 9) (sgwb3)	3	6
C	Advice based on additive thinking (e.g., "2 less each time you go up") (sgwc1)	1	5
	Correct, advice based on rule (e.g., 26 x 26) (sgwc2)	2	6

A similar phenomenon is observed for the Canteen Capers task where the item difficulties ranged from Zone 2 to Zone 8. Recognising and providing a multiplicative explanation for part a (e.g., "It's 6 because for each roll she could have one of the 3 drinks") was located in Zone 8. For item b, determining that there were enough different options for each child in a class of 26 on a systematic basis that suggested use of 2 x 4 x 3, was located in Zone 6. Again, this suggests that noticing the relationship was harder than applying it.

There are a number of possible explanations for the difficulty of these items that warrant further investigation. One is the absence of a familiar multiplicative model, which is known to facilitate multiplicative understanding and calculation (Larsson et al., 2017). However, the fact that multiplicative thinking is elicited by these tasks despite this suggests that something more is needed to support the shift from additive to multiplicative thinking, particularly as models connected to solution strategies can invoke instrumental responses (Skemp, 1976) making it difficult to discern multiplicative thinking.

Apart from the obvious need to offer a broader range of multiplicative tasks and contexts that are not readily connected to students' existing models of multiplication (e.g., Downton & Sullivan, 2017), the analysis here suggests that the "something more" is a disposition to attend to relationships between quantities in ways that look for generalities rather than particulars. In other words, it is about an alertness to and appreciation of mathematical structure (e.g., Mason et al., 2009) and multiplicative structure in particular (e.g., Mulligan, 2002; Vergnaud, 1983).

References

Brown, M., Küchemann, D. E., & Hodgen, J. (2010). The struggle to achieve multiplicative reasoning 11–14. In M. Joubert & P. Andrews (Eds.), *Proceedings of the Seventh British Congress of Mathematics Education* (BCME7, Vol. 30, pp. 49–56). University of Manchester: BSRLM. https://bsrlm.org.uk/wp-content/uploads/2016/02/BSRLM-IP-30-1-07.pdf

Callingham, R., & Siemon, D. (2021). Connecting multiplicative thinking to mathematical reasoning in the middle years. *Journal of Mathematical Behavior, 61*, 1–12. https://doi.org/10.1016/j.jmathb.2020.100837

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in Grades 1-5. *Journal for Research in Mathematics Education, 27*(1), 41–51. https://www.jstor.org/stable/749196

Downton, A. & Sullivan, P. (2017). Posing complex problems requiring multiplicative thinking prompts students to use sophisticated strategies and build mathematical connections. *Educational Studies in Mathematics, 95*(3), 303–328. https://doi.org/10.1007/s10649-017-9751-x

Harel, G. & Confrey, J. (1994). *The development of multiplicative reasoning in the learning of mathematics*. Suny Press

Hilton, A., Hilton, G., Dole, S., & Goos, M. (2016). Promoting middle school students' proportional reasoning skills through an ongoing professional development programme for teachers. *Educational Studies in Mathematics, 92*(2), 193–219. https://doi.org/10.1007/s10649-016-9694-7

Lamon, S. J. (1993). Ratio and proportions: Connecting content and children's thinking. *Journal for Research in Mathematics Education, 24*(1), 41–61. https://doi.org/10.2307/749385

Larsson, K., Pettersson, K., & Andrews, P. (2017). Students' conceptualisations of multiplication as repeated addition or equal groups in relation to multi-digit numbers and decimals. *Journal of Mathematical Behaviour, 48*. http://dx.doi.org/10.1016/j.jmathb.2017.07.003

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10–32. https://doi.org/10.1007/BF03217543

Mulligan, J. (2002). The role of structure in children's development of multiplicative reasoning. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. Thomas (Eds.), *Mathematics Education in the South Pacific: Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia* (pp. 497-503). MERGA.

Siemon D. (2016). Addressing the STEM challenge through targeted teaching: What's the evidence? *Proceedings of the 2016 Research Conference Improving STEM Learning: What will it take?* ACER: Melbourne.

Siemon, D. (2019). Knowing and building on what students know: The case of multiplicative thinking. In D. Siemon, T. Barkatsas, & R. Seah (Eds.). *Researching and using learning progressions (trajectories) in mathematics education* (pp. 6–31). Brill.

Siemon, D. Banks, N., & Prasad, S. (2018). Multiplicative thinking a STEM foundation. In T. Barkatsas, N. Carr & G. Cooper (Eds.), *STEM education: An emerging field of inquiry*. Sense Publications

Siemon, D., Breed, M., Dole, S., Izard, J., & Virgona, J. (2006). *Scaffolding Numeracy in the Middle Years.* https://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/scaffoldnum.aspx

Skemp, R. (1976). Relational and instrumental understanding. *Mathematics Teaching in the Middle School, 12*(2), 88–95.

Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From addition to multiplication ... and back: The development of students' additive and multiplicative skills. *Cognition and Instruction, 28*(3), 360-381. https://doi.org/10.1080/07370008.2010.488306

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 127–174). Academic Press.

Key Shifts in Students' Capacity to Generalise: A Fundamental Aspect of Algebraic Reasoning

Max Stephens
The University of Melbourne
m.stephens@unimelb.edu.au

Lorraine Day
University of Notre Dame Australia
Lorraine.Day@nd.edu.au

Marj Horne
RMIT University
marj.horne@rmit.edu.au

This paper will elaborate five levels of algebraic generalisation based on an analysis of students' responses to Reframing Mathematical Futures II (RMFII) tasks designed to assess algebraic reasoning. The five levels of algebraic generalisation will be elaborated and illustrated using selected tasks from the RMFII study. The five levels will be matched against the eight zones identified in the RMFII study supported by its Rasch analysis. We identify two shifts where students' capacity to generalise appear difficult to navigate. The first being where students move from noticing and describing regularities to formalising these regularities into verbal or symbolic expressions. The second is where students use their understanding of equivalence based on relational thinking to write and recognise equivalent algebraic expressions.

Key ideas implicit in the idea of generalisation as they relate to the algebraic reasoning tasks of RMFII have been presented by authors such as Love (1986) and Mason (1996), who suggested that the generalisation of a pattern, at its core, rests on the capability of noticing something *general in the particular*. Kieran (2007), however, noted that this feature alone may not be sufficient to characterise the *algebraic generalisation* of patterns, arguing that, in addition to seeing the general in the particular, students need to be able to express their generalisation algebraically, drawing on *explicit* reasoning in terms of justification and explanation. These points are directly relevant to the tasks used by RMFII to assess algebraic reasoning in which students were invited to explain their reasoning. Kieran's ideas will feature clearly in the third, fourth and fifth levels of a progression for algebraic generalisation advanced in this paper.

These five levels were enumerated in a previous paper (Stephens et al., 2021). They are: Working with particular instances; Noticing and describing regularities and patterns; Forming expressions—either verbal or symbolic; Using equivalence to examine different expressions of the same relationships and expressions; and Explicit generalised reasoning where students move between the particular to the general and vice versa, are able to identify and describe what varies and what stays the same, and work confidently with generalised expressions including their representation in different forms.

The research in RMFII developed an effective evidence-based learning progression with associated tasks for students' algebraic reasoning (Day et al., 2017). Nearly all tasks are graduated (multi-part) and designed to elicit progressive levels of students' algebraic generalisation, which is a key element of algebraic reasoning. Assessment tasks of this kind are helpful for classroom teachers to focus on the key shifts in students' thinking in order to foster their capability in this area. This paper will firstly show how the existing RMFII tasks, supported by Rasch modelling, align with and illustrate our five-level categorisation of algebraic generalisation. Secondly, the paper will show teachers of mathematics in the middle school years the importance of having *all* students progress at least to the third level of algebraic reasoning.

Drawing on the Rasch modelling (Bond & Fox, 2015) that was used in RMFII to rank the task item difficulty of scored responses across eight zones of algebraic reasoning, the Learning Progression for Algebraic Reasoning (LPAR) is related to the five levels of algebraic

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), pp. 46–49. Launceston: MERGA.

generalisation. In our recent paper (Stephens et al., 2021), several of the RMFII tasks were used to illustrate and validate the five levels of algebraic generalisation and in this paper one task, the Relational Thinking task, is used to exemplify how the LPAR zones relate to the levels of generalisation (Table 1). The Relational Thinking task (ARELS) is comprised of seven task items (ARELS1-ARELS7). The coding in the right column refers to the task items enumerated in Table 1, and to the score obtained for that item. For example, in Table 2, ARELS4.3 refers to the fourth relational thinking task item for which a score of 3 has been obtained.

Table 1
Relational Thinking Task Items and Rubrics

Item no.	Task item	Task item rubric Score
ARELS1	What numbers would go in these boxes to make a true number sentence (the numbers may be different). Explain your reasoning. ☐ + 521 = 527 + ☐	0 No response or irrelevant response 1 Incorrect response but suggest the difference of 6 is recognised in some way (e.g., *add 6 to the right hand side)* 2 Two correct numbers given (e.g., 13 and 7; 527 and 521) but little/no reasoning. 3 Two correct numbers given where the number on the left is 6 more than the number on the right (e.g., 100 and 94) with reasoning that reflects the relationship between 521 and 527 (difference of 6).
ARELS2	Find a different pair of numbers that would make the number sentence above true.	0 No response or irrelevant response 1 A different and correct pair.
ARELS3	Describe how you could find **all** possible pairs of numbers that would make this a true number sentence.	0 No response or irrelevant response 1 Incomplete attempt based on previous answers (e.g., *add 2 more to both*). 2 Statement regarding the difference of 6 (e.g., *number on the left must be six more than the number on the right)* or expression showing the difference (e.g., a + 6, and a)
ARELS4	What numbers would go in these boxes to make a true number sentence (the numbers may be different). ☐ – 521 = ☐ – 527 Explain how you worked it out.	0 No response or irrelevant response 1 Incorrect answer (possibly due to errors in calculation) but recognises relationship between 521 and 527 (difference of 6). 2 Two correct numbers given (e.g., 613 and 619) but little/no reasoning, may include some calculations. 3 A pair of correct numbers given where the number on the right is 6 more than the number on the left (e.g., 600 and 606) with reasoning that reflects the relationship between 521 and 527 (difference of 6).
ARELS5	Find another pair of numbers that would make the number sentence above true.	0 No response or irrelevant response 1 A different and correct pair.
ARELS6	Describe how you could find all possible pairs of numbers that would make this a true number sentence.	0 No response or irrelevant response 1 Incomplete attempt based on previous answers (e.g., *add 10 to both*). 2 Statement regarding the difference of 6 (e.g., *number on the right must be six more than the number on the left*) or an expression showing the difference (e.g., a and a + 6)
ARELS7	What can you say about the relationship between *c* and *d* in this equation? $c \times 2 = d \times 14$	0 No response or irrelevant response 1 Specific solution provided (e.g., *c must be 7 and d must be 1 to make it a true number sentence)* or a general statement *(*e.g., *c is bigger than d*) 2 Statement correctly describes relationship (e.g., *c is 7 times the number d)*

The first level of our classification of algebraic generalisation is working with particular instances, where students find solutions to simple equivalence situations or extending simple growing patterns. For example, in the Relational Thinking task the first part of the task asks students to find two numbers that make the number sentence true (ARELS1), and the second part of the task asks the students to identify a second pair of numbers that also make the statement true (ARELS2).

The second level of our classification of algebraic generalisation is noticing and describing regularities, where students are asked to notice regularities among a sequence of particular cases. In these cases, students attend to quantities that stay fixed and those that vary (Radford, 2006; Rivera, 2013) within the context of the task. This is an important level as the next three algebraic generalisation levels rely upon being able to notice regularities.

Forming expressions, either verbal or symbolic is the third level of algebraic generalisation, which extends the noticing of regularities to expressing these regularities, as constants and variables in formulae that may be articulated verbally or using symbolic language. To obtain all three marks for the ARELS1 task item, students have to provide two correct numbers as well as demonstrate reasoning that showed the difference of six relationship.

Establishing and using equivalence enables students to be able to recognise that generalisations may be represented by different symbolic expressions. Students should be able to show that different expressions can generate the same number where the same variables are used and/or algebraic simplification can be used to show equivalence. It is important for students to be able to distinguish situations where although two expressions may look different from each other, they are in fact equivalent.

The final level of our classification of algebraic generalisation is explicit generalised reasoning. This is where students can move flexibly between the particular and the general and vice versa. Students at this level can identify and describe variables and constants and work confidently with generalised expressions. The Relational Thinking task item (ARELS7) asked students to comment on the relationship between c and d in the equation c x 2 = d x 14. To answer this successfully, students need to understand the equivalent relationship between two product expressions, and to generalise a relationship explicitly between the two variables c and d, using appropriate mathematical language.

Table 2
RMFII Zones and Levels of Generalisation Reported in Stephens et al. (2021)

Item no.	RMFII Zone	Level of algebraic generalisation
ARELS1.1	Zone 1	Level 1: Working with particular instances.
ARELS1.2	Zone 2	Level 1: Working with particular instances.
ARELS1.3	Zone 6	Level 3: Forming expressions – verbally or symbolically.
ARELS2.1	Zone 3	Level 2: Noticing and describing regularities.
ARELS3.1	Zone 5	Level 2: Noticing and describing regularities.
ARELS3.2	Zone 6	Level 4: Using equivalence.
ARELS4.1	Zone 3	Level 2: Noticing and describing regularities.
ARELS4.2	Zone 4	Level 2: Noticing and describing regularities.
ARELS4.3	Zone 7	Level 4: Using equivalence.
ARELS5.1	Zone 5	Level 2: Noticing and describing regularities.
ARELS6.1	Zone 6	Level 3: Forming expressions – verbally or symbolically.
ARELS6.2	Zone 6	Level 4: Using equivalence.
ARELS7.1	Zone 5	Level 2: Noticing and describing regularities.
ARELS7.2	Zone 7	Level 5: Explicit generalised reasoning.

From the examination of the Relational Thinking task, coupled with the analysis of three other RMFII tasks (Stephens et al., 2021) where several responses were located in Zone 8, it appeared that two of the key shifts in students' ability to generalise are difficult for students to navigate. The first of these key shifts is where students move from Level 2 noticing and describing regularities to Level 3 where they formalise this noticing and describing to correctly form algebraic expressions, either verbally or symbolically. This is demonstrated by noticing and describing regularities appearing in Zones 3 and 4 and the beginning of Zone 5 in the LPAR (Table 2), while Level 3, which formalises this in verbal and symbolic algebraic expressions does not appear until Zone 6. The second of the key shifts, which students find difficult to negotiate, is moving from Level 3 to Level 4 drawing on students' understanding of equivalence based on relational thinking and the writing and recognition of equivalent algebraic expressions. This level is evident in Zones 5, 6 and 7 of the LPAR (Table 2).

As these two key shifts are somewhat problematic for students, it is important that teachers provide multiple opportunities for students to identify regularities, identify variables and constants, form and communicate expressions, and use equivalence. One way for teachers to do this is to utilise rich tasks, such as Garden Beds from maths300 (maths300.com), that provide opportunities for students to demonstrate all forms of generalisation. By using several rich tasks within different contexts, teachers can ensure that students are being exposed to these critical steppingstones in the generalisation process. The RMFII Teaching Advice (Day et al., 2018) includes references to rich tasks from well-known sources such as maths300, reSolve (resolve.edu.au) and nrich (nrich.maths.org) at each of the LPAR Zones, which provide teachers with tasks that will assist them to progress students in their algebraic learning journeys.

The algebraic generalisations exemplified in this paper require students to become proficient in using appropriate combinations of language, algebraic representation, and mathematical justification. These forms of reasoning and proof are applicable across many problem-solving situations and explicitly generalised algebraic reasoning will be necessary for students' continuing study of mathematics. Just as important, this paper has drawn attention to assisting all students to navigate successfully Levels 3 and 4 where they learn to form correct algebraic expressions either verbally or symbolically, and subsequently become able to recognise and work with equivalent expressions. Navigating these two key shifts appears essential for students to be able to reason algebraically.

References

Bond, T., & Fox, C. M. (2015). *Applying the Rasch model: Fundamental measurement in the human sciences* (3rd ed.). Routledge.

Day, L., Horne, M., & Stephens, M. (2018). *Reframing Mathematical Futures II*: *Algebraic reasoning learning progression: Teaching advice.* http://www.mathseducation.org.au/wp-content/uploads/2020/11/Algebra-Teaching-Advice-FINAL-30-10-20.pdf

Day, L., Stephens, M., & Horne, M. (2017). Developing learning progressions to support mathematical reasoning in the middle years: Algebraic reasoning. In A. Downton, S, Livy, & J. Hall (Eds.), *40 years on: We are still learning: Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 655–658). MERGA.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester, Jr., (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Information Age Publishing.

Love, E. (1986). What is algebra? *Mathematics Teaching, 117*, 48–50.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bernarz, C. Kieran & L. Lee (Eds.) *Approaches to algebra* (pp. 65–86). Springer. https://doi.org/10.1007/978-94-009-1732-3_5

Radford, L. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In S. Alatorre, J. L. Cortina, M. S´aiz, & A. M´endez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 2–21). Universidad Pedagógica Nacional.

Rivera, F. D. (2013). *Teaching and learning patterns in school mathematics: Psychological and pedagogical considerations.* Springer.

Stephens, M., Day, L., & Horne, M. (2021). An empirically based practical learning progression for generalisation, an essential element of algebraic reasoning. *Australian Journal of Education, 65*(3) 313–328. https://doi.org/10.1177/00049441211044798

Cognitive Flexibility and the Coordination of Multiple Information in Geometry and Measurement

Rebecca Seah
RMIT University
rebecca.seah@rmit.edu.au

Marj Horne
RMIT University
marj.horne@rmit.edu.au

Building from the evidence-based learning progression in geometric reasoning from the RMFII project, this paper presents data from students' solutions to three problems in geometry and measurement situations to identify key components needed to nurture reasoning. To show emerging analytical reasoning students must coordinate multiple pieces of information and demonstrate cognitive flexibility in their use of visualisation, diagrams, language, and symbols.

Understanding, fluency, problem-solving and reasoning are an integral part of becoming numerate. Good problem solvers exhibit cognitive flexibility, the ability to coordinate number skills, visual-spatial and other cognitive processes such as organising multiple pieces of information (Ionescu, 2012). Given the considerable difficulty Australian students face with solving problems and justifying their mathematical thinking (Thomson et al., 2017), we seek to identify key components needed to nurture reasoning. Geometric reasoning is the ability to critically analyse axiomatic properties, formulate logical arguments, identify new relationships, prove propositions, and used geometric knowledge in solving measurement problem situations (Seah & Horne, 2021b). A draft learning progression was developed based on Battista's (2007) exposition of Van Hiele levels of geometric thinking. Analysis of student data produced an evidenced based learning progression comprise of eight thinking zones: Zone 1: Pre-cognition; Zone 2: Recognition; Zone 3: Emerging informal reasoning; Zone 4: Informal and insufficient reasoning; Zone 5: Emerging analytical reasoning; Zone 6: Property-based analytical reasoning; Zone 7: Emerging deductive reasoning; Zone 8: Logical inference-based reasoning. We analyse student work in depth to determine how to nurture increasingly sophisticated reasoning from informal (Zone 3) through to emerging deductive reasoning (Zone 7).

Method

The data source used for this analysis is taken from the Reframing Mathematical Future II project. The results of these findings have been published elsewhere. Our aim here is to identify significant changes in student thinking by finding factors that cause a shift from Zone 3 to Zone 7. We do this by analysing students' responses to three tasks: 1) reasoning about nets (Seah & Horne, 2020), 2) making deductions of angle magnitudes (Seah & Horne, 2021a), and 3) enlarging a logo and determining its area (Seah & Horne, 2021b) (Figure 1). The geometric contexts of these tasks allow students to demonstrate their knowledge and understanding. The reasoning required for the net task is Zones 2, 3 and 5. The angle magnitudes task is Zones 2, 5, 6, and 8. The logo drawing task is Zone 3 and 5. The logo area task is Zones 4 and 7.

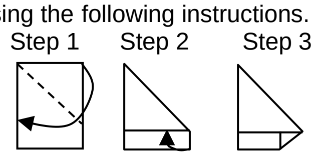
In designing the rubric, we determined that a zero score is given for no response or irrelevant responses. A '1' score denoted some recognition of the concepts but not full application. A maximum score would be given for a correct response with sound reasoning. Scores in between, the number of which depended on the complexity and the context of the task, would be given for partially correct answer and reasoning. For example, GCRD1 requires either a correct or incorrect enlargement logo drawn so the ceiling score is 2. Conversely, it was possible to get some of the angle magnitudes (GANG4) correct and give partial reasoning, thus requiring more gradation with a score of 4 being the ceiling. The data analysed came from students in 12 trial schools and 32 project schools.

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), pp. 50–53. Launceston: MERGA.

GNET 4. Sam thinks he has drawn a net of a cube using six squares but it does not fold up to make a cube. What might Sam's drawing look like? Explain how you know.

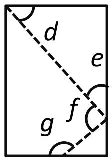
Geometric Angles 2

A four-sided shape is folded from a sheet of A4 paper using the following instructions.



a [GANG3]
What is the name of this shape?
____
Explain your reasoning.

b [GANG4]
Unfold the paper and find the size of each marked angle.
Angle *d* = ____________ Angle *e* = ____________
Angle *f* = ____________ Angle *g* = ____________



Explain your reasoning.

LOGO

A designer draws a triangular logo on grid paper. He wants to enlarge the logo so the sides are twice as long.

a. [GCRD1] Draw his enlarged logo on the graph.

b. [GCRD2]. Write the coordinates of the corners A', B', and C' of the new large triangle**:**

c. [GCRD3] If the area of the original logo is $2.25m^2$, what will the area of the new logo be? Explain how you know?

Figure 1. Sample of assessment tasks on geometric reasoning.

Findings

Overall Results

Students' responses to the tasks reflected not only their ability to reason, but the extent of the task requirement and the exposure they had with the concepts. As shown in Table 1, by the number of no responses and correct responses received, the GNET task was the easiest whereas GCRD3, which required students to find the area of the enlarged shape, was the hardest.

Table 1
Breakdown of Student Responses for Each of the Questions (percentage)

	GNET4		GANG4		GCRD1		GCRD3	
	Trial	Project	Trial	Project	Trial	Project	Trial	Project
Score	*n = 233*	*n = 566*	*n = 157*	*n = 270*	*n = 118*	*n = 328*	*n = 118*	*n = 328*
0	13.2	9.1	38.5	16.3	17.8	30.8	37.3	47
1	11.4	10.7	28.9	27.4	53.4	38.4	52.5	47
2	36.5	30.4	18.6	19.6	28.8	30.8	1.7	2.1
3	38.9	49.8	10.9	22.6			8.5	4
4			3.9	14.1				

In GNET4, 48% of the students used just the information in the question by either drawing six squares that would fold into a cube or drew a correct shape but did not provide a reason. In the trial data, 39% of students gave a correct response. This improved in the project data. Students who gave a correct reason went beyond the information given in the question and called on other knowledge, such as visualising the nets from different perspectives. Compared to GNET task, the number of no response or irrelevant responses was higher in the GANG4. Around 29% of the students showed partial angle knowledge by providing a label (e.g., acute, or obtuse) or recognising one angle magnitude; 19% showed emerging analytical reasoning giving two angles correctly, with some explanation; and 11% trial and 23% project students correctly calculated the angles giving some reasons though often sparce. Logical inference-based reasoning, albeit about a simple situation, was shown by 4% and 14% of trial and project students respectively who reasoned correctly and deduced all angle magnitudes.

In the logo task, 18% of the trial students did not draw an enlarged logo and 37% did not attempt to calculate the area. More than half of the students (53%) operated within the information given in the question by drawing a larger logo in some form although incorrectly either by enlarging one dimension only or a larger logo with no attention to the magnitude of the enlargement. A similar number (52%) gave a response to the area question that was incorrect, often just using the numbers given in the question by doubling 2.25 and did not provide units or gave little reasoning. Around 29% correctly enlarged the logo and just over 2% were able to give a correct area measurement, often using a procedural explanation. Just over 8% were able to reason correctly, giving an explanation that recognised that doubling the length of all the sides quadrupled the area, thus showing emerging deductive reasoning.

Types of Reasoning

Table 2 shows the responses to the three questions. The questions are shown with the score given following the dot so that GANG4.1 means a sore of 1 on the question GANG4.

Table 2
RMFII Zones of Geometric Thinking

Zone 2. Recognition	GNET4.1	GANG4.1		
Zone 3. Emerging informal reasoning	GNET4.2		GRD1.1	
Zone 4. Informal and insufficient reasoning			GRD3.1	
Zone 5. Emerging analytical reasoning	GNET4.3	GANG4.2	GRD1.2	
Zone 6. Property-based analytical reasoning		GANG4.3		
Zone 7. Emerging deductive reasoning			GRD3.2	GRD3.3
Zone 8. Logical inference-based reasoning		GANG4.4		

We can see that student responses to these three questions spread across the zones of reasoning. For GNET, the move to analytical reasoning appeared to occur with a response scored of 3. The two student responses in Figure 2 demonstrate this. Student A used recognition of a taught prototype. Student B used visualisation and then used a combination of diagram and language to explain the image in their mind and hence their reasoning.

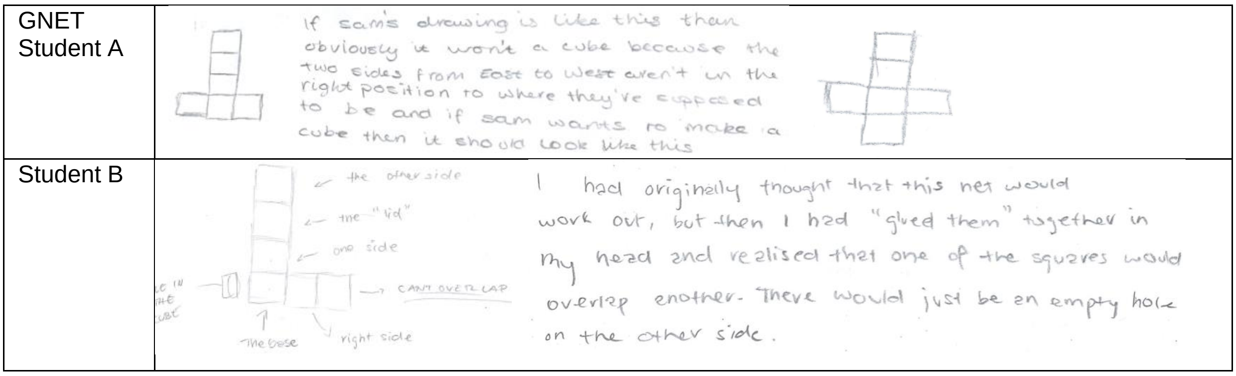


Figure 2. Students' responses on the GNET4 task.

In GANG4, analytical reasoning emerged with a response score of 2 where students gave partially correct answers (usually 45° with no explanation). Some students were starting to make connections but tended to explain using benchmarks such as 90°, as demonstrated here by student C who used no diagrams.

Student C: *d* and *e* has the same size angle as you can see, *f* as everyone knows that it is 90° because it's a right angle and *g* is an obtuse, which is 180° (wrote 45°, 45°, 90°, 180°).

Limited ability to explain, use diagram effectively and present a sequential argument show clearly in the attempts of the students. The few students who were able to reason deductively justified 45° as half of the corner right angle and calculated the 135° either by using the interior angles or the straight angle with 45°. For GCRD3, student 10JW27701 shows an attempt to calculate area but is just using the numbers given in the question rather than demonstrating analytical reasoning in the solution (Figure 3). Meanwhile, student 10YL4700 demonstrates sound deductive reasoning showing explanations both algebraically and in words.

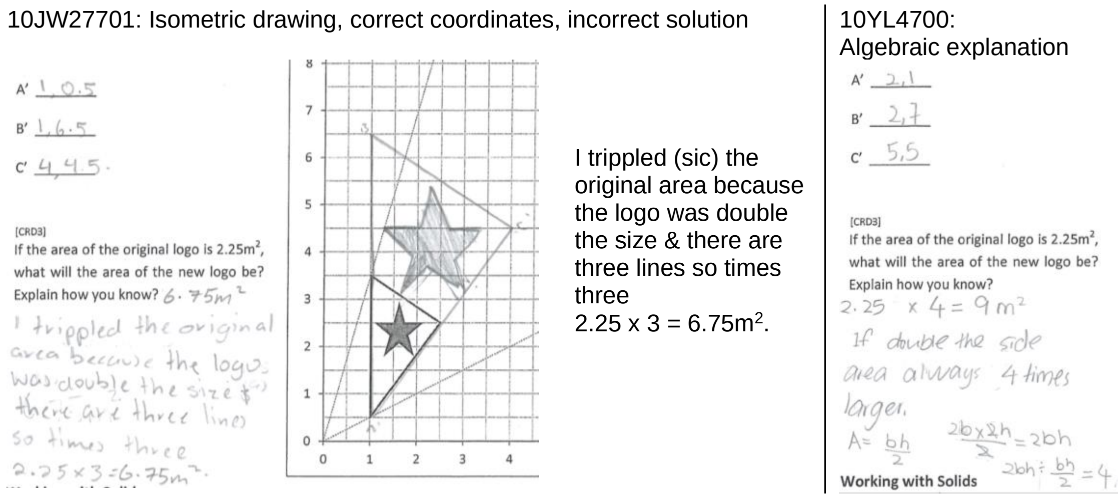


Figure 3. Students' responses on the GCRD task.

To reason analytically or deductively, coordination between the information presented in the question with the network of one's own conceptual understanding is needed. While knowing the mathematical concepts is important, the results here demonstrate that students needed to visualise the problem in situ, coordinate the information in the question with their prior knowledge to obtain a solution and present their argument using diagrams, language, and symbols flexibly. Finally, they need to be able to check that their reasoning is sound. In short, they need cognitive flexibility. These things need to be explicitly in the curriculum. At the moment, visualisation and the flexible use of communication tools is absent.

References

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Information Age Publishing.

Ionescu, T. (2012). Exploring the nature of cognitive flexibility. *New Ideas in Psychology, 30*(2), 190–200. https://doi.org/10.1016/j.newideapsych.2011.11.001

Seah, R., & Horne, M. (2020). The influence of spatial reasoning on analysing about measurement situations. *Mathematics Education Research Journal, 32*(2), 365–386. https://doi.org/10.1007/s13394-020-00327-w

Seah, R., & Horne, M. (2021a). Charting a learning progression for reasoning about angle situations. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in mathematics education: Foundations and pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 345–352). MERGA.

Seah, R., & Horne, M. (2021b). Developing reasoning within a geometric learning progression: Implications for curriculum development and classroom practices. *Australian Journal of Education, 65*(3), 248–264. https://doi.org/https://doi.org/10.1177/00049441211036532

Thomson, S., Wernert, N., O'Grady, E., & Rodrigues, S. (2017). *TIMSS 2015: Reporting Australia's results*. Australian Council for Educational Research.

Facilitating the Shift to Higher-order Thinking in Statistics and Probability

Rosemary Callingham
University of Tasmania
Rosemary.Callingham@utas.edu.au

Jane Watson
University of Tasmania
Jane.Watson@utas.edu.au

Greg Oates
University of Tasmania
Greg.Oates@utas.edu.au

It is increasingly recognised that to be informed citizens and to participate fully in the workforce requires an understanding of statistical data and risk. Such understanding is underpinned by statistical reasoning. It has been shown, however, that students have difficulty moving from concrete representations and procedural mathematical statistics to the context-based appreciation of data drawing on proportional reasoning that is becoming increasingly necessary. Based on the Statistical Reasoning Learning Progression (SRLP), this paper examines the barriers to shifting to higher-order thinking.

Introduction

As statistics and probability began to be acknowledged as a fundamental part of the mathematics curriculum towards the end of the 20th century (Australian Education Council [AEC], 1991), it became important to consider the new challenges for students in mastering this part of the curriculum. Although traditionally the other parts of the mathematics curriculum have claimed to have applications across other school subjects and outside of the classroom, two aspects of statistics and probability add even more potential to the application of the curriculum outside of the mathematics classroom: *uncertainty* and *context* (Callingham et al., 2021). At this point in time, the combination of uncertainty and context is seen starkly in society's experience of the COVID-19 pandemic (Watson & Callingham, 2020). The uncertainty associated with chance events and the confidence associated with decisions in contexts where statistics have been collected, is different from the rest of the mathematics curriculum, which is based on undisputed facts and proved theorems. Further, context is essential to any meaningful data that are collected (Cobb & Moore, 1997), and the entire statistical problem-solving process is based on anticipating, acknowledging, accounting for, and allowing for variability in these data (Bargagliotti et al., 2020). At each step in this process, particular skills and understandings need to be applied and combined to reach the answer to the statistical problem posed.

Students' progress in developing their statistical understanding and reasoning has been described in terms of an 8-zone Statistical Reasoning Learning Progression (SRLP) (Callingham et al., 2019). A question has arisen, however, as to why, as students progress through the middle school years (aged 11 to 16 years), many have difficulty moving to the highest zones in the learning progression but remain around the middle zones, particularly in Zone 4 (Callingham et al., 2019).

Approach

The Statistical Reasoning Learning Progression (SRLP) was developed during the Reframing Mathematical Futures (RMFII) project (Siemon et al., 2018). The SRLP describes an increasingly sophisticated hierarchy in which procedural mathematical statistics, such as calculation of an average or quantifying outcomes from a probability experiment, interact with an understanding of the context of the problem. In Zones 1 and 2, skills are limited to, for

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), pp. 54–57. Launceston: MERGA.

example, reading a value from a graph or offering an opinion about a context with no reference to data. At the higher levels (Zones 7 and 8), students call on proportional reasoning with data integrated with contextual understanding to make decisions and draw informal statistical inferences. Of particular interest here are the middle levels of the 8-zone hierarchy (See handout).

Students in Years 7 to 10 undertook a series of assessments based on statistical reasoning tasks. The student data reported here are taken from the third round of RMFII assessment, (Callingham et al., 2019) and have not been previously reported. Two tasks are used to exemplify the shifts observed in moving across zones, particularly in respect of the difficulties observed in moving up from of Zone 4: one based on probability (STATS) in the context of the interpretation and implications of winning Tattslotto; and the second based in a statistical context (STWN) with students contrasting two different graphical representations of the same data set to tell a story in the context of how long families have lived in a town. Both tasks are based in social contexts with which students are likely to be familiar. The abbreviated titles were used to identify tasks during the analysis and are used here for consistency. The tasks were marked by teachers based on the rubrics provided. The tasks and rubrics are shown in Figure 1.

Findings and Discussion

Table 1 presents the findings from a sample of 581 students in Years 7 to 9 (aged 13 to 15 years) who undertook at least four statistical reasoning tasks (not just the tasks reported here) during the third round (MR3) of assessment. Student responses were Rasch analysed, and the person measures used to determine the distribution of students across the zones.

Table 1
Number and Proportion of Students across SRLP Zones

	n	Zone 1 (%)	Zone 2 (%)	Zone 3 (%)	Zone 4 (%)	Zone 5 (%)	Zone 6 (%)	Zone 7 (%)	Zone 8 (%)
Yr 7	165	19 (11.52)	19 (11.52))	51 (30.91)	48 (29.09)	20 (12.12)	6 (3.64)	2 (1.21)	0 (0.00)
Yr 8	215	13 (6.05)	16 (7.44)	43 (20.00)	56 (26.05)	41 (19.07)	25 (11.63)	19 (8.84)	2 (0.93)
Yr 9	201	26 (12.94)	29 (14.43)	49 (24.38)	38 (18.91)	38 (18.91)	6 (2.99)	14 (6.97)	1 (0.50)
Total	581	58 (9.98)	64 (11.02)	143 (24.61)	142 (24.44)	99 (17.04)	37 (6.37)	35 (6.02)	3 (0.52)

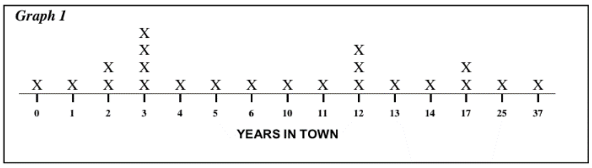
The proportion of students in each zone is very similar to that reported elsewhere (Callingham et al., 2019), and in previous similar studies (Callingham & Watson, 2017). It should be emphasised that this analysis is based on a new and different group of RMFII students, and that the nature of the analysis allows for skewed distributions and is not based on a normal distribution. The very similar patterns shown to previous analyses suggest that the sticking points in the middle zones are not environmental but related to cognitive development.

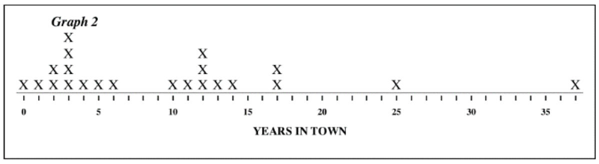
Shifts to Higher Zones

As shown in Figure 1, the rubrics reflect an increasing sophistication and quality of response and their position along the SRLP is based on the Rasch analysis. Across these two different tasks, to reach higher levels of response students need to bring together multiple aspects of reasoning.

STATS	One day Claire won Tattslotto with the numbers 1, 7, 13, 21, 22, 36. So she said she would always play the same group of numbers, because they were lucky. What do you think about this?
Code 1 **Zone 3**	Affirms a belief in being lucky (e.g., *I think it would be lucky I will pick the same number's too; I don't think many numbers are lucky. But I think 4, 7 & 9 are, so I guess I'd agree in a way you can have lucky numbers*).
Code 2 **Zone 4**	Rejects 'luck' (e.g., *There is no such thing as lucky numbers*) or states that numbers were unlikely to occur again, or less likely to occur than other numbers (e.g., *I think she shouldn't go for the group of numbers again because you can't get the same numbers after numbers, you always get different numbers all the time*).
Code 3 **Zone 6**	Implicitly recognises that all combinations of numbers have the same chance of occurring on any draw (e.g., *It was just a stroke of luck because any number could of come up; There is no such thing as a lucky number, things like Tattslotto are picked at random*).
Code 4 **Zone 7**	Explicit recognition that all numbers or combinations of numbers are equally likely, may/may not offer an opinion (e.g., *There is an equal chance for all combinations, but she's already won once, so why keep gambling, why not invest the money, you would get more out of it*).
Code 5 **Zone 8**	Reasoning that recognizes equal chance and interprets Claire's comments relative to context (e.g., *It is a good idea to use the same numbers all the time but there is as much chance as getting any other six numbers*).

A class of students recorded the number of years their families had lived in their town. Here are two graphs that students drew to tell the story.





STWN1: What can you tell by looking at Graph 1?
STWN2: What differences do you notice between Graph 1 and Graph 2?
STWN3: Which graph is better at presenting information and "telling the story"? Explain your answer.

	STWN1
Code1 **Zone 3**	Tautological response (e.g., *The numbers along the bottom tell you how many years; How long people lived in that town).*
Code 2 **Zone 5**	Response refers to one or more specific aspects (e.g., *3 and 12 have the most; 1 family had lived there 37 years, There are 22 kids).*
Code 3 **Zone 8**	Summative or comparative response that reflects some appreciation of information overall (e.g., *They range from all years; Not many families have stayed there for the same time*).

	STWN2
Code 1 **Zone 4**	Incorrect (e.g., *Less people live in the town in Graph 2 than Graph 1; There are more Xs in Graph 2*) or superficial comments related to the appearance of the graph (e.g., *Graph 2 is harder to read because numbers are together, Graph 1 is easier to read because numbers are spread out).*
Code 2 **Zone 5**	Some indication that difference recognised in terms of spread and accuracy (e.g., *Graph 2 goes up in fives and Graph 1 doesn't*).
Code 3 **Zone 7**	Acknowledges that graphs show the same data and describes the difference in terms of the scales used (e.g. *There is no difference from graph 1 to graph 2 except that graph 2 shows the spaces where graph 1 doesn't; graph 2 says all the years between 0 and 37 – while graph 1 only tells the relevant ones*).

	STWN3
Code 1 **Zone 3**	Statistically inappropriate choice (Graph 1) with reasoning that ignores spread (e.g., *Graph 1 because it only has the time it needs*)
Code 2 **Zone 5**	Statistically appropriate choice (Graph 2) with reasoning based on personal preferences (e.g., *Graph 2 because they have set it out better*) or indicates both the same (e.g., *Neither – they tell the same amount of information*).
Code 3 **Zone 7**	Statistically appropriate choice (Graph 2) with reasoning that recognises the importance of seeing all the years (e.g., *Graph 2 because you can see the difference between the years more clearly and the graph is more spaced out; Graph 2 because it has all the years*).

Figure 1. Exemplar items and rubrics.

In the probability item (STATS), it appears that developing the complex concept of random, and the necessity for appreciating the probability of groups of numbers occurring rather than single values, as appropriate for the context of the question, is important in moving responses to the higher zones. The emerging recognition of randomness and the application to groups of numbers is evident in the Code 2 (Zone 4) response but this loses coherence and falls back on individual numbers ("you always get different numbers all the time"). The Code 4 (Zone 7) response, however, is confident about working with groups of numbers but falls back on opinion ("but she's already won once so why keep gambling") to justify the thinking.

The other item (STWN), in its two-part structure (presenting two graphs and requiring a comparison rather than a single description), requires several components of the context, both visual and textual, to be integrated for a higher-level response. Students need to recognise the subtlety of the comparison needed between the graphs and to bring together understanding of the nature of the graphs and the context of the question to reach higher zones. That the lowest levels of the responses (Code 1) appear in Zones 3 and 4 rather than lower down the SRLP indicates that comparing two graphs creates some difficulty for students. The reasoning demonstrated to obtain a Code 1 is procedural, (e.g., Graph 1 is easier to read because numbers are spread out) focussing on aspects of the graph alone, rather than the information each graph conveys. To reach a Zone 7 response, students have to explicitly reason by integrating both the visual appearance of the graph and the nature of the information conveyed (e.g., Graph 2 says all the years between 0 and 37—while Graph 1 only tells the relevant ones).

Conclusion

It appears that coordinating different types of information and bringing together diverse aspects of mathematics and context are critical to shift reasoning to more sophisticated levels of response. This capacity to bring together two or more aspects of knowledge and understanding is important in other areas of mathematics, including in the shift from additive to multiplicative thinking. The inclusion of Statistics and Probability in the Mathematics Curriculum (AEC, 1991) has extended the appreciation of the structure of the multiple understandings required when data and context need to be combined rather than considered separately. It is appreciating this combination that moves reasoning to higher zones.

References

Australian Education Council. (1991). *A national statement on mathematics for Australian schools.*

Bargagliotti, A., Franklin, C., Arnold, P., Gould, R., Johnson, S., Perez, L., & Spangler, D. A. (2020). *Pre-K–12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II).* American Statistical Association. https://www.amstat.org/asa/files/pdfs/GAISE/GAISEIIPreK-12_Full.pdf

Callingham, R., & Watson, J. (2017) Appropriate goals for statistical literacy at school. *Statistics Education Research Journal 16*(1), 181–201. https://doi.org/10.52041/serj.v16i1.223

Callingham, R., Watson, J., & Oates, G. (2019). Statistics and probability: From research to the classroom. In D. Siemon, T. Barkatsas, & R. Seah (Eds.) *Researching and using progressions (trajectories) in mathematics education* (pp. 181–204). Brill Sense.

Callingham, R., Watson, J., & Oates, G. (2021). Learning progressions and the Australian curriculum mathematics: The case of statistics and probability. *Australian Journal of Education 65*(3), 329–342. https://doi.org/10.1177/00049441211036521

Cobb, G. W., & Moore, D. S. (1997). Mathematics, statistics, and teaching. *American Mathematical Monthly, 104*(9), 801–823. https://doi.org/10.2307/2975286

Siemon, D., Callingham, R., Day, L., Horne, M., Seah, R., Stephens, M., & Watson, J. (2018). From research to practice: The case of mathematical reasoning. In Hunter, J., Perger, P., & Darragh, L. (Eds.). *Making waves, opening spaces* (Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia) pp. 40–49. Auckland: MERGA.

Watson, J., & Callingham, R. (2020). COVID-19 and the need for statistical literacy. *Australian Mathematics Education Journal, 2*(2), 16–21