Mathematics Education Research Group of Australasia


# Proceedings of the 45th Annual Conference of the Mathematics Education Research Group of Australasia

**Weaving Mathematics Education From All Perspectives**

**Proceedings of the 45th Annual Conference of the Mathematics Education Research Group of Australasia**

Bronwyn Reid-O'Connor, Elena Prieto-Rodriguez,
Kathryn Holmes & Amber Hughes (Editors)

Mathematics Education Research Group
of Australasia

## Newcastle MERGA 45 Local Organising Committee

Elena Prieto-Rodriguez, School of Education, University of Newcastle (Conference Convenor)

Amber Hughes, School of Education, University of Newcastle (Conference Secretariate)

Andrew Kepert, School of Information and Physical Sciences (Mathematics), University of Newcastle

Corrinne Towns, NSW Department of Education

Megan Quentin-Baxter (Support Officer)



ISBN 978-1-920846-34-3

Published by:

The Mathematics Education Research Group of Australasia Inc.

GPO Box 2747

Adelaide, SA 5001

https://merga.net.au/

# Preface

This is a record of the Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia (MERGA). The conference was hosted by colleagues at the University of Newcastle. The Proceedings were published online at the MERGA website https://merga.net.au/common/Uploaded%20files/Annual%20Conference%20Proceedings/2023%20Annual%20Conference%20Proceedings/MERGA45_Proceedings_2023_Final.pdf

The University of Newcastle, established in 1965, is located in the city of Newcastle and is committed to its role as an academic leader, exerting a significant influence on opportunities for individuals in the region as well as the broader national and international communities. The conference took place at the beautiful Newcastle City Campus, which offers captivating views of the Hunter River, Nobby's Headland, and Beach.

The theme of the conference, "Weaving Mathematics Education From All Perspectives," reflected the intention of the local organising committee to foster discussions and collaboration among researchers from diverse backgrounds and perspectives. The committee recognise the significance of Mathematics Education Research Group of Australasia (MERGA) in creating an inclusive community of scholars who recognise and appreciate the transformative potential of mathematics education.

By choosing this theme, the local organising committee aimed to create a conference environment that would facilitate learning and collaboration among all participants. The conference provided a platform for researchers to share their insights and experiences, enabling attendees to learn from one another. The ultimate goal was to foster enduring connections and relationships within the mathematics education community, weaving a network of support and collaboration that extended beyond the conference itself.

The word "weave" holds special significance in this context as it symbolises the connection to the traditions and heritage of the Awabakal and Worimi peoples, the traditional owners of the land where the conference took place. It recognises the importance of respecting and acknowledging the rich history and knowledge of the Indigenous people who have always regarded the land as a place of learning.

Two plenary lectures were delivered on the theme. In the opening and first plenary lecture, Doctors Michael Donovan and Judy-anne Osborn described the *Indigenising University Maths* project, a long -term collaboration between Indigenous and non-Indigenous practitioners around the world.

The second lecture by Professor Rochelle Gutiérrez drew upon the concept of restor(y)ing mathematics to highlight the ways in which researchers are engaging in Indigenous futurity and what that says about who we are becoming as researchers and persons.

The *Clements/Foyster* lecture was delivered by Professor Colleen Vale and focused on the relationships between teacher adaptive expertise and generative teacher practitioners. The final plenary was presented by Professor Mellony Graven, who shared key aspects of her work on the *Mental Starters Assessment Project* as a means to address an absence of teaching for number sense and persistent unit counting methods.

Entries for the Beth Southwell Practical Innovations Award were invited in both the Early Bird and Main submission rounds of reviewing this year, in order to reinvigorate interest in the award and highlight opportunities for embedding research in practice. Five candidates were identified either by expressing interest, or being nominated by the reviewers. Shortlisted papers were considered by a panel of three expert external assessors who arrived at a worthy winner in 2023. Five shortlisted Early Career Award candidates were judged by a 'mystery shopper' during the conference.

The conference included presentations of symposia, research papers, short communications, and round tables that covered a wide range of topics related to mathematics education in Australasia and other countries. All symposia and research papers were competitively double-blind reviewed by panels of mathematics educators with expertise in the field, and accepted for publication and presentation, or presentation only. All the short communications were reviewed by the organising committee and research-focused abstracts were accepted for presentation. The first Early Bird round of reviewing was designed to give formative feedback, while the Main round was competitive. Reviewing took place online this year for the first time which we hope was a success. There was a high conversion rate of those submitting research papers to the Early Bird round being accepted into the Proceedings.

The published Proceedings include the plenary papers, symposia papers, research papers, and abstracts of research presentations, short communications, and round tables. Other activities during the conference, such as workshops and panel sessions, provided additional opportunities for participation.

Delegates from Australia, Chile, Fiji, Greece, Hong Kong, India, Indonesia, Japan, Mexico, New Zealand, Republic of Korea (South Korea), Singapore, South Africa, Vietnam, United Kingdom, United States of America and Zimbabwe participated in the conference. The Proceedings illustrate the breadth of mathematics education research undertaken in the region and beyond by the MERGA research community. The Editorial Team would like to thank the authors for sharing their research. Gratitude also goes to the Review Panel Chairs and all the reviewers for their professionalism and effort in reviewing the papers and providing constructive feedback. The review process ensured that the high academic standards of the MERGA community were upheld.

Thanks to all the many colleagues who have helped bring MERGA 45 together, the Local Organising Committee, the Proceedings Editors, the MERGA Executive, the School of Education, University of Newcastle, the Mathematical Association of NSW, the City of Newcastle, and our sponsors, Newcastle Coal Infrastructure Group and Essential Assessment. And finally, thanks to all the presenters who have come to the party, in such uncertain times, to celebrate the first face-to-face meeting of our community in several years.

**Elena Prieto-Rodriguez (Conference Convenor) and Amber Hughes (Conference Secretariat)**

Bronwyn Reid-O'Connor, Elena Prieto-Rodriguez, Kathryn Holmes & Amber Hughes (Editors)

# Table of Contents

**PROCEEDINGS OF THE 45TH ANNUAL CONFERENCE OF THE MATHEMATICS EDUCATION RESEARCH GROUP OF AUSTRALASIA**

Weaving Mathematics Education From All Perspectives

# MERGA 45 Reviewers

## Review Panel Chairs

Judy Anderson
Lynda Ball
Ban Heng Choy
Jaguthsing Dindyal
Gregory Hine
Generosa Leach
Tracy Logan
Margaret Marshman
Lisa O'Keefe
Greg Oates

## Reviewers

Jonathan Adams
Amie Albrecht
Judy Anderson
Catherine Attard
Robin Averill
Lynda Ball
Nathan Berger
Janette Bobis
Helen Booth
Alexandra Bowmar
Fiona Budgen
Rebecca Burtenshaw
Scott Cameron
Katherin Cartwright
Michael Cavanagh
Ban Heng Choy
Audrey Cooke
Kate Copping
Mary Coupland
Chelsea Cutting
Lisa Darragh
Melinda Dixon
Ann Downton
Allan Dougan
Emma Every
Bruce Ferrington
Jill Fielding
Louise Fitzgerald
Kym Fry
Lorraine Gaunt
Robyn Gendall
Bronwyn Gibbs
Merrilyn Goos
Peter Grootenboer
Vesife Hatisaru
Gregory Hine
Kai Fai Ho
Katherine Holmes
Kathryn Holmes
Amber Hughes
Roberta Hunter
Dan Jazby
Michael Jennings
Jyoti Jhagroo
Harry Kanasa
Berinderjeet Kaur
Ho Weng Kin
Barry Kissane
Pauline Kohlhoff
Nicola Kovacs
Rox-Anne L'Italien-Bruneau
Kevin Larkin
Generosa Leach
Tracy Logan
Yew Hoong Leong
Tracy Logan
Laurinda Lomas
Susan Mabb
Katie Makar
Mayamiko Malola
Margaret Marshman
Heather McMaster
Carmel Mesiti
Paula Mildenhall
Jodie Miller
Carol Murphy
Priscilla Murphy
Lisa O'Keeffe
Greg Oates
Sitti Maesuri Patahuddin
Cath Pearn
Elena Prieto-Rodriguez
Susilahudin Putrawangsa
Lisa O'Keefe
Kate Quane
Robyn Reaburn
Bronwyn Reid O'Connor
Kaitlin Riegel
Emily Ross
Shaileigh Rowtcliff
Justine Sakurai
Karyn Saunders
Wee Tiong Seah
Max Stephens
Kevin Sullivan
Matt Thompson
Steve Thornton
Dung Tran
Belinda Trewartha
Kristen Tripet
Colleen Vale
Bridget Wadham
Jennifer Way
John West
Wanty Widjaja
Susanna Wilson
Andy Yeh
Joseph Yeo
Ben Zunica

# Clements-Foyster Lecture

The Clements-Foyster Lecture acknowledges an eminent mathematics education researcher from Australia, New Zealand or a South East Asian rim country, who is invited to present a keynote address at the annual MERGA conference. This annual keynote address is named in honour of Ken Clements and John Foyster who initiated and organised the first Mathematics Education Research Group of Australia Conference at Monash University in 1977. This led to the establishment of the organisation now known as MERGA.

# Generative Teacher Practitioners: Enacting Adaptive Expertise in and Beyond the Classroom

Colleen Vale
*Monash University*
colleen.vale@monash.edu

Much of our mathematics education research has focussed on effective teaching practices, yet participation and achievement of Australian students has stagnated and fallen behind other countries of similar standards. Whilst school mathematics leaders, both primary and secondary, support teachers to develop their knowledge and develop whole school approaches much of this work focuses on developing teachers' routine expertise rather than their adaptive expertise. Research on generative teaching focuses on developing students' understanding to improve learning, whereas generative teacher practitioners continually seek to improve their understanding and practice of mathematics teaching by learning to be responsive to their students' mathematics thinking and developing students' mathematical reasoning and problem solving. They are also responsive to the socio-cultural and gender composition of their classrooms. In this presentation, the relationships between teacher adaptive expertise and generative teacher practitioners along with opportunities for further research will be discussed.

Student participation, engagement and achievement in mathematics in Australia is an on-going concern and priority for teachers, leaders and policy makers. As researchers we have focused on a range of aspects of student learning and teaching practice to improve outcomes for students. My current Australian Research Council funded project concerns the development of primary teachers' adaptive expertise in interdisciplinary mathematics and science (Berry et al. 2021). Much of my previous research has focussed on equity, mathematical reasoning and out-of-field teaching and drawn on theories of generative teaching (Carpenter et al. 2004), and developing generative teacher practitioners (Franke et al., 2001; Kemmis et al., 2014; Sherin, 2002). In this paper I will discuss these theories and my research findings and make connections between generative teaching, generative teachers and adaptive expertise (Anthony et al., 2015; Bransford et al., 2005; Timperley & Twyford, 2022; Yoon et al., 2019). In the final section of the paper I will provide some further details of the current ARC project.

## Generative Teaching

Since the turn of the century, much of the research on improving teaching and learning has focussed "mathematics teachers' attention to student reasoning and sense making to develop deep and flexible thinkers" (van Es & Sherin, 2021, p. 17). However, as Liljedahl (2021) has pointed out, developing thinking classrooms is a challenge for many mathematics teachers. Generative teaching is inclusive and responsive to the students' and their cultural context (Anthony et al. 2015). That is, teachers provide opportunities for all students to reason and make sense of mathematics. Given the strong emphasis on external assessment scores rather than innovating their practice to include an emphasis on sense making, teachers tend to focus their attention on developing students' recall of mathematics concepts, facts and procedures.

In my PhD study (Vale, 2001) which investigated gender equity in classrooms using digital technology, one of the teachers I observed designed a task for students to form a conjecture about the sum of exterior angles of polygons using Geometer Sketchpad. At the time, there were no online templates and students had to construct their own polygons using the software. While the teacher's intention was to develop students' geometric reasoning, they spent most of the two lessons helping students use the software. One high achieving girl was frustrated as the teacher had not explained what was meant by "conjecture." An Indigenous student who had finished the task for homework and helped other students during the second lesson, was ignored and told to sit down. The teacher was not at all aware that this student had demonstrated skill with using the software and

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 3–9). Newcastle: MERGA.

understanding of the geometric property. This example highlights the importance of equity and social justice and that all students need to experience a sense of agency to engage in sense making (Schoenfeld, 2016).

Noticing and responding to students in the moment to develop their understanding and reasoning requires capacity to use a range of teacher actions and questioning strategies (Chan, 2021; Choy & Dindyal, 2021). Therefore, effective professional noticing is a generative teacher practice (Pynes, et al., 2020). To do this successfully teachers need a deep understanding of mathematics, flexibility and to reflect on their actions (van Es & Sherin, 2021).

## Generative Teachers

A generative teacher of mathematics takes responsibility for the creation and generation of his or her own knowledge (Carpenter, et al., 2004). Generative practitioners have the knowledge, skills and disposition to continuously seek to improve their knowledge of students, mathematics and mathematics learning, and their practice of mathematics teaching through deliberative reflection in collaboration with colleagues (Kazemi et al., 2009; Prestage & Perks, 2000; Sherin, 2002; Valli, 1997):

> … the deliberative approach to reflection emphasizes decision making based on a variety of sources: research, experience, the advice of other teachers, personal beliefs and values and so forth. No one voice dominates. Multiple voices and perspectives are heard. (Valli, 1997, p. 77)

Generative teachers meet outside the classroom to discuss their students' written work (Herbert et al., 2022; Pynes et al. 2020; Vale & Davies, 2005), reflect on their actions-in-the-moment (Eden, 2020), and to co-plan and reflect on action (Eden, 2020; Yoshida 2012).

Vale and Davies (2005) reported on pre-service teachers' (PST) reflection of implementing problem solving and reasoning tasks with groups of Year 4 students in a local school. Two cases were reported of PSTs who had used the problem-solving task, Eric the Sheep (Maths300, https://www.maths300.com), with their group of students. When reflecting on the student learning, the first case David, indicated that he valued procedural knowledge and the traditional practice of teacher transmission. He made distinctions between "better" and "slower" students and did not value the productive power of interactions between students. He clearly explained the procedures for solving the problem using concrete materials and then was surprised when one girl was able to compute the solution very quickly mentally and then explained the concept to the whole group. Rather than thinking about how he might have introduced the task differently, this PST recommended that the groups in the class should be re-organised by skill level. David seems more intent on developing his efficiency with transmission teaching methods rather than changing his teaching practice to support students to learn with and through each other. In the second case, Nerida, reflected on the productive power of the interactions between the students in her group. She valued the discussion among the students and recorded examples of the questions that they posed each other:

> How do you think we should set up the counters? Should we use the same colour counter? Why are we making Eric a different colour? Let's read the task again and work through it together. What do we do next? Why did you do that? Why? What's next? How should we correct our work to check if we got the answer correct? Are we doing this right Miss? What else can we do? What happens if we try to add more counters and more Eric's? Let's work this out together. (Vale & Davies, 2005, p. 749)

The authority that this PST gave to the sense-making of students' questions and explanations illustrates an openness with regard to her learning and support for a generative approach to student learning. These two cases illustrate that whilst individual reflection is valuable, PSTs need to reflect on their teaching collaboratively if they are to generate new understandings of the mathematics and their students' sense making.

In our mathematical reasoning project (Herbert et al., 2022; Vale et al., 2017; Widjaja et al., 2021) which used design-based research to develop a rubric for formative assessment of mathematical reasoning. The rubric, *Assessing Mathematical Reasoning Rubric* (AMRR, Australian Academy of Science, 2018), was created using demonstration lessons of reasoning tasks and then collaboration with the teachers observing the demonstration lesson to develop and refine the rubric. The rubric focusses on three main reasoning actions used across the mathematics curriculum: analysing, generalising and justifying. As a formative assessment rubric it includes descriptions of these actions at different levels of sophistication, rather than using it as summative assessment by defining reasoning competency for each year level. Pairs of teachers met after the demonstration lesson to analyse student work samples using the AMRR. Two rounds of data collection occurred, so we were able to not only refine the rubric, but also compare the teachers' understanding of reasoning over time. We found that in the first round of post-lesson discussions, the teachers focused on the number of properties or examples that students provided and whether or not they were correct, rather than analysing the quality of their argument. Many of the teachers in the first round used the language of summative assessment and, whilst they did describe students' reasoning using phrases from the rubric, some remained focused on the level of reasoning. In the second round during the post-lesson discussion, the teachers more readily used terms from the AMRR to describe students' reasoning. They interacted with each other to reach agreement on their formative assessment of students' reasoning to demonstrate shared meaning (Kemmis et al. 2014) of mathematical reasoning.

In Lesson Study, teachers undertake generative practices as they co-plan, co-observe and co-reflect. They analyse student work collaboratively and reflect (Yoshida, 2012):

> In order for lesson study to be successful, teachers need to think of lesson study as *a way to improve their own learning* as well as student learning. Spending more time studying mathematical content and curriculum, developing a strong pedagogical content knowledge with colleagues, and establishing a community of learning through lesson study. (Yoshida, 2012, p. 140)

Widjaja, Vale and Doig (2020) reported on the collaborative practices of Lesson Study. In this study, teachers, from three local primary schools formed two cross-school Lesson Study Teams. Each team included teachers with a diversity of teaching experience and expertise: a Year 3, Year 4 and Year 3/4 teacher, and a numeracy leader or coach. We observed the collaborative practices of these two Lesson Study teams and tracked their professional growth. We found that trialling the planned lessons prior to conducting the Lesson Study lesson when lots of teachers and experts would observe the lesson, enabled them to revise their planning of questions to elicit students' thinking. The participants in each of the Lesson Study teams valued the range of expertise among members of the team, including the knowledge and expertise of teachers from the other schools in the project, together with the collaborative processes of planning and trialing lessons and reflecting on these lessons through the post-lesson discussion. One of the teachers in an interview about the Lesson Study process reported on the importance and meaning of collaboration for her own practice and stated:

> [The] big thing that's dawning on our teachers here, is that collaboration doesn't mean, someone else does the work for you and its less work. It's about really challenging each other's thinking and questioning each other, and that's been a big feature of this. (Widjaja et al., 2020, p. 16)

This finding accords with Brodie (2020) who found that for collaborative practices to be generative, "[teachers] need to able to challenge each other's thinking and practices" (p. 40).

## Adaptive Expertise

Bransford and colleagues (2005) identified innovation and efficiency as two main components of expertise. Routine experts work efficiently in standard situations and use known routine tasks and practices, while frustrated novices are keen to innovate but still need to develop efficient practices for standard situations. Adaptive experts exhibit both innovation and efficiency. They quickly become accustomed to unfamiliar, unexpected and complex situations as they purposefully apply professional knowledge, innovation and creativity (Hatano & Inagaki, 1986; Hatano & Oura, 2003).

Anthony, Hunter and Hunter (2015) have led the research on adaptive expertise of prospective teachers within the MERGA community. They, and others, argue that the adaptive expertise of teachers is a critical component of quality teaching (Anthony et al., 2015; Timperley & Twyford, 2022; Yoon et al., 2019). According to Timperly and Twyford (2022) adaptive expertise evolves as the teacher shifts their focus from self to students and from simplicity to complexity. Characteristics of adaptive expertise relate closely to teacher noticing of critical moments for student engagement and learning in the classroom and their actions in these moments (Anthony et al., 2015; Chan, 2021; Choy & Dindyal, 2021; Pynes et al., 2020; van Es & Sherin, 2021).

Adaptive expert teachers also have a propensity to experiment with new and different teaching and learning activities that is, to innovate (Anthony et al., 2015). They may do this individually or collaboratively. One approach to innovation is action research, a form of critical praxis involving reflection that is action-orientated, social and political (Kemmis et al., 2014). It involves participants in planning actions, implementing these plans, monitoring and evaluating the processes and consequences of their action, and re-planning for further action. Hence it involves a series of on-going cycles of action. As practising teachers, two colleagues and I used action research to investigate sexually (gender) inclusive curricula. This project got me interested in curricula and pedagogies to improve student engagement, equity and social justice. Recently my colleagues and I reported on findings from a professional learning project in which school mathematics leaders used the Teaching Sprint model (Breakspeare & Jones, 2020) to conduct an action research cycle (Vale & Delahunty, 2022). These small-scale action research projects (teaching sprints) conducted with small teams of teachers at their school provided the school mathematics leaders with evidence of practices that were effective for their students and worthy of both celebrating and continuing. The Teaching Sprint also provided a collaboration and consultation process that supported them to develop respectful relationships with teams of teachers.

Whilst the model of adaptive expertise used by Bransford and colleagues (2005) included the components of innovation and efficiency, the framework developed by Yoon et al. (2019) has three components: flexibility, deep level of understanding, and deliberate practice. Flexibility is manifested in the teacher's ability to integrate aspects of their knowledge in relation to the teaching act with the goal of improving outcomes while responding to their specific contexts. This may involve acknowledging and recognizing cultural and gender diversity of students as well as knowledge of their prior learning. A deep level of understanding involves the teachers' ability to recognise meaningful patterns quickly, allowing one to attend to deeper-level problem solving and, in turn, facilitating students to perform at a higher level. That is, the ability to act in the moment is a manifestation of a deep level of understanding. Finally, deliberate practice concerns the teacher's ability to engage in reflection during the lesson and when reviewing the lesson individually, or with a co-teacher, to consciously deliberate about students' engagement and learning, their teacher actions, and the use of regulation processes to ensure cognitive engagement of all students.

What's interesting about the Yoon and colleagues' (2019) model of adaptive expertise, is the similarity of the components of adaptive expertise with those of generative teachers. Generative teachers use deliberative reflection to improve their knowledge, planning and teaching. They aim to be flexible when acting-in-the moment and when responding to individual student. They also aim to develop their knowledge of the content and their teaching context, their students and the school's practice architectures (Kemmis et al., 2014).

## Adaptive Expertise in Interdisciplinary Mathematics and Science Project

The aim of the three-year Adaptive Expertise in Interdisciplinary Mathematics and Science project is to improve theoretical and practical understanding of the nature and development of primary teachers' adaptive expertise in interdisciplinary mathematics and science. The key research questions are:

- How can primary teachers' adaptive expertise in interdisciplinary mathematics and science be characterised in terms of components and levels?
- To what extent, and how, does primary teachers' adaptive expertise change and develop during a trajectory across two school years aimed at interdisciplinary mathematics and science in a co-plan, co-teach and co-reflect approach?

The study uses a mixed method longitudinal research design. Pairs of Year 5 or Year 6 teachers from five schools will co-teach the same two sequences of STEM lessons twice, that is, in two of the three years of the study. The two sequences of STEM lessons designed for the study were Keep your Finger on the Pulse and Journey through Space (Hughes et al., 2022). Data will be generated using an initial teacher interview and an online questionnaire constructed using videoed episodes of teachers' practices. These video episodes were collected during the pre-study trial of Keep your Finger on the Pulse. Each of the STEM lessons taught by the co-teachers will be video-taped and co-teacher interviews will explore their reflections on their teaching using video recordings of their teaching, and researcher observations of these lessons. The adaptive expertise framework developed by Yoon and colleagues (2019) has been used to select video episodes and construct items for the video-based questionnaire. This framework will also be used to analyse post-lesson co-teacher interviews.

Research about the development of teacher's knowledge and practice of mathematics teaching is essential for improving student engagement, learning, equity and social justice. We need to continue to develop understanding of students' mathematical thinking as the socio-political context for learning changes and evolves as this research provides resources for teachers' deliberative reflection. We also need to continue to research the socio-political-economic context of schools and systems to ensure that teachers can engage in generative practices and develop adaptive expertise.

## Acknowledgements

I would like to thank my many colleagues for their collaboration on various research projects related to developing generative teacher practitioners: Anne Davies, Sharyn Livy, Sandra Herbert, Leicha Bragg, Wanty Widjaja, Esther Loong, Aylie Davidson, Susie Groves, Brian Doig, Kerryn Driscoll, Carmel Delahunty, Penelope Kalogeropoulos, Jill Cheeseman, Amanda Berry, Jan van Driel, Lihua Xu, Joe Ferguson, Jinny Kim, James Russo and Jennifer Mansfield.

## References

Anthony, G., Hunter, J., & Hunter, R. (2015). Prospective teacher's development of adaptive expertise. *Teaching and Teacher Education, 49*, 108–117. http://dx.doi.org/10.1016/j.tate.2015.03.010

Australian Academy of Science (2018). *Assessing mathematical reasoning rubric*. https://www.resolve.edu.au/assessing-mathematical-reasoning

Berry, A., van Driel, J., Vale, C., Widjaja, W., Xu, L., Ferguson, J., & Kim, J. (2021). Developing adaptive expertise in primary science and mathematics education. ARC Discovery Grant, DP210101171.

Bransford, J., Derry, S., Berliner, D., Hammerness, K., & Beckett, K. L. (2005). Theories of learning and their roles in teaching. In L. Darling-Hammond, & J. Bransford (Eds.), *Preparing teachers for a changing world: What teachers should learn and be able to do* (pp. 40–87). Hoboken, NJ: Jossey-Bass.

Breakspear, S., & Jones, B. R. (2020). *Teaching sprints: How overloaded teachers can keep getting better*. SAGE Publications.

Brodie, K. (2020). Resources for and from collaboration: A conceptual framework. In H. Borko, & D. Potari (Eds.), *ICMI study 25 conference proceedings: Teachers of mathematics working and learning in collaborative groups* (pp. 37–48). Lisbon, Portugal.

Carpenter, T. P., Blanton, M. L., Cobb, P., Franke, M. L., Kaput, J., & McClain, K. (2004). *Scaling up innovative practices in mathematics and science.* Madison, WI; Wisconsin Centre for Education Research.

Chan, K. (2021). Using classroom video-based instruments to characterize pre-service science teachers' incoming usable knowledge for teaching science. *Research in Science and Technological Education*, *41*(4), 1–23. DOI: 10.1080/02635143.2021.1872517.

Choy, B. H., & Dindyal, J. (2021). Productive teacher noticing and affordances of typical problems. *ZDM—Mathematics Education, 53*, 195–213. doi.org/10.1007/s11858-020-01203-4

Eden, R. (2020). Learning together through co-teaching mathematics: The role of noticing in teachers' collaborative inquiry. In H. Borko, & D. Potari (Eds.), *ICMI study 25 conference proceedings: Teachers of mathematics working and learning in collaborative groups* (pp. 300–307). Lisbon, Portugal.

Franke, M. L., Carpenter, T. P., Fennema, E., et al. (2001). Capturing teachers' generative change: A follow-up study of teachers' professional development in mathematics education. *American Educational Research Journal, 38*, 653–689.

Hatano, G., & Inagaki, K. (1986). Two courses of expertise. In H. Stevenson, H. Azama, & K. Hakuta (Eds.), *Child development and education in Japan* (pp. 262–272). New York: Freeman.

Hatano, G., & Oura, Y. (2003). Commentary: Reconceptualising school learning using insight from expertise research. *Educational Researcher, 32*(8), 26–29. https://doi.org/10.3102/0013189X032008026

Herbert, S. Vale, C., White, P., & Bragg, L. (2022). Engagement with a formative assessment rubric: A case of mathematical reasoning. *International Journal of Educational Research*, *111.* https://doi.org/10.1016/j.ijer.2021.101899 https://doi.org/10.1016/j.ijer.2021.101899

Hughes, S., Russo, J., Mansfield, J., Green, A., Jones, D., Vale, C., & Berry, A. (2022). Co-designing integrated mathematics and science lesson learning sequences for primary education. *International Journal of Innovation in Science and Mathematics Education*, *30*(4). DOI: https://doi.org/10.30722/IJISME.30.04.005

Kazemi, E., Lampert, M., & Franke, M. L. (2009). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides. Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1). Palmerston North, NZ: MERGA.

Kemmis S., Wilkinson, J., Edwards-Groves, C., Hardy, I., Grootenboer, P., & Bristol, L. (2014). *Changing practices, changing education*. Singapore: Springer.

Liljedahl, P. (2021). *Building thinking classrooms in mathematics: Grades K–12*. Corwin.

Prestage, S., & Perks, P. (2000). Subject knowledge: Developing the fourth sense. *Paper presented at the British Educational Research Association Conference*, Cardiff University, 7–10 September, 2000.

Pynes, D'A., Empson, S., & Jacobs, V. (2020). Supporting teachers in the development of noticing children's mathematical thinking with a web-based tool. In H. Borko, & D. Potari (Eds.), *ICMI study 25 conference proceedings: Teachers of mathematics working and learning in collaborative groups* (p. 676). Lisbon, Portugal.

Schoenfeld, A. H., & the Teaching for Robust Understanding Project. (2016). *An introduction to the Teaching for Robust Understanding (TRU) framework*. Berkeley, CA: Graduate School of Education. http://map.mathshell.org/trumath.php or http://tru.berkeley.edu

Sherin, M. (2002). When teaching becomes learning. *Cognition and Instruction, 20*(2), 119–150.

Timperley, H., & Twyford, K. (2022). Adaptive expertise in educational leadership: Embracing complexity in leading today's schools. *Australian Education Leader, 44*(1), 8–12.

Vale, C. M. (2001). *Gender and computer based mathematics in selected secondary classrooms* [Doctoral thesis, La Trobe University].

Vale, C., & Davies, A. (2005). Glimpses of generative practice: Constructing pre-service teachers' learning in partnership. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practice. Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia* (pp. 743–750). Geelong, Sydney: MERGA.

Vale, C., & Delahunty, C. (2022). Teaching sprints: Action research led by school mathematics leaders. In N. Fitzallen, C. Murphy, & V. Hatisaru (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of Mathematics Research Group of Australasia* (pp. 33–36). Launceston: MERGA.

Vale, C., Widjaja, W., Herbert, E. S., Loong, E., & Bragg, L. A. (2017). Mapping variation in children's mathematical reasoning: The case of 'What else belongs?' *International Journal of Science and Mathematics Education*, *15*(5), 873–894. DOI 10.1007/s10763-016-9725-y

Valli. L, (1997). Listening to other voices: A description of teacher reflection in the United States. *Peabody Journal of Education, 72*(1), 67–88.

Van Es, E., & Sherin, M (2021). Expanding on prior conceptualizatons of teacher noticing. *ZDM 53*, 17–27. https://doi.org/10.1007/s11858-020-01211-4

Widjaja, W., Vale, C., & Doig, B. (2020). Nothing like planning and reflecting together to build trust. In G. Lloyd, & O. Chapman. (Eds.), *Participants in mathematics teacher education: International handbook of mathematics teacher education* (2nd Edition, Volume 3, Chapter 5, pp. 128–150). Kononklijke Brill.

Widjaja, W., Vale, C., Herbert, S., Y-K. Loong, E., & Bragg, L. (2021). Linking comparing and contrasting, generalising and justifying: A case study of primary students' levels of justifying. *Mathematics Education Research Journal, 33*(2), 321–344. DOI 10.1007/s13394-019-00306-w

Yoon, S. A., Evans, C., Miller, K., Anderson, E., & Koehler, J. (2019). Validating a model for assessing science teacher's adaptive expertise with computer-supported complex systems curricula and its relationship to student learning outcomes. *Journal of Science Teacher Education, 30*(8), 890–905.

Yoshida, M. (2012). Mathematics lesson study in the United States: Current status and ideas for conducting high quality and effective lesson study. *International Journal for Lesson and Learning Studies, 1*(2), 140–152.

# Keynote Presentations

# Collaboration and Partnership; Indigenous Mathematics and Promoting Justice

Judy-anne Osborn
*Monash University*
judy-anne.osborn@monash.edu

Michael Donovan
*La Trobe University*
m.donovan@latrobe.edu.au

We describe the Indigenising University Maths project, which is a long-term collaboration between Indigenous and non-Indigenous practitioners around the world, in this space. We describe some of our conclusions so far in terms of useful possibilities to explore. We indicate how researchers and educators everywhere can help.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 13). Newcastle: MERGA.

# Restor(y)ing Mathematics, Restor(y)ing Ourselves: A Spiritual Turn in Mathematics Education

Rochelle Gutiérrez
*University of Illinois*
rg1@illinois.edu

Partly in response to the movements of Black Lives Matter and Land Back on Turtle Island; land and water rights severely threatened globally (e.g., Brazil and India) as well as proposed constitutions and new laws that name lands and waters as persons who have rights (e.g., Bolivia, Australia, and Chile); climate justice; Truth and Reconciliation processes in Canada and South Africa that have major implications for Indigenous education; a queer and trans movement; and a global pandemic that has shifted education to embrace socioemotional learning and care for one another, the global mathematics education community is situated in a unique moment to reconsider mathematics in helping us to get free and heal. In what way(s) are our research projects taking seriously the idea that the (school) mathematics that got us to this moment is not the (school) mathematics that will get us out? What is involved in radically dreaming towards a mathematical future that can help us to re-attach to each other and to our more-than-human relatives, to weave ourselves together? What are the languages needed to describe relations between various mathematics and mathematical forms as well as between various mathematicians? And, in what ways could this work be considered spiritual? In this presentation, I will draw upon the concept of restor(y)ing mathematics to highlight the ways we are engaging in Indigenous futurity and what that says about who we are becoming as researchers and persons.

We are always in and of relation, both with respect to others and with respect to the work we carry out. So, our research methods reflect not just our desires but also our sense of the futures we believe are possible. Our methodologies offer us the ability to deepen our relations with one another (Tachine & Nicolazzo, 2022), including our more-than-human relatives (Gutiérrez, 2017; 2019). Therefore, as researchers, as we begin to open ourselves to new ontologies (e.g., recognizing humans are not the center of the world), new epistemologies (e.g., embracing acts such as observing, making, storytelling, and dreaming as ways of knowing) (e.g., Barajas & Bang, 2018), and new axiologies (e.g., valuing relations and holistic views over discrete material objects, categorizing, and sorting), we are better able to embrace a spiritual turn in mathematics education, one that can continue the work of our ancestors and begin to put ourselves back together. Elsewhere, I have articulated the basis of this spiritual turn as desire-based research and Indigenous futurity (Gutiérrez, 2022) where I draw upon the concept of restor(y)ing from Bang et al. (2014). A spiritual turn recognizes that we have all been affected by the grief of diaspora and are trying to find our ways home. The work extends a sociopolitical perspective (Gutiérrez, 2013) by highlighting the axiological (e.g., ethical) aspect of our work and by engaging notions of time in non-linear ways. I offer questions we can consider if we aim to embrace this spiritual turn:

- In what way(s) is our research desire-based? And whose desires are centered?
- How are we performing futurity? What are our methods for moving into and inhabiting the next world? (What are our underlying theories of change?)
- What kind of (mathematical) futures are we making? And what does that say about who we are becoming as researchers or persons?

From the perspective of Indigenous futurity (Harjo, 2019), the concept of restor(y)ing allows us to engage with past, present, and future in entangled ways. Restoring focuses on bringing back that which has been erased largely by school systems and colonial scripts handed to us from the academy, as both of these places tend to reflect and perpetuate white supremacy and cisheteropatriarchy. By

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 15–16). Newcastle: MERGA.

recognizing that mathematics is much more than formalized symbols and proofs and instead can be considered more broadly as pattern, relation, structure, and logic, we can honor the ways everyday people and lands/waters recreate known/ancestral patterns, relations, structures, and logics to reattach to others and, therefore, to remake themselves as woven kin. Restorying is an act of radical dreaming that treats the future as a form of the present. By recognizing that we can gather under a tree for which the seed has only just been planted, we honor the ways everyday people and lands/waters constantly invent new patterns, relations, structures, and logics to reflect changes in the world. That is, new forms of mathematics can be created (e.g., a woven proof, non-axiomatic forms of mathematics) to address problem solving and joy. The forms of dehumanization and domination with which we have participated (e.g., privileging school outcomes such as increased tests scores as the only desired forms of success) become the backdrop for new possible futures. That is, whereas previously we might have seen equity work as something we did to "help others" (e.g., students who have been oppressed by society or school systems), we shift to seeing our work as having the potential to "liberate and heal us."

A key aspect of a spiritual turn is recognizing that the work we do affects not just the students, teachers, and communities with whom we work, it affects us as persons. I offer the table *Restor(y)ing Mathematics: Restor(y)ing Ourselves* that reflects the relationship I see between mathematics and us as researchers/persons. With this table, I encourage us to see the ways we, our research, and mathematics are entangled with students, teachers, communities, and our collective futures.

## Restor(y)ing Mathematics: Restor(y)ing Ourselves

FUTURES / PAST / US

| RESTORING (dispossession, erasure → attachment) | | RESTORYING (dehumanization, domination → new futures) | |
|---|---|---|---|
| Mathematics | Ourselves | Mathematics | Ourselves |
| • Ethnomathematics<br>• Place-based mathematics<br>• Patterns, relations, structures, logics for problem solving & joy<br>• Ecology of knowings<br>• Centering our more-than -human relatives<br>• Living mathematx | • Sovereignty<br>• Self determination<br>• Upholding original instructions<br>• Learning our gifts and how to give them<br>• Observing, listening, making, singing, dreaming, pitching in, proper asking, & ceremonies<br>• As we've always done | • Mathematics as story<br>• Who speaks on behalf of mathematics?<br>• Narrating for a particular purpose (e.g., racial capitalism, eugenics, joy, abundance)<br>• Mathematics as literal (place-based) and abstract (aspects that travel)<br>• Mathematics as intervention (mathematx) | • Storied by our relations<br>• Who claims us?<br>• Reflective of & allows us to adapt to changes in world<br>• What we're ready to experience/live<br>• Nepantla, In Lak'ech, reciprocity<br>• Towards becoming a good ancestor |

*We are **in relation to** mathematics; we **are** mathematics.*


Bang, M., Curley, L., Kessel, A., Marin, A, Suzukovich, E. S., III, & Strack, G. (2014). Muskrat theories, tobacco in the streets, and living Chicago as indigenous land. *Environmental Education Research*, *20*(1), 37–55. https://doi.org/10.1080/13504622.2013.865113

Barajas-López, F., & Bang, M. (2018). Toward indigenous making and sharing: Implications for mathematics learning. In I. Goffney, & R. Gutiérrez (Eds.), *Rehumanizing mathematics for Black, Indigenous, and Latinx students* (pp. 13–22). National Council of Teachers of Mathematics.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 37–68. https://doi.org/10.5951/jresematheduc.44.1.0037

Gutiérrez, R. (2017). Living mathematx: Towards a vision for the future. *Philosophy of Mathematics Education Journal*, *32*.

Gutiérrez, R. (2019). Mathematx: Towards a way of being. In J. Subramanian (Ed.), *Proceedings of the tenth international mathematics education and society conference* (pp. 67–110). MES10. https://www.mescommunity.info/proceedings/MES10.pdf

Gutiérrez, R. (2022). A spiritual turn: Towards desire-based research and indigenous futurity in mathematics education. *Journal for Research in Mathematics Education*, *53*(5), 379-388.

Harjo, L. (2019). *Spiral to the stars: Mvskoke tools of futurity*. University of Arizona Press. https://doi.org/10.2307/j.ctvh4zjdg

Tachine, A. R., & Nicolazzo, Z. (Eds.) (2022). *Weaving an otherwise: In-relations methodological practice*. Stylus Publishing.

# Merging Mathematics Research and Development: Connecting Communities in an Emerging Network of Local and National Projects

Mellony Graven
*Rhodes University*
m.graven@ru.ac.za

In this paper I share aspects of the work of my team, over the past twelve years as the incumbent South African Numeracy Chair at Rhodes University in Makhanda, South Africa. This is one of two National Research Foundation Chairs established to merge development with research in the search for sustainable solutions to persistent challenges in primary mathematics education particularly for 'disadvantaged' communities. The funding model stipulated establishing professional development (PD) with a minimum of ten schools, allowing flexibility, and, following rigorous evaluation, up to three five-year terms. This long-term flexible Research and Development (R&D) Chair model enabled a grounded, organically-emerging network of research-informed projects with multiple iterations enabling continual strengthening and gradual up-scaling. Here I share the PD and emergent projects from inception to date. This includes the Mental Starters Assessment Programme (MSAP) developed to address poor number sense and pervasive unit counting for calculating. MSAP emerged collaboratively from the first term experiences of the two Numeracy Chairs. Multiple iterations of design, implementation, and research, with key partners, enabled gradual upscaling in our second term with national piloting leading to roll-out with the Department of Basic Education (DBE) in our current third term. Thus, I hope to challenge dominant funding models that separate research from development in tightly-defined short-term cycles, and rather point to the possibilities within models that enable the development of an ever-improving and expanding network of R&D projects, informed by school and community partnerships.

It is a pleasure and an honour to give a keynote for the MERGA community that I have participated in, alongside my team of research collaborators and students, for the past few years. There are so many similarities between MERGA and our Southern African Association of Research in Mathematics, Science and Technology Education (SAARMSTE) which also holds an annual conference. Both communities provide essential participatory platforms for our ever-evolving, being and becoming, passionate, and engaged mathematics educators and researchers.

As noted in the abstract I draw on the body of work (and network) of my Chair team, and ongoing partnership with former fellow Numeracy Chair Hamsa Venkat, to elaborate on, and illuminate, the power of the *long-term, flexible,* '*development with research*' (R&D) funding model we have had the privilege of working with. [I note that this model of funding is quite different from, for example, Australian Research Council grants in length, flexibility and the ][Foundation (NRF) Research Chairs in its inclusion of *development*]. A key broker in this model of Maths Chairs was Professor Mamokgethi Setati (now Phakeng) who brought together private funders (with particular interest in intervention work) and South Africa's Department of Science and Technology and National Research Foundation to establish six mathematics education research and development chairs. The appetite for this kind of funding emerges from the post-apartheid context with the urgency and ethical need to *respond* to challenges (that particularly impact the poor and historically racially oppressed), rather than to just report on problems. Major post-apartheid curriculum changes did little to address issues of inequality and poor performance in mathematics education. In 2010/2011 four R&D Chairs were appointed in secondary mathematics education (including Professor Jill Adler) and two in numeracy education (referring to mathematics in the early grades of schooling). The Chairs' brief was to work with the teachers of at least ten 'previously disadvantaged' schools to search for (research) sustainable scalable solutions to challenges encountered in mathematics education with an emphasis on reciprocal relationships between researchers and teachers where mutual benefits are negotiated (Setati, 2005). In design, the chairs ethically responded to the need to move away from reporting deficit research findings that simply feed into the problem (Graven, 2012), towards speaking about possibilities within the widely noted mathematics education 'crisis'.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 17–24)). Newcastle: MERGA.

A key aspect of the power of long-term R&D programmes with flexible trajectories is that they allow repeated iterations of research and development programmes, improving and building capacity along the way, and thus strengthening the likelihood of success when scaling up (Venkat & Graven, 2022). In the talk I share a somewhat historical perspective on the emergence of a rich network of projects (beginning local and expanding nationally) to illuminate how long-term R&D funding, predicated on establishing local partnerships responsive to school needs (thus requiring flexible outcomes and deliverables), allows rich learning opportunities *for all* participants in the network (i.e. myself as Chair, researchers, teachers, learners, parents, teacher-educators, DBE partners) (Graven, 2019). I use this as a counterpoint to dominant funding models that work against such grounded longitudinal work. In last years' MERGA post-presentation discussion of our paper (Jorgensen & Graven, 2022), delegates expressed frustration with funding models that constrain opportunities for community partnerships to inform the direction and pace of research, particularly frustrating in the case of working with marginalised, indigenous and/or remote communities where research 'norms' are challenged. Thus, an aim of this paper is to highlight opportunities arising when funding enables flexible, responsive, community-informed, long-term research that is explicitly paired with development. While I acknowledge such funding opportunities are thin on the ground, and this model emerged from particular circumstances, I believe communities such as MERGA and SAARMSTE are well-placed to advocate for, and lobby private and public funders to support such endeavours.

## The Context

The term 'crisis' is often paired with South African education, particularly mathematics education (e.g., Fleisch, 2008; Soudien & Harvey, 2021; Cosser, 2023). While our Grade 5 performance on TIMSS 2019 (Reddy et al., 2022) shows modest improvement since we began participating, we remain third from the bottom of 58 participating countries. In regional comparative studies we similarly perform worse than many of our poorer neighbours (Spaull & Kotze, 2015). We also have among the highest inequality in performance among wealthier and poorer learners. While we might challenge aspects of the validity of international comparative studies, our performance on our own annual national assessments (ANAs) last written in 2014 (as teachers refused to continue participating in them), was no better. Only 3% of Grade 9 learners got over 50% (DBE, 2014). 'Problems' have been noted to begin in the early grades of schooling with only 16% of Grade 3 learners performing at grade level and a learning gap of three grade levels 'between the poorest 60% of students and the wealthiest 20%' (Spaull & Kotze, 2015, 13). A widely noted cause of poor performance is the absence of number sense and the persistent use of the unit counting for calculations well beyond what is sensible or appropriate for the grade or number range (Schollar, 2008; Graven et al., 2013). Furthermore, learner access to mathematical meaning making is challenged when a fifth of Grade 1 children learn in English, increasing to almost 80% in Grade 4, this despite less than 10% speaking English as their first language, and despite a Language in Education policy that advocates home language instruction for at least the first four years of schooling (Robertson & Graven, 2019, 2021).

Teachers are often blamed for the crisis and yet the nature of the support they receive tends to involve short-term information dissemination sessions by district authorities aimed at curriculum coverage and compliance (Jita & Mokhele, 2012). Teachers have been sceptical about participating in such programmes due to a perceived lack of benefit, and particularly sceptical of those programmes involving research as this so often meant colluding in producing negative narratives about their work. Increasingly however attention is turning to initial teacher education given evidence that most teachers are not developing the knowledge required for primary mathematics teaching during their studies (Bowie et al., 2019).

In the South African Numeracy Chair Project (SANCP) we were determined to challenge deficit discourses by i) exploring opportunities for excellence in mathematics teaching and learning in marginalised, under-served communities and ii) communicating researched possibilities through multiple platforms. To do this, I developed ongoing partnerships with schools, teachers, district advisors, teacher educators and researchers, where jointly we could explore ways forward to challenges faced and create opportunities and share productive narratives of what is possible in South African mathematics teaching and learning.

## Learning Through the Creation of Communities of Practice

The professional development (PD) programmes and the SANC research undertakings were developed with a Community of Practice (CoP) perspective of learning, drawing from the work of Wenger (1998) and Jaworski (2005). The latter's focus on critical inquiry communities is reflected in the names of three of the four PD programmes: early Numeracy-, Numeracy-, and Mathematics Inquiry Community of Leader Educators for Grade 1&2 (eNICLE), Grade 3&4 (NICLE), and Grade 4-7 (MICLE) programmes, respectively. We called the additional Grade R (pre-Grade 1) programme Early Number Fun (ENF), to highlight the emphasis on play-based learning pedagogies. It was the only single-grade PD CoP, a necessary arrangement, given that Grade R is a relatively new introduction to South African schooling with many teachers un- or under-qualified and few opportunities for PD tailored to the specialised nature of this important transition grade.

Wenger's (1998) CoP learning theory, based on his earlier work with Lave (Lave & Wenger, 1991) in adult learning communities, argues that the location of learning is in processes of co-participation in CoPs involving changes in members' ways of being and becoming in relation to practice. Building on Wenger (1998, 214), all of SANC's PD CoPs were designed as living contexts, giving newcomers (and all members) access to competence and inviting "a personal experience of engagement by which to incorporate that competence into an identity of participation". In these CoPs our "history of mutual engagement around a joint enterprise" provides a context for "leading-edge learning, which requires a strong bond of communal competence along with a deep respect for the particularity of experience". As Wenger notes "When these conditions are in place, communities of practice are a privileged locus for the creation of knowledge." (p. 214)

With this perspective, enabling effective professional learning requires providing access to quality resources and ongoing interactions between experienced and newer members of communities, as well as to information, resources (physical, knowledge, and other), and opportunities for full participation (Lave & Wenger, 1991). All of SANC's PD programmes established partnerships with teachers in which a deep respect for the teachers' particularity of experience was the basis for engagement with a range of research- and practice-informed resources and emerging initiatives. All programmes were long-term (minimum 18 months) with regular engagement (once or twice per month) with more than 14 partner schools (over 120 teachers and DBE partners across programmes). A staggered approach to implementation of programmes occurred over the ten-years of the first two funding terms such that, by the end of 2020, teachers of all grades in our partner schools had had the opportunity to participate in a programme with many teachers participating in more than one programme. In the current third and final term of funding we meet with teachers across all programmes on a bi-annual basis. Our PD orientation documents emphasize partnerships where *all* are co-learners, bringing different experiences and expertise to the joint enterprise of searching for productive possibilities in mathematics teaching and learning.

SANC's PD activities, including those for use in classrooms and after-school clubs, drew on Kilpatrick et al.'s (2001) definition of mathematical proficiency as encompassing five interconnected strands: conceptual understanding; procedural fluency; strategic competence; adaptive reasoning, and productive disposition. These strands would be developed through facilitating active engagement with research- and practice-informed resources in social settings, where language and various modes of communication were considered essential tools for understanding (Vygotsky & Cole, 1978). Although there were some variations in the key representations and activities emphasised in different grade level programmes, the driving idea across programmes was the development of number sense with emphasis on the following (considered high-leverage) priorities: sense making and reasoning; developing a structural understanding of number; using key manipulatives and representations; progressions from concrete; enabling home practices and second sites of learning; developing productive dispositions and growth mindsets; stimulating maths talk (language & multilingualism) as a key resource; and developing fluency and efficient strategies.

Our research CoP comprises post-doctoral, doctoral and masters' students as well as research and development collaborators in our Faculty of Education and beyond. We similarly engage regularly around the joint enterprise of researching sustainable ways forward to challenges in mathematics education, jointly attend writing retreats, and overlap with multiple research and professional CoPs through our participation at multiple conference and stakeholder forums. Researchers are actively involved in PD programmes and development projects. While there are constant newcomers and departures also from our research community, the funding allows for long-term relationships and a relatively stable 'core team' as many researchers move from master to doctoral to post-doctoral studies and in some cases to colleagues. This has allowed for a socially cohesive and dynamic research team with growing research expertise and impact.

## Emergent Projects

Through SANC's partnerships with schools and teachers, the need for supplementary learner-focused and home-based programmes emerged. The challenge of most learners being one or more grades behind curriculum level expectations required increased opportunities to learn beyond the limited time in school, often further challenged by disruptive events (e.g., protests, lack of water supply, extreme weather conditions). Furthermore, school and after-care centre communities requested opportunities for SANC's engagement with learners and families outside of school hours. A range of learner- and community-focused programmes thus emerged (Table 1). In the presentation I will elaborate on these community-focused projects. Here, however, I focus on only the two learner-focused projects that have evolved to national scalability.

**Table 1**

*Emergent Learner-focussed and Community-focused Projects*

| Learner-focused Projects | Community-focused Projects |
|---|---|
| • After School Maths Clubs<br>[National scaling of clubs with NGO & DBE district partners since 2015]<br>• Mental Starters Assessment Programme (MSAP)<br>[National scaling with DBE provinces since 2021] | • Family Maths days<br>• Family Maths Storytime programme<br>• Math/Science camp with after-care centre learners and facilitators<br>• Community resource drive: Local newspaper maths activity supplements; online and Facebook resources expanded during Covid-19 |

The After School Maths Clubs were designed to be informal supportive learning spaces in which groups of learners and facilitators/teachers would explore maths, solve problems and play maths games free from the constraints of large class teaching with the pressure of curriculum conformity. Emphasis is placed on talking maths, asking questions, being free to be messy, playing fluency and strategy games, and being an active (rather than passive) learner (Graven, 2015). Following the first pilot in 2011, multiple clubs were set up at local partner schools and after-care centres (catering for vulnerable learners). The clubs were run by researchers in the Chair team (along with teachers and facilitators), and provided us the opportunity to work directly with learners and to trial several research-informed resources and activities prior to engaging with them in PD CoPs. They also provided an empirical field for several post-graduate research projects. Some researchers ran their own clubs or supported others to run clubs. For example, three SANC researchers who were also teacher advisors in the DBE, researched the expansion of clubs as part of their district work with teachers, each working with ten teachers to set up and run clubs (see Stott et al., 2017). Following multiple presentations by our club researchers across various platforms (e.g., Baart, 2019), we began receiving requests from various mathematics education NGOs working across multiple provinces for support in setting up clubs. To date, hundreds of Maths Clubs have been set up across grades and provinces and we (our Chair and various NGOs, particularly OLICO) have formed a Maths Clubs Collective of partners collaborating to provide quality club resources and support to those willing to run clubs (see mathsclubs.co.za). The scalability of the Maths Clubs thus emerged organically from SANC's local work, expanding nationally in partnership with DBE teacher educators and NGOs.

The Mental Starters Assessment Programme (MSAP) emerged from the grounded experience of working with teachers and learners in SANC's first five-year term. It was jointly conceptualised in 2016 by our two Numeracy Chairs, with key partners, as a possible national intervention aimed at addressing the pervasive persistence of unit counting for calculating and an absence of number sense. The opportune 'classroom space' identified for implementation of targeted teaching for number sense and progression away from unit counting was the 10-minute 'mental maths warm-up' start of lessons. Research had indicated that this space was being ineffectively used in many classrooms (Venkat & Naidoo, 2012). The curriculum space identified was mental strategies. The policy space, as pointed out by our partner Dr Marc Chetty (DBE's Acting Director: National Assessment), was a call focusing on diagnostic assessments. In terms of teacher professional development, we saw MSAP as an ideal space for promoting the use of key representations that support developing a structural understanding of number. These included the empty number line and part-part-whole diagrams, representations that were not visible in many classrooms. The relatively low-stakes lesson starters and diagnostic assessments had the advantage that they could be run concurrently with other DBE initiatives making it less likely they would be derailed by possible future policy initiatives. While beyond the scope to elaborate upon here, the MSAP teacher guides and print masters (available on the DBE platform www.education.gov.za/MSAP2022.aspx) provide Grade 3 teachers with six units each focused on a calculation strategy (bridging through ten, jump strategy, doubling and halving, reordering, rounding and adjusting, and linking addition and subtraction), with a brief pre-test (with rapid recall, strategic calculating and strategic thinking items), 8 scripted (and QR code demonstrated) lesson starters, followed by a post-test. The pre- and post-testing focuses on learner gains rather than raw marks. (See, e.g., the QR code, below, for the support video for Jump Strategies Lesson Starter 2). Following multiple iterations of trialling, gradually scaling up from Chair-run local feasibility trials to DBE-run provincial trials, the project is now being rolled out nationally by the DBE. (See Graven & Venkat, 2021; Askew et al., 2022, for discussion on the evolution of MSAP and research results).

## Network of Interconnected CoPs and Projects Engaging Locally and Beyond

The long-term flexible funding model for the Chairs thus enabled an innovative network of mathematical research and development projects that provided members (i.e., learners, teachers, teacher educators, families, researchers) opportunities to engage in ongoing activities with members from different communities, providing multi-directional learning opportunities for all members. Figure 1 shows the way in which various projects (indicated in the triangles) connected members from the different communities. Opportunities for members to participate in multiple projects and communities enabled them to broaden their engagement with mathematics education and/or research activities and to develop leadership trajectories along the way (Graven, 2019). Many teachers, teacher educators, and researchers thus expanded their engagement to become centrally participating members in multiple CoPs they had not previously 'belonged' to. Teachers, for example, became researchers participating in conference CoPs, and became involved in supporting districts in teacher 'training', and providing input into national initiatives. Researchers were inducted into multiple national and international conference communities, and some became teachers/facilitators (in clubs) and teacher educators (in PD CoPs) (Graven, 2019). Figure 1 shows how flexibility of funding enabled the Chair to go far beyond its mandated PD triangle. In terms of the research mandate, our Numeracy Chairs grew South Africa's field of primary mathematics research. To date our SANC, Rhodes team of researchers have contributed over 170 peer-reviewed papers published in conference proceedings, book chapters and local and international journals. Comparing the decades 2003-2012 and 2013-2022, Morrison et al.'s (in press) review of South African research in mathematics in the early grades (Grade R,1,2, & 3) found that *journal* articles increased more than fivefold (with an increase from 1 to 20 in top international journals). Analysis of this growth points to substantive contributions from the two Numeracy Chair research teams.

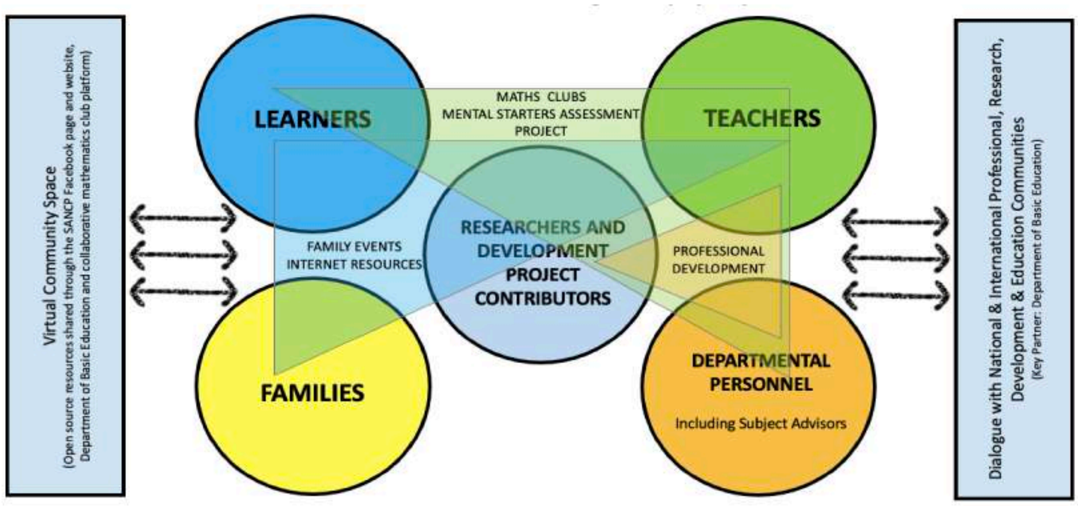



Figure 1. Communities interconnected through project spaces (adapted from Graven et al., 2022).

## Concluding Remarks: Returning to the Funding Model

It has been an enormous privilege to have the opportunity to support both research and development in primary Mathematics as the South African Numeracy Chair at Rhodes University since 2011. In this paper I have pointed to the way in which three key aspects of the Chair funding model: long-term engagement, research *with development*, and flexible 'deliverables', enabled ethical, grounded work that supported the emergence of a powerful network of communities. Multiple learning opportunities emerged from this network far exceeding the sum of those available within each individual project or community. The network provided a momentum for learning that grew with time and allowed key stakeholders in mathematics teaching and learning to work together in ways that supported dialogue and mutual collaborative learning. The longitudinal timeline allowed multiple stakeholders to shape the work and allowed the ever-improving and expanding iterations of programmes to support capacity-building across different communities thereby increasing the opportunity for successful scalability and sustainability.

## Acknowledgments

I thank all those I have had the privilege of collaborating with on this SANC journey and acknowledge the support of the NRF (Grant No. 74658).


## References


Askew, M., Graven, M., & Venkat, H. (2022). From what works to scaling up: Improving mental strategies in South African Grade 3 classes. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 27–34). PME: Alicante, Spain.

Baart, N. (2019). Plenary: Investigating the nature of grade six after school mathematics club learners' shifts in mathematical number sense and procedural fluency. In J. Naidoo, & R. Mudaly (Eds.), *Proceedings of the 25th annual national congress of the Association for Mathematics Education of South Africa* (pp. 11–24). AMESA

Bowie, L., Venkat, H., & Askew, M. (2019). Pre-service primary teachers' mathematical content knowledge: An exploratory study. *African Journal of Research in Mathematics, Science and Technology Education*, *23*(3), 286–297, DOI: 10.1080/18117295.2019.1682777

Cosser, M. (2023). *Daring solutions are needed to solve South Africa's maths teaching crisis*. Daily Maverick, Daily Disaster Education Op-Ed, 22nd. https://www.dailymaverick.co.za/article/2023-01-22-daring-solutions-are-needed-to-solve-south-africas-maths-teaching-crisis/ (20/04/2023)

Fleisch, B. (2008). *Primary education in crisis: Why South African schoolchildren underachieve in reading and mathematics*. Juta and Company Ltd.

Graven, M. (2012). Changing the story: Teacher education through re-authoring their narratives. In C. Day (Ed.), *The Routledge international handbook of teacher and school development* (pp. 127–138). Routledge.

Graven, M. (2015). Strengthening maths learning dispositions through 'maths clubs'. *South African Journal of Childhood Education*, *5*(3), 1–7.

Graven, M. (2019). Building multidirectional learning opportunities between researcher, teacher, and teacher educator communities. In *International handbook of mathematics teacher education* (Vol 3, pp. 241–264). Brill.

Graven, M., & Venkat, H. (2021). Piloting national diagnostic assessment for strategic calculation. *Mathematics Education Research Journal*, *33*(1), 23–42.

Graven, M., Venkat, H., Westaway, L., & Tshesane, H. (2013). Place value without number sense: Exploring the need for mental mathematical skills assessment within the Annual National Assessments. *South African Journal of Childhood Education*, *3*(2), 131–143.

Graven, M., Vale, P., Hokonya, W., & Long, R. (2022). A decade of the South African Numeracy Chair Project: Out-of-school learner-focused interventions. In H. Venkat, & N. Roberts (Eds.), *Early grade mathematics in South Africa* (pp. 134–151). Oxford University Press.

Jaworski, B. (2005). Learning communities in mathematics: Creating an inquiry community between teachers and didacticians. *Research in Mathematics Education*, *7*(1), 101–119.

Jita, L. C., & Mokhele, L. M. (2012). Institutionalising teacher clusters in south Africa: dilemmas and contradictions. *Perspectives in Education*, *30*(2), 1–11.

Jorgensen, R., & Graven, M. (2022). Playing the "research game in marginalised fields". In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 322–329). MERGA.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* National Academy Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Morrison, S., Graven, M., Venkat, H., & Vale, P. (in press). A 20-year review of South African early grade mathematics research articles. To appear in *African Journal of Research in Mathematics, Science and Technology Education.*

Reddy, V., Winnaar, L., Harvey, J., Hannan, S., Isdale, K., Arends, F., & Juan, A. (2022). *The South African TIMSS 2019 grade 5 results: Building achievement and bridging achievement gaps*. HSRC Press.

Robertson, S. A., & Graven, M. (2019). Exploratory mathematics talk in a second language: A sociolinguistic perspective. *Educational Studies in Mathematics, 101*(2), 215–232.

Robertson, S. A., & Graven, M. (2020). Language as an including or excluding factor in mathematics teaching and learning. *Mathematics Education Research Journal, 32*(1), 77–101.

Schollar, E. (2008). *Final report: The primary mathematics research project 2004–2007—Towards evidence-based educational development in South Africa.* Eric Schollar & Associates.

Soudien, C., & Harvey, J. (2021). The crisis of COVID-19 and opportunities for reimagining education. *Journal of Education,* (84), 169–185.

Setati, M. (2005). Researching teaching and learning in school from "with" or "on" teachers to "with" and "on" teachers. *Perspectives in Education*, *23*(March), 91–102.

South Africa. Department of Basic Education (2014). *Report on the annual national assessments: Grades 1 to 6 & 9*. DBE.

Spaull, N., & Kotze, J. (2015). Starting behind and staying behind in South Africa. *International Journal of Educational Development, 41*, 13–24.

Stott, D., Graven, M., Baart, N., Hebe, G., & Mofu, Z. (2017). After school maths clubs: Investigating learner progression in an expanding intervention model. In *Proceedings of 23rd annual national congress of AMESA* (pp. 313–325). AMESA.

Venkat, H., & Graven, M. (2022). Bringing the mental starters assessment project to scale in foundation phase: A 'building your timber' approach. In Spaull, N., & Taylor, S. (Eds.), *Early grade interventions in South Africa* (pp. 229–244). Oxford University Press.

Venkat, H., & Naidoo, D. (2012). Analyzing coherence for conceptual learning in a grade 2 numeracy lesson. *Education as Change*, *16*(1), 21–33.

Vygotsky, L. S., & Cole, M. (1978). *Mind in society: Development of higher psychological processes.* Harvard University Press.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge University Press.

# Beth Southwell Practical Implications Award

The Beth Southwell Practical Implications Award was initiated and sponsored by the National Key Centre for Teaching and Research in School Science and Mathematics, Curtin University, Perth, Western Australia. Curtin sponsored the "Practical Implications Award" (PIA), as it was then known, for the first 10 years. The Australian Association of Mathematics Teachers (AAMT) now sponsors the Award. In 2008, MERGA was honoured to be able to rename the PIA as the Beth Southwell Practical Implications Award, in honour of MERGA's and AAMT's esteemed late member, Beth Southwell. The award is designed to stimulate the writing of papers on research related to mathematics teaching or learning or mathematics curricula. Application for the award is open to all members of MERGA who are registered for the conference. Applications for the PIA are judged against specific criteria set by a four-member panel. The panel consists of two members from MERGA, two from AAMT, and is chaired by the MERGA Vice President (Development).

# Throw Away the Script: Examining the Introduction of a Guided Mathematical Inquiry Unit

Jill Fielding
*University of New England*
jill.fielding@une.edu.au

Saidat Adeniji
*University of New England*
sadenij2@une.edu.au

Penelope Baker
*University of New England*
pep.baker@une.edu.au

Explicit teaching, an instructional method promoted by Australian Education Departments over student-centred methods, entails teacher-led instruction and sequenced breakdown of student tasks. Inquiry-Based Learning is often considered an opposing, student-centred approach, which explicit teaching proponents argue lacks guidance and support. In this paper, we examine one experienced teacher's introduction of a Guided Mathematical Inquiry (GMI) unit to provide insight into key teacher practices and illustrate the approach in practice. The findings suggest that GMI establishes, guides and supports contextual and discipline knowledge in mathematics from the start of the lesson. Teacher considerations for implementation of GMI are provided.

Engelmann (1967) believed that students' learning is determined by the teacher and is independent of student characteristics, leading to the introduction of a teacher-centred approach to instruction commonly referred to as explicit instruction. Explicit instruction entails a teacher-controlled, highly scaffolded and sequenced breakdown of tasks for learners with ample opportunities for student repetition and practice (Archer & Hughes, 2011). By contrast, constructivist and socio-cultural theorists promote student-centred instruction, such as Inquiry-Based Learning (IBL), to maximise learning outcomes. Student-centred approaches emphasise the roles of students in exploring, modelling, guiding, collaborating, questioning, and carrying out activities that enhance the development and ownership of mathematical ideas. There are research-identified limitations and benefits to both teacher-centred and student-centred instructional approaches. As such, teaching through only one method is likely to maximise the limitations inherent with that method and minimise the opportunities associated with the other.

While explicit instruction is promoted in Australian schools, IBL is less so (e.g. NSW Centre for Education Statistics and Evaluation, 2020). Part of the reason for this is the challenge imposed on teachers through lack of structure and difficulties in getting started (Makar & Fielding-Wells, 2011). To counter this, we examined ways in which an experienced teacher of Guided Mathematical Inquiry (GMI) introduced an inquiry unit to develop insights into, and provide an illustration of, key components to be considered. The research question addressed was:

- What insight into GMI can be gleaned from analysing the introduction phase of a GMI unit.

## Literature

### *Explicit Instruction*

Explicit instruction refers to a teaching approach where the teacher provides clear statements of objectives, instruction, and practice to achieve the intended outcomes of a lesson. When introducing a mathematical concept, the teacher gains students' attention, presents the objectives and relevance of the lesson, and then proceeds to teach the concept by breaking tasks down into small steps and providing significant teacher guidance (Archer & Hughes, 2011). A rationale for using this approach is to effectively manage and maximise students' cognitive resources (Sweller, 2012). However, since its' inception, explicit instruction has received significant criticism. In the Australian

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 27–35). Newcastle: MERGA.

mathematics context, explicit instruction has been associated with non-participation and disengagement; passive approaches to learning; and reduced opportunities for students to "access or use the language of mathematics to express their mathematical ideas" (Ewing, 2011, p77). Classroom interactions are largely a one-way process and memorisation, rehearsal and rote learning are the outcome. An assumption that underpins this approach is that mathematical knowledge can be handed from teacher to student (Ewing, 2011), which has implications for student transfer of knowledge to new applications.

### *Guided Mathematical Inquiry (GMI)*

GMI is an approach to IBL which maintains a student-centred focus whilst incorporating scaffolding of the learning process by the teacher—hence the term *Guided.* While the extent of guidance can vary, here it is taken to mean that the teacher provides students with the support necessary to enable them to engage with an inquiry problem while scaffolding the process and supporting student direction (Fielding & Makar, 2022).

IBL approaches to learning involve students addressing a complex problem or question, identifying or developing a method or strategy to solve it, and then evaluating the outcome and approach (e.g., Bruder & Prescott, 2013). During IBL, "students are cognitively engaged in sense making, developing evidence-based explanations, and communicating their ideas" (Hmelo-Silver et al., 2007, p. 100). Through these activities, students are encouraged to engage with mathematical concepts in depth, promoting a deep understanding of the subject matter (Carpenter et al., 2012), increased transferability of learning (Duffy & Raymer, 2010), improved motivation and perceived relevance of mathematics (Bruder & Prescott, 2013), and a critical stance (Goos, 2004). However, IBL is not without difficulties, including: extended time demands limiting curriculum content coverage; inadvertent privileging of students who have prior knowledge (Kazemi & Franke, 2004); student struggle and frustration (Sengupta-Irving & Enyedy, 2015); and, the need for teacher preparation and skill at implementation, which can be challenging for some teachers to acquire (Makar, 2007).

**Table 1**

*Model of Knowledge Domains in Inquiry-Based Learning (Fielding-Wells, 2016)*

| Knowledge Domain | Knowledge Sub domain | Code | Description |
|---|---|---|---|
| Context Knowledge: | Prior Experience | CK:P | Past involvement with the context at school or otherwise. |
| | Understanding | CK:U | Knowledge of the context from experience, but may also derive from teaching, reading or discussion. |
| | Discourse | CK:D | Underlying language and terminology required to support understanding and discussion. |
| | Affect | CK:A | Feelings or intuitively held beliefs about a context. |
| Mathematical Knowledge: | Knowledge | MK:K | Conceptual and procedural knowledge associated with the discipline and curriculum. |
| | Discourse | MK:D | Using the language and terminology of the discipline to express and comprehend ideas. |
| | Community Practices | MK:C | Mathematical practices that increasingly approximate the authentic practices of mathematicians. |
| | Affect | MK:A | Disposition and willingness to engage in mathematics and use the knowledge of the discipline. |

To engage with IBL, students are required to draw on a number of knowledge domains, including, but not limited to, Context Knowledge and Mathematical Knowledge. These domains have been conceptualised, along with a number of sub-domains, through classroom research of IBL in practice (see Table 1, Fielding-Wells, 2016). It is important to note that context in this model usually does not have an immediately visible mathematical underpinning, for instance, a sport, a children's book or game, or a common practice such as a meal.

Researchers have identified a number of issues when supporting teachers to teach using IBL, particularly if they are accustomed to more teacher-centred approaches (e.g., Makar, 2007). Whereas teachers might be accustomed to planning a lesson in detail, teaching through IBL requires some uncertainty about the direction students may take and potentially the content that may be covered, necessitating deeper consideration of the content by the teacher (Makar, 2007). Teachers must also know how to support a classroom culture conducive to inquiry, including students' familiarity with ambiguity, propensity to risk-take, collaborative skills, and comfort with not 'being told' (Goos, 2004; Makar et al., 2013).

## Method

The findings reported in this paper are taken from a larger study which involved repeated interventions into the development of argument-based mathematical inquiry with children. The larger study adopted a Design-Based Research (DBR) (Cobb et al., 2003) approach as DBR is typified by practitioner and researcher collaboration to plan successive learning episodes, reflecting on progress and making progressive adjustments in planning with the intent of improving learning approaches.

The participants reported in this paper included a Year 3 class (~8 years of age) from a high-average socio-economic suburban public school in Australia. The teacher was experienced in IBL in mathematics, having used this method of teaching mathematics for approximately 3 years. The teacher and researcher identified a goal of addressing observed student equiprobability bias by engaging students with a problem context in which outcomes would have unequal likelihood of being obtained. They worked together to develop a game of addition bingo with the inquiry question: *What is the best card for a game Addition Bingo?* In this version of Addition Bingo, each possible combination of the sum of two numbers (1 to 10) is placed in a box. Players have a card with a 5 x 5 array of self-selected numbers. As each paper is drawn (e.g., 9+6) from the box, players mark off the sum (in this case, 15) if it appears on their card. The winner is the first to mark off all of their numbers.

Data included student work samples, transcribed videotapes of each lesson, and field notes including pre-and post-lesson discussions with the teacher. The focus was an analysis of the approach the teacher took to introducing the inquiry prior to presenting the inquiry question. Accordingly, the introductory lesson was viewed in full by all researchers, transcribed in full by the first researcher, and linked to relevant representations. Thematic analysis was carried out following a process described by Braun and Clarke (2022) with initial codes derived from the Domains of Knowledge Framework (refer Table. 1, Fielding-Wells, 2016).

Two of the authors independently analysed the transcript by applying the initial codes and noting any lesson aspects that could not adequately be coded to these initial codes. They then compared their coding and discussed any discrepancies until coding was agreed, referring back to research field notes as warranted. The coding focus was on teacher 'moves' and this included both the teachers' initiation of activity or discussion as well as her responsiveness to opportunities presented by students' discourse/activity.

## Findings

The introduction lasted for 100 minutes and was bounded by the commencement of the first lesson and the point at which the teacher introduced the inquiry question: *What is the best card for a game of Addition Bingo?*. Context Knowledge sub codes were most prevalent in the initial stages of the lesson before a shift to a high proportion of Mathematical Knowledge sub codes. Below, we briefly highlight key aspects of the lesson with excerpts to illustrate. Numbers in parentheses refer to line numbers in the excerpts. The excerpts are numbered sequentially for ease of discussing, however, not all interactions are reported. Ellipses (…) indicate omissions.

The teacher commenced the inquiry by ascertaining student familiarity with Bingo. Even though all students stated prior experience, she proceeded to question the students' familiarity with the game and asked it to be explained [CK-P]. She then had students play two games to ensure that they all have common contextual experience [CK-P], and as a means of developing common contextual language [CK-D].

In the excerpt below [1-3], the teacher is introducing the game to be played [CK-U]. After which the students design their own Bingo cards, as distinct from using an existing game cards. This was deliberate [field notes] to provide experiences which would focus students' attention on the need to consider the parameters of the set of possible outcomes to be called.

| | | |
|---|---|---|
| 1 | Teacher: | … before I start I'm going to tell you something. These are all the numbers from 1 to 50. … and these are going to be my numbers for calling out. What I want you to do is choose any numbers you like between 1 and 50 and write them in the first square—in the first bingo card. …So how many numbers will you have in there? |
| 2 | Students: | 25 |
| 3 | Teacher: | … So, I want you to choose 25 numbers between 1 and 50…and fill up that box. |

The teacher moved between the students as they created their cards, observing and answering direct questions, noting errors but not correcting or providing feedback [MK-C]. A sample of the interactions below [4-5] illustrates. These reflect the teachers' desire for students to learn experientially rather than being told, and was observed frequently through the lesson.

| | | |
|---|---|---|
| 4 | Student: | Can you have the same number twice? |
| 5 | Teacher: | If you would like, you can. You decide whether you think that would be a good idea to have the same number twice. |

The teacher proceeded to call the Bingo numbers, using language often associated with Bingo for authenticity (e.g., "legs eleven"). She observes students' emerging realisations of their errors but does not discuss these yet. When the first student to call Bingo does so erroneously (duplicated a number on their card), she mentions this but defers it, "Oh you had number three two times, I will get back to that Troy" [MK-C]. As the games progress, students began realising their errors and calling out (e.g. "I had 50 twice"; "oh …there's two eights"). The teacher attempts to continue; however, the students continue to comment about duplicates, so she addresses the issue [6-7], using the opportunity to draw students' attention to possible outcomes while also prompting the language of chance [MK-K, MK-D].

| | | |
|---|---|---|
| 6 | Teacher: | If I had all the numbers from 1 to 50 and then all the numbers from 1 to 50 again, so if I had two of each, how many threes could you have written down then? … would it be possible for you to get three twice? |
| 7 | Students | Yeah |

The teacher continued calling numbers until a student legitimately won. She then used students' newly developed collective experience and understanding [CK-P; CK-U] [8] to begin to focus on the mathematical content of the lesson, drawing on students' prior knowledge while focussing attention on the need to consider possible outcomes [MK-K]. She elicits students conceptions about chance [MK-K][8-16] The teachers' intention for the students to learn experientially [MK-C] is further apparent in this exchange.

| | | |
|---|---|---|
| 8 | Teacher: | Ok Duncan has got Bingo. If we play this game again, using exactly the same numbers, will Duncan win again? |
| 9 | Students: | [calling out simultaneously] No; possibly; 50:50; even; yes. |
| 10 | Teacher: | 50:50 [acknowledging this response]? |
| 11 | Justine: | The same numbers called can be called again or they might not. |
| 12 | Teacher | So, 50:50 means and even chance? Is there an even chance that I could call out those numbers again? Like 50:50 |
| 13 | Students: | Yes |
| 14 | Justine: | Like you've got 20 numbers…25 numbers, so that's half of 50… but you've still got 25 numbers that haven't been called out so they might be the ones that are being pulled out this time. |
| 15 | Teacher: | That is a really interesting way of explaining 50:50. You think that because we've got 25 numbers that we've written down and there are 50 tickets to pull out …then 50:50. [no judging of response by T—just accepting] |
| 16 | Student: | Not 50-50, 25-25 |

Following on from this interaction, the teacher began to address the issues that she had previously observed by drawing on students' experiences to support them to challenge their practices/ideas [MK-C]. The first issue addressed was a reiteration of the duplication of numbers beyond what was supported by the possible outcomes [MK-K]. The second was the use of numbers that did not occur within the range of possible outcomes [MK-K][17-22]. In both instances the teacher was guiding students to developing awareness of the importance of considering the outcomes possible.

| | | |
|---|---|---|
| 17 | Teacher: | Clay, what happened with you? What did you have on your card that you said, I can't win? |
| 18 | Clay: | Zero |
| 19 | Teacher: | You had a zero? |
| 20 | Clay: | I had a zero and I had numbers twice |
| 21 | Teacher: | So, you had two things on yours which stopped you from being able to win?... So, you had a zero. Why is that a problem? |
| 22 | Clay: | Only 1 to 50 |

The teacher then proceeded to elicit potential (mis)conceptions related to the topic that had not yet arisen through game play, including the notion of 'luck' in respect to numbers being drawn [MK-K] [23-26]. Once again, these were elicited but not corrected [MK-C].

| | | |
|---|---|---|
| 23 | Teacher: | Is there a number that you think is a lucky number that comes up all the time? We don't know yet. I suppose we've only had one game. |
| 24 | Jess: | I think 11 |
| 25 | Teacher: | Why do you like 11? |
| 26 | Jess: | Well I like 11 because one time like I played a game with one of my cousins and one time they called out 11 and I had 11 on my bingo card |

The teacher had students design a second Bingo card, with the same requirements, reminding them to focus on what they had learned in the first round. During this second round, she took multiple opportunities to check students' understanding of the likelihood of occurrences and their use of the language of probability (e.g., likely, unlikely, highly likely, highly unlikely, uncertain, certain, possible, impossible), through having students identify and justify their number selection [MK-K; MKC; MK-D]. Once satisfied with students' use of terminology, she encouraged them to think about probability in numerical terms [27-36].

| | | |
|---|---|---|
| 27 | Teacher: | What is the chance. What is the likelihood of my next number being a one-digit number? |
| 28 | Paul: | Possible or unlikely |
| 29 | Teacher: | Possible or unlikely. Why do you say that? |
| 30 | Paul: | Because 3, 4, no 5 of the one-digits have been called out |
| 31 | Teacher: | So how many are left to be called out? |
| 32 | Paul: | 4 |
| 33 | Teacher: | So, there are 4 left to be called out. So, what's the likelihood of me pulling out a single-digit number. |
| 34 | Leah: | Likely |
| 35 | Teacher: | We have pulled out 26 numbers all together. There are only 4 single digit numbers left, so what is the likelihood of me pulling out a single digit number? …Can you base it on any maths? |
| 36 | Student: | Fifty-fifty |

In the above exchange, the teacher tested the students comfort with representing or thinking about probability as an informal numerical representation (i.e. four out of 26) [31-36][MK-K]; however, the students were not ready to move into this practice and the lesson ended.

In the following lesson, the teacher repeated the Bingo game with a 3 x 3 Bingo card and a number set from 1-20 inclusive. After giving the instruction to the children regarding the new size and number set, she instructed them to create a new Bingo card. The new arrangement allowed her to see if the students were transferring their understanding of outcomes to an altered context [MK-

K]. After the cards were made, the teacher again drew whole class attention to taking a numerical approach to chance [37-40].

| | | |
|---|---|---|
| 37 | Teacher: | So, if we pick one number out, what is the chance of it being your number? |
| 38 | Paul: | 1 out of 20. No. It would be 9 out of 20. |
| 39 | Teacher: | OK. So, the first time I pull out a number, you have a one in 20 chance of it being your number? |
| 40 | Paul: | No. 9 out of 20 still because you've got 9 numbers still. |

Despite this discussion involving the whole class, the remainder of the class continued with the language of probability rather than numerical representations. About 20 minutes into the lesson, Paul again mentioned a probability in numeric terms [44]. This time, other students began to use fractional language in their comments [MK-K][MK-D] [45] and the teacher capitalised on the opportunity. However, she also validated the use of non-numerical terms [47] to support students who were not yet ready to make the shift.

| | | |
|---|---|---|
| 41 | Teacher: | There are three numbers left, what is Bruce's chance of getting his number. Do you know another way of saying it without using the language of likelihood? |
| 42 | Student: | Probably |
| 43 | Teacher: | That is the language of likelihood. What is another way we could say it? |
| 44 | Paul: | 9 out of 10 |
| 45 | Bethany: | I think about 7 out of 10 |
| 46 | Teacher: | [Many hands up] Write down what you think, you are all itching to tell me. |
| 47 | Teacher | If you've got something written down, that's good. If you've got nothing written down you can use the language of likelihood to tell me. |

The teacher continued on to use the students' responses to demonstrate a fraction, commencing with the 1/3 identified by many students, naming its parts and reminding the students of the names and meanings of numerator, denominator and vinculum. She then moved on to introduce the inquiry question to the students.

## Discussion & Conclusion

In this paper, we examined how an experienced GMI teacher introduced her students to an inquiry problem. We did so in order to illustrate how a teacher might introduce a GMI question. Proponents of explicit instruction argue the need to provide a clear focus on what is to be learnt, through clear identification of the mathematics to be focussed upon and the necessary procedures, which are practiced (Sweller, 2012). Conversely, in GMI, the teacher provides and supports involvement with tasks and discussions so that the student is able to notice and identify what is important in the task experientially and through real-life or life-like contexts, thereby demonstrating the relevance and application of mathematics.

We do not dispute that explicit instruction could have reached the objectives of learning more quickly. Students could have been told the importance of considering the elements of a set, and instructed not to use numbers outside of the set or to duplicate numbers. By distinction, this teacher took 100 minutes, over almost two lessons to establish the contextual and mathematical background she felt necessary to ensure students had sufficient understanding to discover the importance of these aspects for themselves. Later in the lesson, this experience facilitated the students' decisions to construct a variety of representations to determine frequency for Addition Bingo—including frequency charts—which they had not previously had experience with but rather 'invented' for themselves.

The notion that students may have discomfort with not being 'told' the processes or procedures necessary to address the problem remains a consideration. However, exploring and even negotiating the boundaries of problems, discovering errors and limitations, and testing these within a context are the ways of knowing the discipline of mathematics and being 'mathematicians' (Dewey, 1938). In the excerpts above, the teacher facilitates students' involvement with a mathematical inquiry question by establishing and enhancing students' knowledge of a context in which the mathematics (or statistics) becomes relevant. Rather than privileging students with prior knowledge of the context (Kazemi & Franke, 2004), she seeks to establish common prior knowledge through engagement with the context before introduction of the problem. This includes understanding and contextual discourse.

The teacher also took time to establish prior mathematical knowledge, using the context to achieve two purposes: 1) supporting the students to discover and explore important concepts for themselves (a key shortcoming of explicit instruction (Kuhn et al., 2000), including the importance of considering the boundaries of a set and the frequency of the numbers in that set; and, 2) exploring students' mathematical knowledge and advancing potential understanding (e.g., readiness to shift from probabilistic language to fractional representations). In this way, she was able to determine the level of support and challenge that might be required to balance engagement and frustration (Sengupta-Irving & Enyedy, 2015). The teacher can also be seen to change the context slightly, from a 5x5 Bingo card with 50 calling numbers, to a 3x3 Bingo card with 25 calling numbers. By doing so, the teacher is testing for transfer—have these concepts been understood—or have the students merely become aware of their relevance as related to a single contextual instance? Again, a shortcoming of explicit instruction lies in the potential difficulty with transfer of learning (Ewing, 2011). Yet here we see transfer intentionally built into lesson design from the earliest stages.

The proposal is not that all learning must take place through GMI and we recognise that there are limitations to both IBL/GMI and explicit instruction. However, in a time when explicit instruction is supported as a sole approach, the potential for GMI to counter limitations associated with this explicit instruction must be considered. As such, we sought to identify some key practical considerations for teachers wishing to establish GMI activities with their classes: the need to support students to have common context experiences, understandings and discourse; and, the need to explore and build the boundaries of mathematical knowledge, community practices and discourse, prior to engagement with the Inquiry Question.

## References


Archer, A., & Hughes, C. A. (2011). *Explicit instruction: Effective and efficient teaching*. Guilford Press.

Braun, V., & Clarke, V. (2022). *Thematic analysis: a practical guide*. Sage.

Bruder, R., & Prescott, A. (2013). Research evidence on the benefits of IBL. *ZDM Mathematics Education, 45,* 811–822. https://doi.org/10.1007/s11858-013-0542-2

Carpenter, T., Fennema, E., Franke, M., Levi, L., & Empson, S. (2012). *Children's mathematics: cognitively guided instruction*. Heinemann.

Centre for Education Statistics and Evaluation (2020). *What works best: 2020 update.* CESE.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13. https://www.jstor.org/stable/3699928

Dewey, J. (1938). *Logic: The theory of inquiry.* Holt.

Engelmann, S. (1967). Relationship between psychological theories and the act of teaching. *Journal of School Psychology, 5,* 93–100. https://doi.org/10.1016/0022-4405(67)90022-2

Ewing, B. (2011). Direct instruction in mathematics: Issues for schools with high indigenous enrolments: A literature review. *Australian Journal of Teacher Education, 36*(5). http://dx.doi.org/10.14221/ajte.2011v36n5.5

Fielding, J., & Makar, K. (2022). Challenging conceptual knowledge during mathematical inquiry: Provoking and maintaining collective disequilibrium. *Instructional Science, 50*, 35–61. https://doi.org/10.1007/s11251-021-09564-3

Fielding-Wells, J. (2016). "Mathematics is just 1 + 1 = 2, what is there to argue about?". In White, B., Chinnappan, M., & Trenholm, S. (Eds.), *Opening up mathematics education research. Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia* (pp. 214–221). Adelaide: MERGA.

Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education, 35*(4), 258–291. https://doi.org/10.2307/30034810

Hmelo-Silver, C. E., Duncan, R. G., & Chinn, C. A. (2007). Scaffolding and achievement in problem-based and inquiry learning: A response to Kirschner, Sweller, and Clark (2006). *Educational Psychologist, 42*(2), 99–107. https://doi.org/10.1080/00461520701263368

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*(3), 203–235. https://doi.org/10.1023/B:JMTE.0000033084.26326.19

Kuhn, D., Black, J., Keselman, A., & Kaplan, D. (2000). The development of cognitive skills to support inquiry learning. *Cognition and Instruction, 18*(4), 495–523. https://doi.org/10.1207/S1532690XCI1804_3

Makar, K. (2007). Connection levers: Supports for building teachers' confidence and commitment to teach mathematics and statistics through inquiry. *Mathematics Teachers Education and Development, 8,* 48–73.

Makar, K., Bakker, A., & Ben-Zvi, D. (2013). Scaffolding norms of argumentation-based inquiry in a primary mathematics classroom. *ZDM Mathematics Education, 47*(7), 1107–1120. https://doi.org/10.1007/s11858-015-0732-1

Makar, K., & Fielding-Wells, J. (2011). Teaching teachers to teach statistical investigations. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Teaching statistics in school mathematics-challenges for teaching and teacher education* (Vol. 14, pp. 347–358). Springer.

Sengupta-Irving, T., & Enyedy, N. (2015). Why engaging in mathematical practices may explain stronger outcomes in affect and engagement: Comparing student-driven with highly guided inquiry. *Journal of the Learning Sciences*, *24*(4), 550–592. https://doi.org/10.1080/10508406.2014.928214

Sweller, J. (2012). Human cognitive architecture: Why some instructional procedures work and others do not. In K. R. Harris, S. Graham, T. Urdan, C. B. McCormick, G. M. Sinatra, & J. Sweller (Eds.), *APA handbooks in psychology* (Vol. 1, pp. 295–325). American Psychological Association.

# Research Symposia

# Symposium: Big Ideas in School Mathematics

Chair: Yew Hoong Leong
*Nanyang Technological University*
yewhoong.leong@nie.edu.sg

The "Big Ideas in School Mathematics" (BISM) is a Research Project funded by the Ministry of Education, Singapore, and administered through the Office of Educational Research, National Institute of Education, Nanyang Technological University. The project began in 2020 and its aim is to investigate various areas in relation to teaching towards mathematical Big Ideas in Singapore schools. The study has currency in so far as "Big Ideas" were introduced in the latest Syllabus Revision by the Ministry of Education. There are three sub-studies in the project: the first is on the development of instruments to measure knowledge of BISM among primary- and secondary-level students and teachers; the second is on professional development work for secondary-level teachers on BISM; the third is similar to the second but for primary-level teachers. The papers in this symposium report information and findings on all these sub-studies.

## Overview of the Symposium Papers and Presenters

Presenters: Associate Professor Leong Yew Hoong (Chair), Associate Professor Toh Tin Lam (Paper 1), Mr Mohamed Jahabar Jahangeer (Paper 2), Assistant Professor Choy Ban Heng (Paper 3), Professor Berinderjeet Kaur (Paper 4)

Paper 1: Overview of the research project on Big Ideas in School Mathematics

Authors: Toh Tin Lam, Tay Eng Guan, Berinderjeet Kaur, Leong Yew Hoong, Tong Cherng Luen

This paper provides a brief overview of the entire research project and the component sub-studies.

Paper 2: Assessment of Big Ideas in School Mathematics: Exploring an Aggregated Approach

Authors: Mohamed Jahabar Jahangeer, Toh Tin Lam, Tay Eng Guan, Tong Cherng Luen

This paper reports on developments under Sub-study 1. An item from the student BISM instrument will be discussed. It argues for the use of an "aggregated approach" in considering the scores of the student responses.

Paper 3: From Inert Knowledge to Usable Knowledge: Noticing Affordances in Tasks Used for Teaching Towards Big Ideas About Proportionality

Authors: Choy Ban Heng, Yeo Boon Wooi Joseph, Leong Yew Hoong

This paper reports on developments under Sub-study 2. Part of the professional development under this project involved teachers designing their own instructional materials to foreground a targeted Big Idea. Snippets of tasks in these instructional materials will be discussed.

Paper 4: Primary School Teachers Solving Mathematical Tasks Involving the Big Idea of Equivalence

Authors: Berinderjeet Kaur, Tong Cherng Luen, Mohamed Jahabar Jahangeer

This paper reports on developments under Sub-study 3. An item from the teacher BISM instrument will be discussed. Some data on teachers' responses to the item will be shared. There are thus implications to teacher professional development on the Big Idea of Equivalence.

2023. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Symposium in the proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 39–55). Newcastle: MERGA.

# *Overview of the Research Project on Big Ideas in School Mathematics*

Tin Lam Toh
*Nanyang Technological University*
tinlam.toh@nie.edu.sg

Eng Guan Tay
*Nanyang Technological University*
engguan.tay@nie.edu.sg

Berinderjeet Kaur
*Nanyang Technological University*
berinderjeet.kaur@nie.edu.sg

Yew Hoong Leong
*Nanyang Technological University*
yewhoong.leong@nie.edu.sg

Cherng Luen Tong
*Nanyang Technological University*
chernglue n.tong@nie.edu.sg

Big Ideas in school mathematics can be seen as overarching concepts that occur in various mathematical topics in a syllabus. Although there has been much interest recently in the understanding of Big Ideas, there is little research done in the assessment of Big Ideas thinking. In this paper, we discuss our research on Big Ideas in School Mathematics. The study consists of three sub-studies: the first sub-study on developing an instrument to measure Big Ideas; two sub-studies on measuring students' and teachers' Big Ideas at test-points before and after a professional development on Big Ideas for primary and secondary school teachers and students.

In the recent mathematics curriculum revision conducted by the Singapore Ministry of Education (MOE), there is a new emphasis on the disciplinarity of mathematics and Big Ideas that are central to the discipline so as to bring coherence and connections between different topics. The objective of this new emphasis is to develop in students a deeper and more robust understanding of mathematics and better appreciation of mathematics (MOE, 2018; MOE, 2019). Each Big Idea connects various concepts and understanding across topics, strands and levels.

The definition of a Big Idea was proposed by Charles (2005) as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p.10). Prior to Charles' definition, the notion of Big Ideas in mathematics education became prominent when it was highlighted by the National Council of Teachers in Mathematics (NCTM) in 2000 that "[t]eachers need to understand the Big Ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise. Their decisions and their actions in the classroom—all of which affect how well their students learn mathematics—should be based on this knowledge." (p. 17).

From our collective classroom experience, the presentation in school mathematics syllabuses as discrete strands and topics could have led teachers and students to view mathematics as a collection of topics with weak connections. Thus, Big Ideas illuminate the interconnectedness between topics across strands and this aids the robustness of understanding mathematics. The depth of understanding is dependent on the number and strength of the connections (Hiebert & Carpenter, 1992, p. 67).

## Challenges in Teaching for and Measuring Big Ideas

Researchers have affirmed the existence of real challenges in the mathematics classroom for teaching Big Ideas in schools from both teachers' and students' perspectives (e.g., Hsu, Kysh, Ramage & Resek, 2007; Askew, 2013; Schoenfeld, 2019). Teachers in schools may not possess the relevant content knowledge pertaining to Big Ideas in mathematics. Lack of such knowledge is manifested in their teaching, for example, in their inability to realize that the generation of the exponent rules is traceable to the definition for positive integral exponents and that the distributive property is a Big Idea understanding for combining like terms and multiplying binomials (Hsu, Ramage & Resek, 2007).

Their deficiency of such knowledge often translates into their lack of explicit attention to Big Ideas underpinning mathematics taught in schools. Consequently, this results in students' acquisition of compartmentalized mathematical content knowledge (Askew, 2013). Lack of appropriate professional development for teachers associated with Big Ideas in mathematics, coupled with lack of time for professional development add to the challenges of teaching for Big Ideas (Hsu, Ramage & Resek, 2007; Askew, 2013).

To date, there has been little research on the assessment of Big Ideas. This could be attributed to three major reasons: firstly, researchers have different classifications of Big Ideas (e.g., Charles, 2005; Niemi et al., 2006; Singapore Ministry of Education, 2018, 2019). Secondly, the lack of clarity on the intent of the assessment. Furthermore, any additional instrument to measure Big Idea would mean an additional load to the already heavy high-stake national examinations. Thirdly, it is difficult to create items that assess thinking which link numerous mathematical understandings that cut across topics.

## Conceptualization of the Research Project Big Ideas in School Mathematics

In addressing the challenges of teaching and measuring Big Ideas, a team of researchers (the authors of the papers in this symposium) conceptualized a research project Big Ideas in School Mathematics (BISM). Broadly, the aim of BISM is twofold: firstly, to develop assessment items to measure of Big Ideas in school mathematics for assessing how teachers and students connect numerous mathematical understandings into a coherent whole over multiple points of their respective developments. To date, there is a dearth of such an instrument. The second aim is to study the development of Primary and Secondary mathematics teachers' and students' knowledge of BISM across a period of time during which teachers participate in professional development about BISM. The research project consisted of three sub-studies: (1) Measures of Big Ideas in School Mathematics (BISM Measures); (2) Big Ideas in Secondary School Mathematics; and (3) Big Ideas in Primary School Mathematics.

Sub-study 1: Measures of Big Ideas in School Mathematics. This sub-study involved the development of instruments for use in sub-studies (2) and (3). The aim of this sub-study was to develop, pilot and validate instruments to measure the knowledge of Big Ideas in School Mathematics (BISM) for primary / secondary school teachers and students.

Initially, we studied the few existing instrument for the measure of Big Ideas by Niemi et al. (2006). Their items consist of three main types of tasks to measure Big Ideas in mathematics: basic computation tasks, partially-worked problems (with or without explanations), and explanation tasks. Basic computation tasks aim to assess whether students could recognize tasks representing specific Big Ideas. They could then apply the relevant Big Ideas and successfully complete the task. The designed tasks are simple and well-defined. Partially worked problems require students to fill in one to three boxes for missing numbers or symbols in the problem solution, or fill in a complete problem solving step. For an explanation task, a fully worked example is given before those partially worked examples. The selected worked example usually involves no more than 3 to 4 steps, and the fully worked examples are from similar mathematics topics but not the same topic used for assessment. The explanation tasks are based on partially worked examples with justifications. Students, in this case, need to understand the steps solved by others, and must be able to provide the principles for one of the steps. Just like the partially worked example tasks, the explanation tasks follow a fully worked example which covers a similar topic but not the same topic for real assessment.

Our approach to the assessment of Big Ideas draws on the PISA experience of assessing mathematical literacy (Stacey & Turner, 2015) in general and in Tout and Spithill's (2015) writing of items to test mathematical literacy in particular. Our overarching principle in the development and validation of items or tasks is fitness-for-purpose because because the notion of Big Ideas can be contentious at its boundaries. Also, we expected the conceptualisation of Big Ideas to be complex and cut across school mathematics content. As such, the assessment items must be accessible to students and teachers. In addition, all the assessment items are designed for computer-based testing. For details about the instrument, refer to Jahangeer et al. (2023), which occurs as a research paper in this conference proceeding.

In this study, we focused on two Big Ideas Equivalence and Proportionality. Each item, consisting of five parts, tests on only one of the two Big Ideas. Part 1 to Part 3 each consists of a selected response question focusing on the same Big Idea and are from the same topic. To facilitate thinking beyond topical content and procedural knowledge, Part 4 seeks to assist participants to look for the link connecting the three parts. Part 4 also seeks to trigger students' Big Idea concepts, if any. The participants then attempt Part 5, a question that focuses on the same Big Idea but based on a different topic. Part 5 assessed the participant's ability to transfer the knowledge of Big Idea across a different topic. We also rode on the affordance of this sub-study to address the real issue of assessment fatigue among students. This is our attempt to balance between maintaining the validity of the instrument (students must answer sufficiently many types of problems); and not over-testing the students (to avoid assessment fatigue of students, aligned to the increasing emphasis on the mental well-beings of students). This will be reported in Paper 2.

Sub-study 2: Big Ideas in Secondary School Mathematics. This sub-study aimed to study the trajectory growth in (a) secondary school teachers' knowledge of BISM in relation to their involvement in professional development related to BISM; and (b) lower secondary school students' knowledge of BISM through their two years' schooling at the lower secondary level. The findings we have obtained so far for this sub-study is presented in Paper 3.

Sub-study 3: Big Ideas in Primary School Mathematics. This sub-study is an analogue of Sub-study 2, with the focus on primary school mathematics teachers and upper primary students at Primary 5 and Primary 6. The findings we have obtained so far for this sub-study is presented in Paper 4.

The instrument developed in sub-study 1 was administered to the teacher and student participants in sub-studies 2 and 3 at various chronological points between the two years' schooling. The first test-point, administered prior to the commencement of the teachers' professional development, provided the baseline information on the state of the teachers' knowledge of BISM prior to formal participation in professional development, and the students' knowledge of BISM prior to their teachers being officially cognizant of BISM. It also guided the researchers in designing the professional development interventions for the participating teachers.

## Conclusion

This study will inform how teachers understand the rationale for teaching towards Big Ideas, their belief and appreciation in the value to teach towards Big Ideas, and how these are translated into their teaching practices in their efforts to develop in students a greater awareness of the disciplinarity of mathematics, the ideas that are central to the discipline, and bring coherence and connection between different topics and across levels. In view of this, most importantly, the study will inform how students are able to better learn new mathematical knowledge with an appreciation of Mathematics as a discipline and its applications in the world.

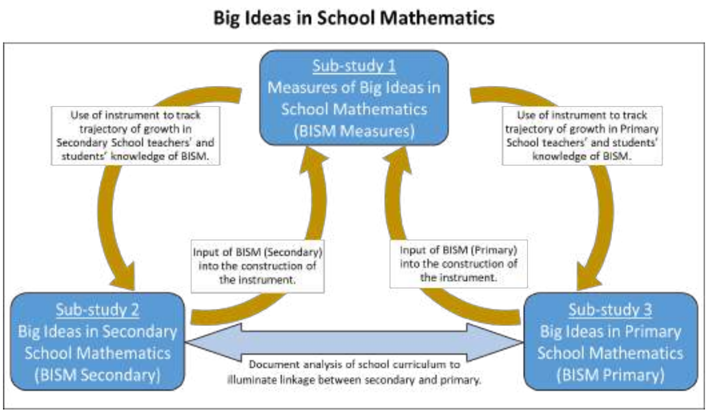



Figure 1. The relation between the three sub-studies in BISM project.

## Acknowledgements

The study reported here has been funded by the Singapore Ministry of Education under the research project OER/31/19BK which is managed by the National Institute of Education, Singapore.

## References

Askew, M. (2013). Big ideas in primary mathematics: Issues and directions. *Perspectives in Education, 31*, 5–18.

Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education Leadership, 7*(3), 9–21.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York: Macmillan.

Hsu, E., Kysh, J., Ramage, K., & Resek, D. (2007). Seeking big ideas in algebra: The evolution of a task. *Journal of Mathematics Teacher Education, 10*, 325–332.

Ministry of Education (MOE) (2018). *2020 Secondary mathematics syllabuses (draft)*. Author.

Ministry of Education (MOE) (2019). *2021 Primary mathematics syllabuses (draft)*. Author.

National Council of Teachers of Mathematics (NCTM). (2000). *Curriculum and evaluation standards for school mathematics*. NCTM.

Niemi, D., Vallone, J., & Vendlinski, T. (2006). *The power of big ideas in mathematics education: Development and pilot testing of POWERSOURCE assessments*. CSE Report 697, National Center for Research on Evaluation, Standards, and Student Testing (CRESST).

Schoenfeld, A. H. (2019). Reframing teacher knowledge: A research and development agenda. *ZDM*, 1–18.

Stacey, K., & Turner, R. (2015). *Assessing mathematical literacy: The PISA experience*. Springer International Publishing.

Tout, D., & Spithill, J. (2015). Chapter 7: The challenges and complexities of writing items to test mathematical literacy. In K. Stacey, R. Turner (Eds.), *Assessing mathematical literacy: The PISA experience* (pp. 145–171).

# *Assessment of Big Ideas in School Mathematics: Exploring an Aggregated Approach*

Mohamed Jahabar Jahangeer
*Nanyang Technological University*
jahabar.jahangeer@nie.edu.sg

Tin Lam Toh
*Nanyang Technological University*
tinlam.toh@nie.edu.sg

Eng Guan Tay
*Nanyang Technological University*
engguan.tay@nie.edu.sg

Cherng Luen Tong
*Nanyang Technological University*
cherngluen.tong@nie.edu.sg

In this paper we report our development of instruments to measure Big Ideas in school mathematics. In tackling the issue of assessment fatigue among students, we present an aggregated approach to measure students' knowledge of Big Ideas.

There has been little research done on how the knowledge and understanding of Big Ideas can be assessed. In one of the rare examples we could find, Niemi et al. (2006) suggested three main types of assessments to measure Big Ideas in mathematics: basic computation tasks, partially-worked problems (with or without explanations), and explanation tasks. Charles' (2005) definition of a Big Idea in Mathematics as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p.10) implies the need to contrast a task across more than one topic to be able to tease out the use of a Big Idea in the task. We followed this basic principle in constructing an instrument to assess Big Idea 'ability'. An example of an item, consisting of five parts, on Equivalence is shown in Figure 1. We have piloted some of the items which we have constructed. The dimensionality of these items are reported in Jahangeer et al. (2023), a separate paper in this conference. An important consequence from a Rasch analysis was that we could only use Part 5 as a reliable measure of Big Idea 'ability' since within an item, Parts 1 to 3 violate the item independence requirement of a Rasch scale.

## Assessment Fatigue

Assessment has always been an integral part of teaching and learning. Analysis of assessment performance is used for a variety of purposes including placement, selection and certification. In many countries, standardised and high stakes assessments are put in place at milestone grades to determine placement and selection of students to the next course of their education. Well-designed assessment tools and analysis can provide accurate information regarding student learning.

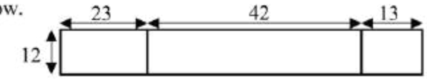
Inaccuracies or deviations from what students have mastered could have been contributed by the students themselves. In particular, the cognitive demand required on students may contribute to them experiencing cognitive fatigue, which naturally affects their overall performance. According to Ackerman and Kanfer (2009), "[a]nticipations of subjective fatigue may lead some individuals to avoid tasks altogether" (p. 176). The duration of an assessment may result in unwilling students not committed to performing to the best of their abilities, affecting the validity of the responses. Thus, a balance between the reliability and validity of the assessment and the duration of assessment without causing a negative anticipation of cognitive fatigue, is an area of worthwhile concern for educators and researchers.

Returning to our attempt to assess Big Idea 'ability', the same consideration of duration of assessment in relation to test validity and reliability arises. Each item of ours necessarily consists of parts to enable a Big Idea to surface across different topics. However, just two items would require at least 30 minutes. A valid Rasch scale would require at least six items to cover a significant range of ability. We derived this based on Andrich's work which, when describing the invariance of appropriate comparison on measures using the Rasch model, used a six-item questionnaire for an example (Andrich, 1988, p. 22).

**Part 1:** The diagram below shows three rectangles joined together to form a bigger rectangle.

You may use the diagram to fill in the blanks below.

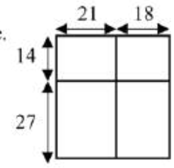
23 x 12 + 42 x 12 + 13 x 12 = ? x 12




**Part 2:** The diagram below shows four rectangles joined together to form a bigger rectangle.

Fill in the missing numbers below.

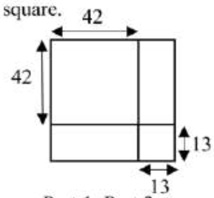
14 x 21 + Δ x 18 + 27 x 21 + 27 x 18 = ♥ x 39




**Part 3:** The diagram below shows 2 squares and 2 rectangles joined together to form a bigger square.

Fill in the missing numbers below.

55 x 55 = 42 x 42 + 13 x 13 + 13 x Δ x ♥




**Part 4:** Which of these following statements best describes the common mathematical idea across Part 1, Part 2 and Part 3?

a) I used diagrams for the parts

b) I used Equivalence for the parts

c) I used Guess and Check for the parts

d) I used Proportionality for the parts

e) Others (Please elaborate)

**Part 5:** The shaded area of the figure below can be used to show a mathematical statement. Which of the following statements matches the shaded part of the figure?

a) (1 + 6) + (2 + 5) + (3 + 4) … + (6 + 1) = 6 x 7

b) 1 + 2 + 3 + … + 6 = (6 x 7 ) ÷ 2

c) 1 + 2 + 3 + … + 7 = (7 x 8) ÷ 2

d) 1 + 2 + 3 + … + 8 = (8 x 9) ÷ 2

e) 1 + 2 + 3 + … + 7 = (7 x 8)

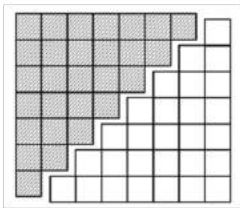


Figure 1. An item comprising 5 parts.

## An Aggregated Solution

We base our solution to the conundrum on the methodology and raison d'etre of sampling, i.e., to understand a population, there is no need for every student to complete the entire instrument. For example, the Programme for International Student Assessment (PISA) carried out international standardized testing every three years across various domains. Each domain consists of items which are subdivided into smaller blocks and each student involved in the assessment will be given a booklet made up of a few blocks. PISA made use of 'plausible values' to determine a student's performance and to give a population score instead of an individual score. The successful computation of plausible values, however, requires a deeper knowledge of mathematics which is not accessible to educators, generally. We are proposing a simpler structure that is mathematically easier and can be implemented by educators in schools.

We propose an aggregated structure which involves the creation of a 'Super-Student' (SS). Each SS is made up of four students of similar ability in Mathematics. A random grouping of students to form a SS will likely confound the results—a strong student in the grouping could have solved a difficult item and a weaker student in the grouping could not solve an easy item assigned. Thus, the SS would be invalid due to the misfit in responses. Although no study has been done to assess the correlation between mathematics ability and big idea thinking, Schoenfeld (2019) mentioned in his study that high performing individuals are able to see and use Big Ideas in problem solving. We thus make a reasonable assumption that students of similar ability may have the same level of Big Idea thinking. In this light, we propose to constitute an SS, with all four students having identical ability (ideally but impossible in practice), by rank ordering students based on their past semestral assessment marks as a proxy of their mathematical ability. Going down the list, every four students are grouped into an SS and given a new SS ID. For example, in a school of 320 students, the top four students will constitute SS01, the next four SS02 and the last four students in the ordered list will be SS80. Triangulation can be carried out with teachers to validate that the students grouped together are indeed of similar ability. To differentiate the students within each SS, a suffix is added, e.g., SS01a, SS01b, SS01c and SS01d for the four students that constitute SS01. This is done to facilitate the correct distribution of the items.

We envisage a final instrument for a Big Idea consisting of eight items (each with five parts). The eight items are split into eight testlets, T1 to T8 as shown in Table 1. Each testlet is only made up of two items and each student attempts only one of the testlets. Table 1 shows how the testlets are distributed to the students as well as to each SS. Since each testlet has only two complete items, it can be administered easily within a much shorter duration and will reduce cognitive fatigue.

**Table 1**

*Matrix Distribution of Items to Two SS Comprising a Total of Eight Students*

| | T1 | T2 | T3 | T4 | T5 | T6 | T7 | T8 |
|---|---|---|---|---|---|---|---|---|
| I1 | SS01a | | | | | | | SS02d |
| I2 | SS01a | SS02a | | | | | | |
| I3 | | SS02a | SS01b | | | | | |
| I4 | | | SS01b | SS02b | | | | |
| I5 | | | | SS02b | SS01c | | | |
| I6 | | | | | SS01c | SS02c | | |
| I7 | | | | | | SS02c | SS01d | |
| I8 | | | | | | | SS01d | SS02d |

As a result of the SS structure and distribution of testlets, each SS will have taken the entire set of items while each student only attempts two items. Thus, the duration required to complete the test is only 25% of the time required to complete all the eight items. The score collated will be for each SS instead of for every student in the school. This SS structure can be used not only in obtaining an aggregated score for assessing group performance on an instrument, but it can also be used for validating an instrument during its initial item creation stage.

## Conclusion

While assessments are important to monitor learning, too many high stakes assessments will reduce available time for teaching, erode the joy of learning and cause a high level of worry and stress about exams and results. However, assessment remains crucial to monitor if learning has taken place and is an important feedback mechanism to improve teaching as well as learning. In place of high stakes assessments, an aggregated structure as proposed may gather sufficient information regarding learning without increasing student cognitive load nor take up too much precious curriculum time. This may be a worthwhile contribution towards the joy of learning.

One of the main issues that arise from the SS structure is the validity of the SS itself. How similar are the four students within each SS? With no prior research done on the relationship between math ability and Big Idea thinking, it is difficult to validate the structure we have proposed. At this juncture, we have piloted the items and the SS structure is due to be tested and analysed later. We intend to explore and analyse the performance of the SS using two different approaches.

The first approach is to study the misfit of SS scores using Rasch analysis. In the development of the instrument, the items would be calibrated and validated using Rasch model. Using the same Rasch model analysis, we will be able to do a fit analysis by looking at person (SS) misfit information, if any. In the event of any person misfit cases, we hope that the misfit is due to the individual students doing the two items erratically, and not caused by the different students within the SS, e.g., the misfit is due to SS01a getting items with higher difficulty correctly while SS01c answering items with lower difficulty incorrectly. The second approach is by comparing a super-student score against the scores of each of the four students forming the super-student structure based on plausible values created for each student. The technique to calculate the plausible values can be found in Von Davier et al. (2009). We will collect our data from July 2023 and report the results thereafter.

## Acknowledgements

The study reported here has been funded by the Singapore Ministry of Education under the research project OER31/19BK which is managed by the National Institute of Education, Singapore.

## References

Ackerman, P. L., & Kanfer, R. (2009). Test length and cognitive fatigue: An empirical examination of effects on performance and test-taker reactions. *Journal of Experimental Psychology. Applied, 15*(2), 163–181. https://doi.org/10.1037/a0015719.

Andrich, D. (1998). *Rasch models for measurement* (Vol. 68). Sage.

Jahangeer, M. J. et al. (in press). Exploring the dimensionality of big ideas in school mathematics. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia*. Newcastle: MERGA.

Niemi, D., Vallone, J., & Vendlinski, T. (2006). *The power of big ideas in mathematics education: Development and pilot testing of POWERSOURCE assessments.* CSE Report 697, National Center for Research on Evaluation, Standards, and Student Testing (CRESST).

Schoenfeld, A. H. (2019). Reframing teacher knowledge: A research and development agenda. *ZDM*, 1–18.

# *From Inert Knowledge to Usable Knowledge: Noticing Affordances in Tasks Used for Teaching Towards Big Ideas About Proportionality*

Ban Heng Choy
*National Institute of Education*
banheng.choy@nie.edu.sg

Joseph B. W. Yeo
*National Institute of Education*
josephbw.yeo@nie.edu.sg

Yew Hoong Leong
*National Institute of Education*
*yewhoong.leong@nie.edu.sg*

Teaching towards big ideas provide opportunities for teachers to think deeply about content and pedagogy in order to support their students to see connections in mathematics. However, teachers may not always activate or mobilise their knowledge in classroom situations. This paper looks into how a teacher, Peter, think about the tasks in his instructional materials he crafted to uncover what he may notice about the affordances of the tasks for teaching proportionality.

Teaching towards big ideas, a recent initiative included in the 2020 Singapore Mathematics Syllabus (Ministry of Education-Singapore, 2019), provides opportunities for teachers to think more deeply about what and how they teach in order to support their students to see connections in mathematics (Choy, 2019). Doing this requires teachers to pay attention to the mathematics embedded in the curriculum, discern the details of the big ideas, and perceive the affordances in tasks for bringing out these ideas (Choy, 2019). A key enabler is the mathematical knowledge for teaching (Ball et al., 2008) that teachers can activate during classroom instruction. This suggests a key distinction between inert knowledge (Renkl et al., 2010) and usable knowledge, or what they mobilise during teaching. Kersting et al. (2012) hypothesized that "teachers with more usable knowledge are able to apply that knowledge to the design and improvement of instruction in their classrooms" (p. 573). Furthermore, as Choy and Dindyal (2021) had pointed out, it is not trivial for teachers to notice the affordances of tasks and harness them to improve instruction. Here, we explore how teachers can be supported, through professional learning (PL) sessions, to transform their inert knowledge into usable knowledge through the discussion and design of instructional materials. This paper is guided by the following research question:

- How does a PL session that focuses on the design of instructional materials activate his inert knowledge of a big idea in mathematics?

## Contexts and Methods

The six teacher participants in the study reported here is part of a larger project on "Big Ideas in School Mathematics", which focused on the notion of teaching towards big ideas in Singapore. These six teachers participated in a series of professional learning (PL) sessions to unpack big ideas about proportionality (Yeo, 2019) so that they can design instructional materials and lessons for teaching the topic of ratio and rates in Secondary One. In the first session, the second author discussed the idea of proportionality from a few perspectives: when one quantity is multiplied by $n$, the other quantity is also multiplied by $n$ (which we will call proportional reasoning), the equality of two ratios (e.g. $\frac{y_2}{y_1} = \frac{x_2}{x_1}$ for direct proportion), the rate $\frac{y}{x}$ is constant for direct proportion, and the product $xy$ is constant for inverse proportion. Two main approaches to solving problems involving proportionality were shared: proportional reasoning via the unitary method and using the constant rate $\frac{y}{x}$ directly. In the next two sessions, the second and third authors facilitated discussions on the use of these two approaches, as well as others (Weinberg, 2002), to solve problems involving constant rates and supported the teachers in thinking about the design of instructional materials to incorporate proportionality in questions involving ratio, percentage, currency exchange and speed. Of interest in this paper is the instructional material shared by Peter (pseudonym), one of the teachers, during the fourth PL session, which was facilitated by the first author. Data collected include video and voice recordings of the PL session, and the draft instructional material designed by Peter. For this paper, the findings were generated from Peter's sharing on his thinking behind the design of the instructional material used for teaching rate, as well as the interactions between him and the other teachers in the PL session. Analyses were guided by the following questions:

- What knowledge on proportionality did Peter utilise in his design?
- What inert knowledge on proportionality did Peter activate during the PL session?

## Three Short Snippets of Peter's Thinking

In this section, we begin by describing three short snippets of Peter's thinking, juxtaposed with what the other teachers said in response to the questions or prompts by the first author (BH). We then unpack Peter's thinking behind his design or choice of tasks put into instructional material before we characterise his understanding of proportionality in terms of what he knew inertly (Renkl et al., 2010) and what he was able to access and use—usable knowledge (Kersting et al., 2012)—through his interactions during the PL session.

### *Snippet 1: Shampoo Investigation Task*

Peter began by describing the investigation task he placed at the beginning of the instructional material (see Figure 1). He had wanted the students to rely on their intuition and explain how they solve the problem *before* teaching them about the concept of rate.

**Investigation!**

- Which bottle of shampoo is more **value for money** (cheaper for the same volume of shampoo)?
- How do you determine that?

| Bottle A | Bottle B |
|---|---|
|  |  |
| 400 mL: $5.00 | 500 mL: $6.00 |

Figure 1. Shampoo problem.

When asked about how students might respond to the task, Peter responded that "some of them might choose to ignore the idea of same volume and just superficially choose the cheapest" [38:26]. Teacher M then shared that "they would use the unitary method" to obtain the cost of shampoo for 100 mL and subsequently 1 mL [38:52]. With more prompting, Peter highlighted that students could "change the volume to 2 litres" [39: 47] and compare. Teacher N also offered a similar size-change strategy (Weinberg, 2002) by changing the price to $30. Building on this discussion, the first author highlighted that these different methods (without using rate explicitly) were all based on the overarching idea of proportionality.

### *Snippet 2: Fastest Typist Problem*

After the investigative task, Peter defined rate as "a quantity per (one) unit of another quantity" and selected a series of tasks, meant for students to compute rates in his instructional material. One such task is given as follows: Jayden can type 720 words in 6 minutes, Ithiel can type 828 words in 18 minutes and Zhi Rui can type 798 words in 19 minutes. Who is the fastest typist?

Peter had intended the task to be used merely for computation. At this juncture, the first author highlighted the possibility of "looking more closely at the numbers used" and modify the numbers to bring out the idea of proportionality more explicitly. The first author suggested Peter to consider how the numbers can be changed to provide opportunities for students to exercise their proportional reasoning. In addition, he highlighted to Peter that the current set of numbers did not require students to do deliberate calculation using "proportional reasoning"; instead, students would just need to mentally estimate that Jayden has to be the fastest typist because he could type around 700 words within 6 minutes, as compared to what the other two could type in a much longer time (18 or 19 minutes). Of course, students could have multiplied 720 by 3 (proportional reasoning) to compare Jayden's typing speed against the other two. Through the discussion, Peter became aware of how the item could be used to emphasise different aspects of proportionality.

### *Snippet 3: Exchange Rate Problem*

The rest of the instructional material focused on providing opportunities for students to calculate per (one) unit rates instead of looking out for opportunities to highlight the "power of proportionality" to make sense of comparisons between two quantities. For instance, Peter went through Example 2 (See Figure 2) as merely computational without noticing the alternative solution to part (b) of the question. When the first author prompted the teachers to look more closely at the answer to part (b), Peter realised that students could simply divide 1256 SGD (given in the stem of the question) by 10 using the idea of proportionality.

**Example 2**

On 13 June 2018, Cheryl exchanged 800 Euros (EUR) for 1256 Singapore dollars (SGD).

**(a)** Find the exchange rate, correct to 4 significant figures if it is not exact, between Euros and Singapore dollars in

**(i)** SGD/EUR, **(ii)** EUR/SGD.

**(b)** Cheryl spent 80 EUR on some gifts for her family. Find the price of the gifts in SGD.

**Solution**

**(a) (i)** 800 EUR = 1256 SGD ÷800

$1 \text{ EUR} = \frac{1256}{800} \text{ SGD}$

$= 1.570 \text{ SGD}$

∴ the exchange rate is 1.57 SGD/EUR.

**(ii)** 1256 SGD = 800 EUR ÷1256

$1 \text{ SGD} = \frac{800}{1256} \text{ EUR}$

$= 0.6369 \text{ EUR}$ (correct to 4 s.f.)

∴ the exchange rate is 0.6369 EUR/SGD.

**(b)** Price of the gifts = 80 EUR

$= 80 \times 1.57 \text{ SGD}$

$= 125.60 \text{ SGD}$

Figure 2. Exchange rate problem.

## Discussion

Taken together, the three snippets detailed in this short paper suggest that while Peter and the other teachers were aware of the ideas of proportionality (as seen in Snippet 1), he might not always be able to *notice* these ideas and *harness the affordances* of the tasks embedded in the instructional materials he had designed (Choy & Dindyal, 2021). As seen from the three snippets, he was able to articulate the knowledge about teaching proportionality, especially the idea of providing opportunities for students to reason proportionally using different solution strategies (Weinberg, 2002). Yet, he did not always notice affordances of these tasks to bring out the idea of proportionality and instead focused on emphasising a fixed way of finding rate and solving missing value questions. In other words, it is not trivial for teachers to activate their inert knowledge about teaching proportionality to generate usable knowledge that can potentially enhance students' understanding of this big idea when designing instruction materials. What matters is not simply what the teachers know, but how they can learn to mobilise their knowledge in actual classroom situations (Ball et al., 2008; Kersting et al., 2012).

These snippets not only highlight the complex and perennial issue of knowledge activation in the act of teaching but also provide insights into how professional learning activities can be structured to support teachers to bridge the gap between their knowledge and classroom practice. First, such professional learning can be structured around discussion of lessons and more specifically, the design of instructional materials. Designing lesson materials provide an avenue for teachers to transform their knowledge into something usable, and hence enhance the possibility of them mobilising their inert knowledge. Second, we see the need for teachers to learn to notice affordances for using tasks and other instructional materials because doing this provides opportunities for teachers to generate new possibilities that can potentially change practices. Lastly, the role of a knowledge facilitator to support teachers to notice new possibilities in their design of instructional materials, in the context of professional learning sessions, should not be under-estimated. How such sessions could be facilitated remains under-studied and could be a fruitful area for future research.

## Acknowledgements

The study reported here has been funded by the Singapore Ministry of Education under the research project OER/31/19BK which is managed by the National Institute of Education, Singapore.

## References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes It special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Choy, B. H. (2019). Teaching towards big ideas: Challenges and opportunities. In T. L. Toh,, & B. W. J. Yeo (Eds.), *Big ideas in Mathematics* (pp. 95–112). World Scientific Publishing Co. Pte. Ltd.

Choy, B. H., & Dindyal, J. (2021). Productive teacher noticing and affordances of typical problems. *ZDM—Mathematics Education*, *53*(1), 195–213. https://doi.org/10.1007/s11858-020-01203-4

Kersting, N. B., Givvin, K. B., Thompson, B. J., Santagata, R., & Stigler, J. W. (2012). Measuring Usable Knowledge. *American Educational Research Journal*, *49*(3), 568–589. https://doi.org/10.3102/0002831212437853

Ministry of Education-Singapore. (2019). *2020 Secondary Mathematics Syllabus*. Curriculum Planning and Development Division.

Renkl, A., Mandl, H., & Gruber, H. (2010). Inert knowledge: Analyses and remedies. *Educational Psychologist*, *31*(2), 115–121. https://doi.org/10.1207/s15326985ep3102_3

Weinberg, S. L. (2002). Proportional reasoning: One problem, many solutions! In B. Litwiller, & G. Bright (Eds.), *Making Sense of Fractions, Ratios, and Proportions* (pp. 138–144). National Council of Teachers of Mathematics.

Yeo, B. W. J. (2019). Unpacking the big idea of proportionality: Connecting ratio, rate, proportion and variation. In T. L. Toh, & B. W. J. Yeo (Eds.), *Big ideas in Mathematics* (pp. 187–218). World Scientific Publishing Co. Pte. Ltd.

# *Primary School Teachers Solving Mathematical Tasks Involving the Big Idea of Equivalence*

Berinderjeet Kaur
*Nanyang Technological University*
berinderjeet.kaur@nie.edu.sg

Cherng Luen Tong
*Nanyang Technological University*
chernglueen.tong@nie.edu.sg

Mohamed Jahabar Jahangeer
*Nanyang Technological University*
jahangeer.jahabar@nie.edu.sg

The primary school mathematics syllabuses in Singapore as of the year 2020 reinforces that Big ideas are central to the learning of mathematics. In support of the push to teach for big ideas, a research study is presently underway. A part of it is on the professional development (PD) of primary school mathematics teachers. As part of the PD teachers attempted a mathematical task as measure of the big idea, Equivalence, in an online environment at the start of their PD. Data from the task show that teachers, were generally not cognisant of the big idea of equivalence when solving the task. They were also unable to distinguish between a heuristic (diagrams) and a mathematical idea about relationships, specifically equivalence as in the mathematical task.

The revised school mathematics curriculum, in Singapore, as of 2021 has placed emphasis on learning mathematics as a body of connected knowledge (Ministry of Education, 2019). Four themes, namely properties and relationships, representations and communications, operations and algorithms, and abstractions and applications together with six big ideas have been emphasised for the teaching of mathematics in primary schools. A "big idea is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (Charles, 2005, p. 10). The six big ideas are diagrams, equivalence, invariance, measures, notations, and proportionality. A research study, Big Ideas in School Mathematics (BISM) is presently underway in Singapore and a part of it is on professional development (PD) of primary school mathematics teachers related to the enactment of Big Ideas in their mathematics instruction. Research has documented that teachers' lack of relevant content knowledge of Big Ideas in mathematics translates into their lack of explicit attention to Big Ideas underpinning mathematics taught in schools and results in developing isolated compartments of mathematical knowledge in their students (Askew, 2013). The study reported in this paper draws on part of the data from the BISM project. It attempts to uncover if teachers drew on the big idea of equivalence when solving mathematical tasks that encompass equivalent relationships at the beginning of their PD.

## The Study

### *Participants and Instrument*

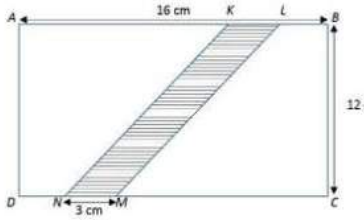
All the mathematics teachers in two primary schools, P1 and P2, participated in the PD (see Kaur et al. 2021; 2022). The PD was spread over two years. In the first year 24 teachers from school P1 and 32 teachers from school P2 and in the second year 23 teachers from school P1 and 33 teachers from school P2 participated in the PD. Due to teacher movement in and out of schools, in the second year there was one less teacher in school P1 and one more teacher in school P2.

Each year during the first session of the PD teachers attempted a set of three mathematical tasks in an online computer environment. These tasks were part of a collection of tasks that were being put together as measures of two big ideas, namely equivalence and proportionality. In the first-year teachers attempted 2 tasks on proportionality and 1 on equivalence, and in the second year they attempted 1 task on proportionality and 2 tasks on equivalence. We limit the data in this paper to the item on equivalence that teachers in School P1 attempted during the first session of their PD in the first year.

Figure 1 shows the equivalence task the teachers attempted in the first session of their first year. The task had 5 parts. Parts 1, 2 and 3 were tasks independent of each other that involved geometrical shapes and measurement. Similar tasks are found in end of school examinations for primary 6 in Singapore schools. Part 4-1 prompted the teachers to review their solutions to Parts 1, 2 and 3 and reflect on any common idea they may have drawn on whilst working on their solutions. Part 4-2 offered some options for teachers to consider about what may have guided their solutions in Parts 1, 2 and 3. Part 5-1 was yet another task on geometry and measurement that teachers had to attempt. Following Part 5-1 was Part 5-2, where teachers were again asked to review their solutions for Parts 1, 2, 3 and 5-1 and consider what may have guided their solution process.

Part 1

The figure below shows a rectangle $ABCD$.
The lines $KN$ and $LM$ are parallel.

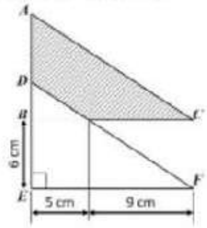



Find the shaded area.

● 18 cm$^2$ ● 36 cm$^2$ ● 156 cm$^2$ ● 192 cm$^2$ ● Others

Part 2

In the diagram below, $ABC$ and $DEF$ are two identical right-angled triangles. Triangle $DEF$ is partly placed on top of triangle $ABC$.

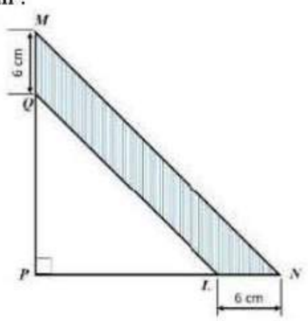



Find the shaded area.

● 42 cm$^2$ ● 57 cm$^2$ ● 75 cm$^2$ ● 84 cm$^2$ ● Others

Part 3

In the diagram $QPL$ and $MPN$ are two right-angled isosceles triangles. $QPL$ is placed on top of $MPN$ such that $QM = LN = 6$ cm. Area of the shaded part is 144 cm$^2$.




Find the length of $PL$.

● 21 cm ● 24 cm ● 27 cm ● 30 cm ● Others

Part 4-1

What is a common mathematical idea in Part 1, Part 2, and Part 3?

| *To help you recall click on the links below to see the parts* | | |
|---|---|---|
| Part 1 | Part 2 | Part 3 |

Enter your response in the space below.

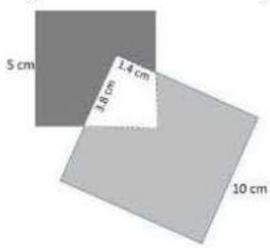
Part 4-2

Which of the following statements best describes a common mathematical idea across Part 1, Part 2 and Part 3?

- o I used diagrams for the parts.
- o I used equivalence for the parts.
- o I used guess and check for the parts.
- o I used proportionality for the parts.
- o Others (Please elaborate)

Part 5-1

The diagram shows a lighter square with side 10 cm placed on top of part of a darker square with side of length 5cm. Their common region is unshaded.




Find the difference between the area of the lighter region and the area of the darker region? (You may want to consider drawing a model)

● 5 cm$^2$ ● 5.2 cm$^2$ ● 5.32 cm$^2$ ● 70 cm$^2$ ● Others

Part 5-2

Choose one of the statements that best describes what is common about Part 1, Part 2, Part 3 and Part 5?

| *To help you recall click on the links below to see the parts* | | | |
|---|---|---|---|
| Part 1 | Part 2 | Part 3 | Part 5-1 |

- o In all these parts I used diagrams.
- o In all these parts I used equivalence.
- o In all these parts I used guess and check.
- o In all these parts I used proportionality.
- o Others (Please elaborate)

Figure 1. Example of mathematical task illuminating equivalence as a big idea.

## *Data and Discussion*

Table 1 shows the performance of 24 teachers from School 1 on the mathematical item shown in Figure 1.

**Table 1**

*Performance of Teachers on Mathematical Task Shown in Figure 1*

| Task | Response | n (%) |
|---|---|---|
| Part 1 | 36 $cm^2$ (correct answer) | 21 (87.5) |
| Part 2 | 57 $cm^2$ (correct answer) | 18 (75.0) |
| Part 3 | 21 cm (correct answer) | 18 (75.0) |
| Part 4-1 | Others* | 24 (100) |
| Part 4-2 | I used diagrams for the parts. | 7 (29.2) |
| | I used equivalence for the parts. | 3 (12.5) |
| | I used guess and check for the parts. | 2 (8.3) |
| | I used proportionality for the parts. | 9 (37.5) |
| | Others (Please elaborate) | 3 (12.5) |
| | Use algebra / Cut-outs and diagrams / Use algebra and part-whole relations | |
| Part 5-1 | Others (75 $cm^2$–correct answer) | 7 (29.2) |
| Part 5-2 | In all these parts I used diagrams. | 7 (29.2) |
| | In all these parts I used equivalence. | 3 (12.5) |
| | In all these parts I used guess and check. | 2 (8.3) |
| | In all these parts I used proportionality. | 12 (50.0) |
| | Others (Please elaborate) | 0 (0.0) |

*Responses of the teachers were coherent with Part 4-2.

It is apparent from the data in Table 1 that at least 18 (75%) of the teachers managed to work through Parts 1, 2 and 3 of the task and arrive at the correct answer. 12 of them mentioned using diagrams, equivalence and part-whole relations as mathematical ideas in their solutions. To resolve Part 1, as shown in Figure 2, one may find the area of the shaded portion by finding the difference between the areas of rectangles with sides 16 cm by 12 cm and 13 cm by 12 cm. Similarly for Parts 2 and 3, teachers may have 'used diagrams' to illuminate relationships. It appears that some teachers were using diagrams as a heuristic to illuminate a mathematical idea which many failed to name as equivalence. This may have been due to a lack of 'vocabulary' in their mathematics discourse.

However, for Part 5-1 it appears that teachers were challenged when trying to construct a relationship using diagrams. The hint provided could have led them to make equations such as:

- area of lighter region + area of overlap = 100 $cm^2$
- area of darker region + area of overlap = 25 $cm^2$

and observe a relationship, but many appear to have failed at it. It is not clear what teachers meant by 'used proportionality' in their responses to Parts 4-1, 4-2 and 5-2. As teachers were not interviewed about their responses to the parts of the task, we are unable to decipher what they meant by this.

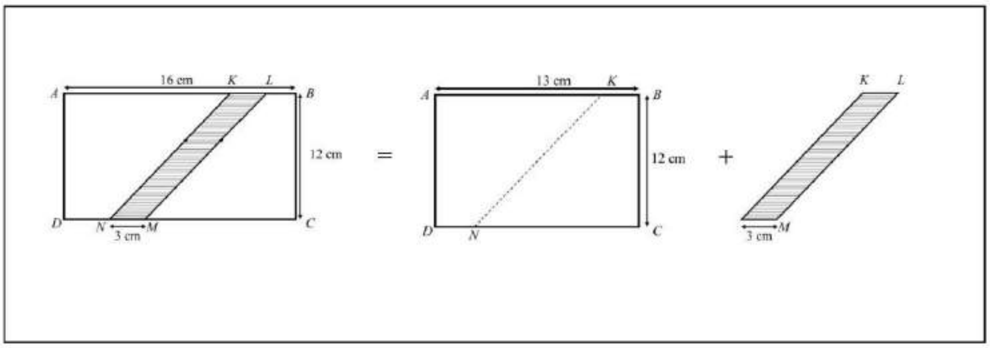



Figure 2. Equivalent relationship of parts in Part 1 of task.

## Conclusion

It is apparent from the teachers' responses to the parts in Figure 1 that generally they were not cognisant of the big idea of equivalence which is stated as follows in the mathematics syllabus for primary schools (Ministry of Education, 2019, p. 15):

> Equivalence is a relationship that expresses the 'equality' of two mathematical objects that may be represented in two different forms. The conversion from one form to anther equivalent form is the basis of many manipulations for analysing, comparing, and finding solutions. In every statement about equivalence, there is a mathematical object (e.g. a number, an expression or an equation) and an equivalence criterion (e.g. value(s) or part-whole relationships).

The findings of the study reported here were critical in shaping the following PD sessions as teachers' lack of relevant knowledge of Big Ideas translates into their lack of explicit attention to them in their instruction (Askew, 2013). During the second session of the PD, teachers shared how they had attempted to resolve Parts 1, 2 3, and 5-1. The whole group discourse together with inputs from the experts (University professors) created a shared vocabulary for Big Ideas and specifically—equivalence and how such an idea facilitated solutions of mathematical tasks similar to the ones in Figure 1 and others in the school mathematics curriculum.

## Acknowledgements

The study reported here has been funded by the Singapore Ministry of Education under the research project OER/31/19BK which is managed by the National Institute of Education, Singapore.

## References

Askew, M. (2013). Big ideas in primary mathematics: Issues and directions. *Perspectives in Education*, 31, 5–18.

Charles, R. I. (2005). Big ideas and understandings as the foundations for elementary and middle school mathematics. *Journal of Mathematics Education Leadership*, 7(3), 9–24.

Kaur, B., Jahabar, J. M., & Tong, C. L. (2021). Perceptions of big idea of equivalence amongst mathematics teachers in primary schools. In Inprasitha, M., Changsri, M., & Boonsena, N. (Eds.), *Proceedings of the 44th conference of the International Group for the Psychology of Mathematics Education* (p. 152). Khon Kaen, Thailand: Thailand Society of Mathematics Education.

Kaur, B., Jahabar, J. M., & Tong, C. L. (2022). Big ideas of equivalence and proportionality in a grade six mathematics lesson. In Fernandez, C., Llinares, S., Gutiérrez, A., & Planas, N. (Eds.), *Proceedings of the 45th conference of the International group for the Psychology of Mathematics Education* (p. 246). Alicante, Spain: University of Alicante.

Ministry of Education. (2019). 2021 Primary mathematics teaching and learning syllabus. Singapore: Author.

# Symposium: Embodied Learning in Early Mathematics

Co-chair: Jennifer Way
*University of Sydney*
jennifer.way@sydney.edu.au

Co-chair: Katherin Cartwright
*University of Sydney*
katherin.cartwright@sydney.edu.au

In this symposium we present some of the findings from Phase 1 of a three-phase project (2021-2024) titled *Embodied Learning in Early Mathematics and Science* (ELEMS). The project aims to translate embodied cognition research from the fields of neuroscience, psychology and education into evidence-based classroom teaching strategies, and to produce professional learning materials for teachers. The overall research design for the project is a three-phase structure, guided by design-based research principles and utilising mixed methods of data collection and analysis (Refer to Way & Ginns, 2022 for a project rationale). The underlying premise for the project is that the haptic modes (gesture, touch-tracing, body-movement and drawing) of embodied learning are under-utilised for mathematical representation, and as thinking and communicating tools in the development of mathematical understanding.

Phase 1 of the project involved a year-long collaboration with seven teachers in one NSW school, and their classes of Preschool to Year 2 children. The school has 340 students, with an additional 38 students in an attached preschool. The students come from a diverse range of cultures and 78% of students are from Non-English-Speaking Backgrounds (NESB). The researchers supported the teachers in their explorations of interpreting the research-based key ideas about embodied learning provided by the researchers, into teaching-learning activities for their students. Each of the three papers in this symposium reports a specific aspect selected from the broad range of research outcomes.

Paper 1: Connecting Mathematical Processes and Conceptual Body Movement—Katherin Cartwright & Jennifer Way

Paper 2: Finger Tracing, Noticing Structures and Drawing—Jennifer Way & Katherin Cartwright

Paper 3: Changes in Year 2 Children's Drawings of a Subtraction Story—Jennifer Way & Katherin Cartwright

## Acknowledgements

The ELEMS project is funded by the NSW Department of Education, Strategic Research Fund (2021-2024).

We acknowledge the collaboration of the leadership team and teachers of the Phase 1 research school.

The University of Sydney research team: Dr Jennifer Way (Mathematics), Dr Paul Ginns (Educational Psychology), Dr Christine Preston (Science), Dr Amanda Niland (Early Childhood), Dr Jonnell Upton (TESOL), Dr Katherin Cartwright (Mathematics).

## References

Way, J., & Ginns. P. (2022). A call for translational research in embodied learning in early mathematics and science education: The ELEMS project. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Symposium in the proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 538–545). Launceston: MERGA.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Symposium in the proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 57–69). Newcastle: MERGA.

# *Connecting Mathematical Processes and Conceptual Body Movement*

Katherin Cartwright
*University of Sydney*
katherin.cartwright@sydney.edu.au

Jennifer Way
*University of Sydney*
jennifer.way@sydney.edu.au

Conceptual body movement in education is an external representation mode that research suggests can support children's learning about mathematical phenomena. Children's learning and understanding of mathematical concepts and processes, such as number structure and relationships, number sequencing, position, or geometric properties, may be supported by experiences using their own body movements. The aim of this paper is to share classroom activities trialled within the Embodied Learning in Early Mathematics and Science project in 7 classrooms focusing on conceptual body movements. The results share what was trialled, and what was observed in relation to children's learning of mathematical processes. Findings revealed that body movement is a helpful mode through which young children can learn and communicate mathematical understanding.

Children's use of body movement supports the development of egocentric spatial frames of reference (Dackermann et al., 2017) as they explore the physical environment around them relative to their own perspective. In the context of mathematics, spatial frames of reference are important to model, and then visualise, structural aspects that are mathematically important such as the equal spacing of numbers on a number line, or the conceptual differences in positioning between 'on top', 'under' and 'next to'. *Conceptual* body movement differs from "movements for the sake of movement" (Shoval, 2011, p. 454) that are simply physical in nature, for example, children running on the spot while counting. *Conceptual* body movement involves whole-body movement that is task-specific, where actions relate directly to conceptual understandings. For example, physically jumping forward to model the process of adding-on 3 on a number line. These more purposeful actions Shoval (2011) calls 'mindful movement' and are for "the purpose of learning" (p. 454). Shoval's research highlights that learning can be enhanced when children participate in co-operative action and re-enactment using their bodies in small groups. Garrett et al. (2018) propose that "embodied representations of concepts create pedagogical opportunities to support student learning" (p. 6). The classroom provides an opportunistic space from which to observe these research claims.

During Phase 1 of the ELEMS project, the seven teachers involved in the research at the school were provided with three days of professional learning (PL) across the year. The first PL day focused on the research behind the embodied learning principles (including conceptual body movement) and examples of classroom activities the teachers could trial or adapt. This paper presents a selection of the activities the teachers' trialled or created that embedded conceptual body movement, discusses how they connect to mathematical processes and concepts, and offers ideas about student learning as identified by the teachers during these lessons.

The following questions guided the analysis of these activities: How do teachers incorporate conceptual body movement in mathematics lessons? and What potential mathematical learning connections were identified by teachers when using body movement?

## Approach

The activities were implemented by teachers in Preschool, Kindergarten, Year 1 and Year 2 classrooms. The data related to the activities was either self-reported (by the teachers via the SeeSaw classroom journal, https://web.seesaw.me/, or during post-Phase 1 teacher interviews) or observed by the researchers (during weekly visits to the classrooms where some activities were co-designed by the mentoring researcher and teachers). The selected activities presented in this paper are from the Kindergarten and Year 1 classrooms ($n = 4$ teachers, $n = 77$ students). Activities linked to a range of curriculum areas such as data, position, number sense, patterning, time, and mass, see Table 1.

**Table 1**

*Body Movement Activities Aligned to Mathematical Concepts*

| Number and algebra | Measurement and space | Statistics |
|---|---|---|
| Being the pattern—children used whole body movement to make a 'two pattern'.<br>*Leaping number line*—children stood in place of numbers to act out addition and subtraction number sentences.<br><br>*Ten frame hustle*, and *Act it out*—children stood in the frame to make numbers to 10, and to depict addition and subtraction scenarios<br><br>Making numbers—children used their bodies to make numbers 1 to 10 on the floor in pairs or groups.<br>*Number track counting*, and *Before and after*—children walk along forwards or backwards<br> | Being a clock—children represented the numbers around the clock face and two children were the hands<br><br>Miming heavier or lighter—children acted out what it might be like carrying something heavy or light.<br>*Stand where?*—children locate themselves in a particular box to match instructions given in relation to left, right, forwards, backwards.<br> | Being the data—children used whole body to be the data points in a column graph.<br> |

Titles in *italics* in Table 1 indicate activities that have been refined and are included in the PL package being developed. Some of the activities were explored by the teachers over multiple lessons within a programmed unit of work. Several of the accompanying images within this paper are from when activities were 'recreated' in collaboration with the researchers and teachers as part of the development of the PL package that will be utilised in Phase 2.

# Findings

## *Incorporating Conceptual Body Movement in Mathematics*

Teachers reported in post-Phase 1 interviews that they "used a lot of gesture and a lot of body movement" [Lauren] as they were "the most naturally occurring in our classroom" [Lauren]. Melissa commented that she "did do a lot of body movement … I'm using it as a trigger for things … body movement would trigger–a memory of a learning." Crystal referred to it as "full body movement" in her interview stating it was an opportunity to "just do different activities where their full bodies involved". Of the lessons teachers reported on in SeeSaw, the use of body movement as an embodied learning mode was mentioned the most (in 19 of the 40 lessons). Teachers connected body movement to a wide range of mathematical processes and concepts: "to see how the students could read, describe and interpret results" [Rosa], "to make the patterns using their bodies" [Crystal], "to find the total then move that many steps forward" [Crystal], "to explore ten frames and addition using ourselves as counters" [Melissa], "to be directed to a number and move backwards and forwards" [Rhonda], "to make the numbers 1-4 using our bodies" [Melissa], "to walk like they were heavy or light" [Melissa], "to move with their bodies towards and away from positions including forward, backward, left and right" [Isla], "to create a clock using our bodies [Crystal].

## *Learning Connections Identified by Teachers*

In the interviews, teachers self-identified ways in which connecting mathematical processes to conceptual body movement was impacting their students' learning in positive ways. Rosa reflected that:

> You could just definitely see the improvements in them [the students] … I mean they're kids, to have them sit still for a long time, it doesn't work. So if they are up, using their body, they seem to make that connection.

Teachers provided examples of potential learning connections children made when lessons focused on conceptual body movement. Teachers reported learning gains, where "body movement enabled students to gain a deeper understanding of patterns, that they can be more than just colours repeated" [Crystal], and building children's conceptual development "to understand the concept of being straight and facing the number directly for us to be able to read the time" [Crystal]. Teachers were also able to identify potential misconceptions, "interestingly we had to correct some reversals (a huge focus on class) within body positioning" [Melissa], where assistance was needed, "students needed guidance and support to make the number line round. Spacing was mentioned by a few students" [Crystal], and a shift in confidence, "we used this line to do subtraction as well. Students who don't normally respond to questions were able to confidently answer the questions" [Crystal].

An interesting additional finding was the positive impact the project was having on teachers' pedagogical practice. Melissa reflected on how the project allowed for time to try new practices:

> I mean, we knew there's more than one way, but there's actually more than two ways and more than three ways and that that it doesn't have to be so regimented with the teaching. It gave us a little bit more freedom to experiment with new things. And you know, things like that, doing body movement for maths, is not usually something typically we might tie together, but it kind of opened that scope.

Teachers also reflected on their own understanding and interpretations of the embodied learning principles. Crystal discussed a lesson using number lines:

> Rosa and I went outside to trial a body movement lesson. Students were given a simple equation and had to find the total then move that many steps forward. Students enjoyed moving but we realised this movement was not embodied learning.

## Discussion and Conclusion

Teachers were able to easily incorporate conceptual body movement as a mode of representation into daily classroom lessons. Teachers themselves reported that body movements were one of the embodied learning modes they could repeatedly enact on a regular basis. These activities were an opportunity to identify misconceptions such as number reversals by Melissa, or areas that need further explicit teaching such as number spacing related to clocks mentioned by Crystal. Incorporating body movement was an opportunity to notice conceptual understanding, to assess knowledge, and to build confidence in reluctant speakers as reported by Crystal when exploring equations on the number line. Teachers found ways to weave the embodied learning principles (specifically body movement) into curriculum lessons utilising environmental spaces inside and outside the classroom as well as making use of physical mathematical structures such as number tracks, number lines, ten frames, and grid-structured classroom mats to assist students in developing spatial frames of reference.

Two teachers questioned whether or not the activities they were implementing aligned with conceptual body movement. Crystal's reflection in the number line lesson she and Rosa completed together is evidence of this self-reflection. The students' movement in the positive direction *is* related to the mathematical process of addition, therefore the activity *does* relate to conceptual boy movement. Nonetheless, observing teachers wrestle with the concept of conceptual body movement indicates their attention to making the connections between mathematical process and conceptual body movement correct, and explicit.

Input from the teachers was invaluable in creating and refining the activities to ensure they aligned to age-appropriate classroom practice as well as the theoretical framing of conceptual body movement. Similar to Garrett et al.'s (2018) findings, implementing activities that focused on conceptual body movement, "impacted teachers' pedagogical practices in various ways" (p. 9), where immediate changes were voiced by the teachers themselves. These initial findings may show "significant promise for improving students' learning engagement in mathematics as well as professional renewal for teachers" (p. 16) through the use of embodied learning principles. A future research direction might include observing when/if students use impromptu body movements as a thinking tool about the mathematical concepts, or do they choose to initiate body movements, without prompt by the teacher, as a communication tool.

## References

Dackermann, T., Fischer, U., Nuerk, H. C., & Cress, U. (2017). Applying embodied cognition: From useful interventions and their theoretical underpinnings to practical applications. *ZDM Mathematics Education, 49*(4), 545–557. http://doi.org/10.1007/s11858-017-0850-z

Garrett, R., Dawson, K., & Meiners, J. (2018). Creative body-based learning: Redesigning pedagogies in mathematics. *Journal for Learning Through Arts, 14*(1). http://doi.org/10.21977/D914136982

Shoval, E. (2011). Using mindful movement in cooperative learning while learning about angles. *Instructional Science, 39*(4), 453–466. http://doi.org/10.1007/s11251-010-9137-2

# *Finger Tracing, Noticing Structures and Drawing*

Jennifer Way
*University of Sydney*
jennifer.way@sydney.edu.au

Katherin Cartwright
*University of Sydney*
katherin.cartwright@sydney.edu.au

This paper presents an initial analysis of 10 Preschool children's responses to a look-draw-trace-draw task. The findings suggest that figure-tracing helped half the children to produce a more accurate representation of the geometric figure presented to them, in their second drawing.

The development of children's mathematical drawing capabilities is largely determined by developmental factors that span several years. Natural development of drawing abilities from playful scribble to realistic representations of imaginings and external objects takes time (Machón, 2013), and is linked with both cognitive and motor factors. Hand-motor control is a crucial component of drawing skill and develops over time in young children (Cohen, Bravi, & Minciacchi, 2021). Cognitive flexibility and associated drawing flexibility (ability to adapt and change familiar figurative schemas) increases over time and with age (Ebersbach, & Hagedorn, 2011). To be able to enhance mathematical drawing of children, particularly those developing at slower rates than expected, teachers need strategies that produce positive outcomes in a shorter timeframe.

In mathematics education, children might be asked to use drawing as a representation of their thinking (an external representation of an internal representation), or to produce a record of tangible objects (external representation of an external representation). Representing a visible, external model through drawing is a different task to drawing an object from an internal image or graphic schema. To reproduce the appearance of an object, say a geometric figure (e.g., a 2D shape), the child needs to give attention to, or notice, the key characteristics of the figure. Therefore, strategies that help the child focus their attention and raise their awareness of task demands are likely enhance the child's drawing performance (Morra, 2005; Sutton & Rose, 1998). This line of thinking suggests that increasing children's 'noticing' might have an immediate effect on children's drawing reproduction accuracy, if other developmental factors are sufficiently advanced. Pointing and finger tracing techniques have been shown to increase performance in particular mathematical tasks in older children (E.g., Hu, Ginns & Bobis, 2015) and may assist children to attend to spatial or structural features of a figure. While pencil-tracing might also be helpful, finger-tracing evokes the genetically driven visual-attention response to pointing (Hu, Ginns & Bobis, 2015), and contact with the surface activates the sense of touch and hence a different part of the brain to 'looking' only.

For this paper we pose the question, what changes in the preschool children's drawings occur after finger-tracing a figure?

## Method

### *Context and Participants*

This study was imbedded within the Embodied Learning in Early Mathematics and Science (ELEMS) project which involved Preschool to Year 2 teachers and their classes in one school. Although data for this tracing-drawing study was collected from all four cohorts, only the preschool data has been tentatively analysed at this point. The 10 Preschool children (approx. 4 ½ years) with parental permission to participate are the focus of this paper.

### *Procedure*

a) In an individual task-based interview, the child was shown a geometric figure (see Figure 1) and invited to look carefully then draw what they saw. As soon as the child began drawing, the card was turned facedown so the figure was hidden.

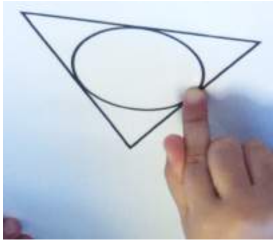


Figure 1. Child finger-tracing the geometric figure.

b) When the drawing was completed, the figure was again placed in front of the child and the interviewer asked them to trace around the shapes with their finger. To ensure the child understood the instruction, the interviewer demonstrated by pointing to the top left corner of the triangle and touch-tracing across the top of the triangle, around the corner (clockwise direction), then invited the child to trace it themselves. If the child did not automatically also trace around the circle, they were prompted to do so, again beginning the trace at the top and moving clockwise.

c) The child was then invited to draw the figure again, on a new sheet of paper, and the figure was again hidden from view.

Analysis was exploratory and open-ended and used several approaches to examine both the product (finished drawings) and process (video of drawing actions). The pairs of drawings for each child were compared for changes and annotated with arrows and numbers to indicate the drawing process. Observation notes were added to capture some key changes, features, or additional information from the videos. The pairs of drawings were grouped according to the magnitude of change between Drawings 1 and 2.

## Findings

The drawings were idiosyncratic, both in process and product, with few patterns identifiable in the small sample. Some observations are:

- Half the pairs of drawings show a definite change in structure and detail in the second drawing (See Figure 2). A notable change for P107 is from drawing two separate shapes to one shape enclosed inside another. Another significant change is from a single stroke to a closed shape (P116).
- Three pairs of drawings showed minimal changes, but the first drawings were already well formed. Small changes were a slightly larger circle or slightly 'pointier' triangle corners. (See Figure 3).
- Two children produced highly idiosyncratic pairs of drawings that were very different the drawings of the other children (See Figure 4).
- All children drew a closed shape (P116 only after tracing), and most were recognisable as a triangle. Most drew a recognisable circle inside.
- No child succeeded in drawing a circle that touched all three sides of the triangle, though P103 tried to make such an adjustment in her second drawing.
- All children except P112 drew the outer shape (triangle) first using a continuous line.
- The starting point and direction of drawing varied. Although some children changed this in the second drawing, it did not seem to be influenced by the tracing sequence modelled by the interviewer.
- Two children persisted with drawing the triangle upside-down relative to the figure presented to them (P213, P109).

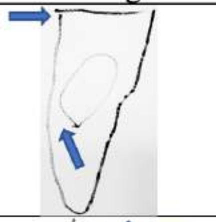
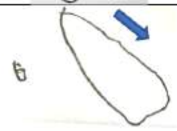
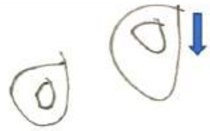
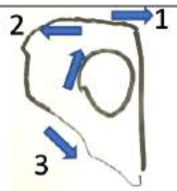
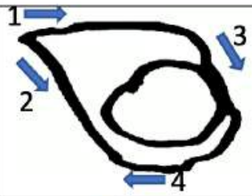
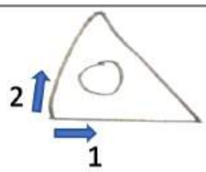
| Student code | Look then draw. Drawing 1 | After finger trace Drawing 2 | Notes |
|---|---|---|---|
| P103 |  |  | Explained scribble by saying, "I wanted it to touch". |
| P107 |  |  | From two separate shapes to one shape enclosed by another. Continued to fill the page with the same figure, |
| P110 |  |  | Awkward pen grip. Very slowly drew separate pieces. Orientation unclear. Inner circle enlarged. |
| P116 |  |  | Poor pen grip. Needed coaching to trace all around. From a single stroke to a closed shape |
| P213 |  |  | Immediately said, "It's like a triangle with a circle", but drew 4-sided shape. Changed from 4 sides to 3 sides. Incorrect orientation of triangle. |

Figure 2: Before and after drawings that show change.

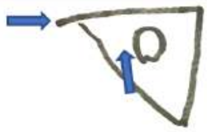
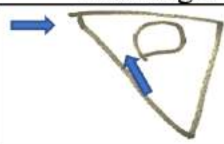
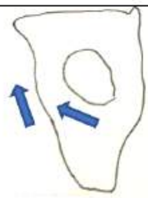
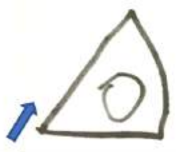
| Student code | Look then draw. Drawing 1 | After finger trace Drawing 2 | Notes |
|---|---|---|---|
| P104 |  |  | Minimal change |
| P108 |  |  | Started in middle of left side. Drew quickly. Lack of straight lines. Change to sharper corners. |
| P109 |  |  | Maintained incorrect orientation of triangle. Circle larger. |

Figure 3. Before and after drawings with minimal change.

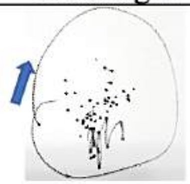
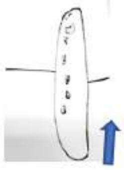
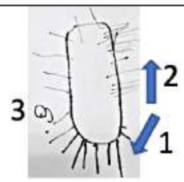
| Student code | Look then draw. Drawing 1 | After finger trace Drawing 2 | Notes |
|---|---|---|---|
| P112 |  |  | Drew circle first, then enclosed with 'triangle'. Needed prompting to trace circle, but did so anticlockwise. Quickly drew a large circle then giggled |
| P113 |  |   | Could not follow sides of triangle all the way around when tracing. Drew large shape, then small 'curl' outside, then added strokes. |

Figure 4. Highly idiosyncratic drawings.

## Discussion and Conclusion

With the limitations of this small exploratory study in mind, we offer some speculative interpretations of the findings. One explanation for the minimal-change in drawings (Figure 4) is that these children held stable mental images of triangles and circles and the task evoked existing schema (Ebersbach, & Hagedorn, 2011) useful for drawing the composite figure. P110 and P116 (Figure 3) struggled with a lack of fluidity in hand movement but the role this played in how they responded to the tracing experience is unclear (Cohen, Bravi, & Minciacchi, 2021). Both P107 and P116 exhibited a remarkable change in geometric structure, in a topological sense, by moving from separate shapes to enclosed shapes, and a single line to a closed shape respectively. It seems likely that the act of tracing stimulated this change somehow. The odd second drawing produced by P112 can be accounted for as playfulness. P113 produced drawings that are classic examples of the drawing stage of exploring combinations of forms, typical around age 3 years (Machón, 2013), which suggests the child was not ready for the type of drawing task used in this study.

In conclusion, the 'self-correction' of drawings by half the children suggests that the finger-tracing may have supported these children's noticing of the structure of the figure presented to them. The preliminary findings from this small sample provide encouragement for continuing the exploratory analysis with the data from the 5- to 8-year-olds and refining the analysis techniques in preparation for further studies, in which the role of memory in 'hidden figure' tasks should be considered.

## References

Cohen, E., Bravi,R., & Minciacchi, D. (2021). Assessing the development of fine motor control in elementary school children using drawing and tracing tasks. *Perceptual and Motor Skills, 128*(2), 605–624.

Ebersbach, M., & Hagedorn, H. (2011). The role of cognitive flexibility in the spatial representation of children's drawings. *Journal of Cognition and Development, 12*, 32–55. doi:10.1080/15248372.2011.539526

Hu, F. T., Ginns, P., & Bobis, J. (2015). Getting the point: Tracing worked examples enhances learning. *Learning and Instruction, 35*, 85–93. doi:10.1016/j.learninstruc.2014.10.002

Machón, A. (2013). Children's drawings: The genesis and nature of graphic representation: A developmental study. Madrid: Fibulas.

Morra, S. (2005). Cognitive aspects of change in drawings: A neo-Piagetian theoretical account. *British Journal of Developmental Psychology, 23*, 317–341.

Sutton, P. J., & Rose, D. H. (1998). The role of strategic visual attention in children's drawing development. *Journal of Experimental Child Psychology, 68*, 87–107.

# *Changes in Year 2 Children's Drawings of a Subtraction Story*

Jennifer Way
*University of Sydney*
jennifer.way@sydney.edu.au

Katherin Cartwright
*University of Sydney*
katherin.cartwright@sydney.edu.au

There is an educational expectation that children's natural drawing will develop into proficient mathematical representations and formal diagrams, yet there is little research available to guide the assessment and development of children's mathematical drawing skills. The aim of this paper is to explore how Year 2 children (approx. 7 years) chose to represent their interpretations of a simple story that is suggestive of the take-away subtraction process, and what changes occurred when the drawing task was repeated 6 months later. Analysis of 13 pairs of drawings revealed changes in what the children drew (categories of number representations) and how they drew it (style). The findings suggest that substantial change in children's representational ability can occur in within 6 months.

Children's representational competence in drawing has been linked with cognitive maturity and flexibility (Brooks, 2009), particularly regarding mathematical development. Children's drawing is also a source of evidence for internal "processes of notational competence and representational change" (Karmiloff-Smith, 1990, p. 58). Although drawing is a naturally developing ability in young children (Brooks, 1990) it can also be influenced by environmental factors including adult interactions (Malanchini et. al., 2016), making drawing development pertinent to teaching practice. Indeed, supporting children's development of drawing schemas, particularly dynamic schematisation (depicting movement and change) can enhance both drawing skill and mathematics comprehension (Poland & van Oers, 2007).

In the context of the Embodied Learning in Early Mathematics and Science project, the Pre-school to Year 2 teachers at one school explored supporting the development of children's drawing through increasing the opportunities for children to draw, discuss their drawings and experience some teacher-modelling of ways of drawing mathematical objects and processes. Pre-school to Year 2 students completed the 'Birds drawing task' in May 2022 (Time 1) as part of a larger assessment of drawing development requested by the teachers. In December 2022 (Time 2), an opportunity arose to repeat the drawing task with participating students. The 'Birds drawing task' is a very brief story used as a provocation to draw (Way, 2018), which is suggestive of a subtraction process.

This paper is focused by the questions: How do Year 2 children represent through drawing, the subtraction process implied by a simple 'take-away' story? and What changes in drawings are evident after 6 months?

## Procedure

The task instructions for the 'Birds drawing task' were delivered verbally to the group of children.

> Say: 'Listen to this little story. Then I'm going to ask you to draw what happened.'
> 'Five birds sat in a row along the top of a fence. Two birds flew away.'
> Repeat the story, then ask them to, 'Draw what happened in the story'.

Only the data from Year 2 students is used in this paper, as an initial development of the analysis technique. In one of the Year 2 classes, 13 students were present for both Time 1 and Time 2 of the drawing task, and these 26 drawings are the subject of this paper.

The modelling of the 'take-away' subtraction process can be described as a sequence of three steps: 1. Represent the original quantity in a group, 2. Separate or 'take-away' the relevant number items, 3. Determine the number of items remaining. Steps 1 and 2 are dynamic—requiring movement of some type. Step 3 implies some form of acknowledgement of the result of the process.

Two approaches were used in the analysis of the drawings. The first approach involved sorting the Time 1 drawings into categories according to whether they depicted one, two or three steps in the subtraction process, or no steps. The process was repeated for the Time 2 drawings. The second approach involved comparing the two drawings produced by each student and examining the nature of changes in the style of the drawing.

## Findings

### *The Drawing Categories*

The categories are presented in order of the completeness of the depiction of the subtraction process, ranging from non-depiction of any step in the subtraction process, to depiction of all three steps.

*Category 1 non-depiction.* The drawing does not depict any recognisable numerical information from the 'story', nor suggest any part of the subtraction process (Figure 1).

Multiple items that do not match any of the group quantities. (2C01-Time 1)

Figure 1. Example of non-depiction.

*Category 2 one step.* One step of the 3-step process of take-away subtraction is drawn: the 5 original birds, or the 2 that flew away, or the 3 that remained, with sub-categories identified based on the specific number of birds depicted (Figure 2)

Group of 2 (2C13–Time 1)

Group of 3 (2C20-Time 1)

Figure 2. Examples for each sub-category of one step drawings.

*Category 3 two steps*. Two of three steps are drawn depiction either 5 birds and the 2 that flew away (total of 7 birds), or the group 5 birds is partitioned into groups of 3 and 2. The partitioning is typically represented by separation of the subgroups by distance but may involve crossing out of 2 birds or arrows/lines indicting movement away (Figure 3).

*Category 4 three steps*. Some drawings included a strategy for focusing on the remaining 3 birds, as well as depicting the original group of 5, and the 'taking away' of 2 birds, even though the 'story' did not mention the remaining group of three, nor ask for 'how many left?'. Three-step drawings complete the operation of take-away subtraction and could be construed as also representing the equation 5 - 2 = 3 (Figure 4).

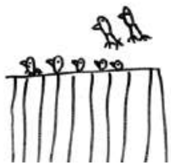
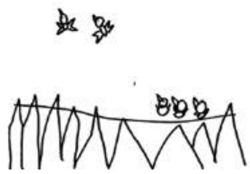
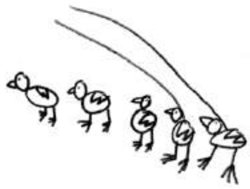
|  5 birds and the 2 that flew away (2C03-Time 2) |  5 birds ‘partitioned’ into 3 and 2 (by distance) (2C21-Time 2) |  5 birds ‘partitioned’ into 3 and 2 (by ‘arrows’) (2C05) |
|---|---|---|

Figure 3. Examples of two steps drawings.

|   “Now there is three birds” Use of words to represent the third step (2C04-Time 1) |  A 3-part story board: 5 birds, flying away, 3 left (2C15-Time 2) |
|---|---|

Figure 4. Examples of strategies for depicting the three steps.

### *Changes in Category and Style*

No child drew the same drawing both times with changes in style and/or changes in the parts of the subtraction process they chose to depict (category change). Table 1 shows the distribution of students’ drawings across the categories. Examining the table to match the student codes in the Time 1 and Time 2 columns reveals changes in categories by individual students. Most drawings from both Time 1 and Time 2 depicted the two steps in the story, and hence two steps in the subtraction process.

**Table 1**

*Distribution of Students’ Drawings Across the Categories, for Time 1 and Time 2*

| Approach Category | Sub-Category | Time 1 Drawings—May (Student codes) N=13 | Time 2 Drawings—December (Student codes) N=13 |
|---|---|---|---|
| Non-depiction | | 2C01* | |
| 1 step | 5 birds | 2C14* | 2C14 |
| | 2 birds | 2C13 | |
| | 3 birds | 2C10 2C20 | |
| 2 steps | 5 and 2 (total 7) | 2C05 2C06* 2C15 | 2C03 2C06* 2C04 |
| | 3 and 2 (partition) | 2C08* 2C17 2C21* | 2C01 2C08 2C13 2C20 2C05 2C10 2C17 2C21 |
| 3 steps | 5 - 2 = 3 | 2C03 2C04 | 2C15 |

*Signifies an incorrect number of items drawn (e.g., 6 birds rather than 5).

A noticeable shift in distribution in Time 2 is towards 2-step drawings that show the partitioning of the group of 5 into subgroups of 3 and 2, rather than representing two quantities specified in the story (5 and 2), or only one of the groups from the story.

Only three students did not substantially change their style of drawing. About half the students produced changes in both category and style (For example, Figures 5 & 6).

Most changes in style involved a more mature representation of the birds, showing some distinctive characteristics such as body shape, as can be seen in Figure 5. The most striking change in style was produced by student 2C01 (Figure 6) with a change from a drawing lacking any features of the story, to a drawing that shows five birds (circles) with two crossed out.

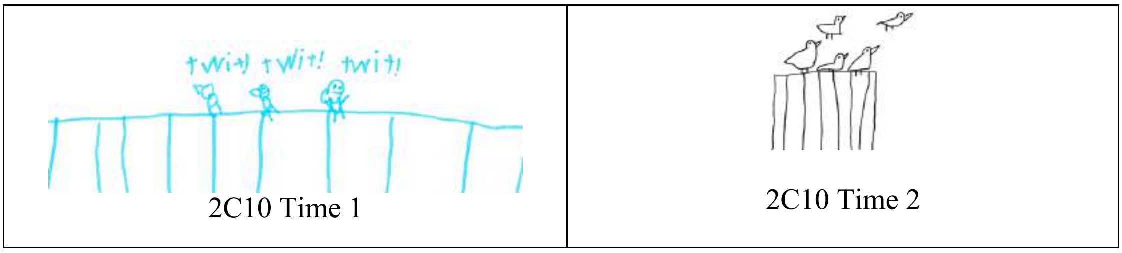



Figure 5. Example of change in both style and category.

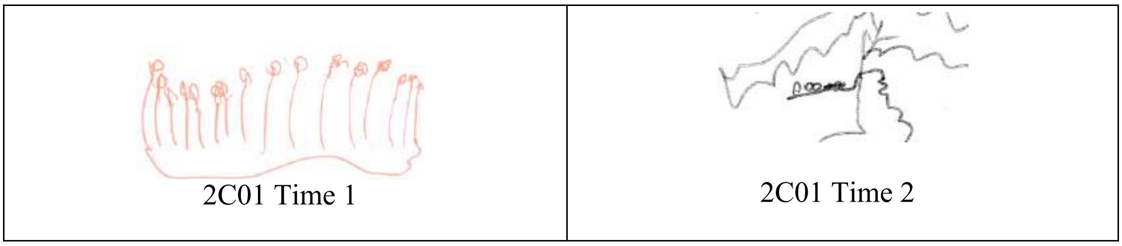



Figure 6. Example of change in both style and category.

## Discussion and Conclusion

It is important to note that the story-task was not intended as an assessment of the children's knowledge of subtraction, but rather and an opportunity to study how they responded. Using the categories related to the steps in the take-way process revealed that, in 6 months, the children had an increased tendency to mathematise the story and represent the partitioning of a group of five into groups of two and three. Comparing the pairs of drawings showed a shift in representational maturity. These findings contrast with the relative stability in 'human figure' drawing over 6 months found by Malanchini et.al (2016). Although no direct claim can be made about the role played by the ELEMS project teachers' increased attention to drawing development, the results do illustrate that substantial development in mathematical drawing skill can occur within 6 months. The analysis procedure will now be applied to the full collection of Preschool to Year 2 drawings to explore age-related patterns and other relationships between Time 1 and Time 2. Further research is needed to develop drawing tasks and interpretation guidelines that teachers can use to monitor their students' drawing development and support development of mathematical drawing ability.

## References

Brooks, M. (2009). Drawing, visualisation and young children's exploration of "big ideas." *International Journal of Science Education, 31*(3), 319–341. DOI:10.1080/09500690802595771

Karmiloff-Smith, A. (1990). Constraints on representational change: Evidence from children's drawings. *Cognition, 34*, 57–83.

Malanchini et.al. (2016). Preschool drawing and school mathematics: The nature of the association. *Child Development 87*(3), 929–943. DOI: 10.1111/cdev.12520

Poland, M. M., & van Oers, B. (2007). Effects of schematizing on mathematical development. *European Early Childhood Education Research Journal, 15*(2), 269–293. DOI: 10.1080/13502930701321600

Way, J. (2018). Two birds flew away: The 'jumble' of drawing skills for representing subtraction pre-school to Year 1. In Hunter, J., Perger, P., & Darragh, L. (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 98–101). MERGA.

# Research Papers

# Tertiary Students' Understanding of Sampling Distribution

Adeola Ajao
*University of Tasmania*
adeola.ajao@utas.edu.au

Noleine Fitzallen
*University of New South Wales*
n.fitzallen@unsw.edu.au

Helen Chick
*University of Tasmania*
helen.chick@utas.edu.au

Greg Oates
*University of Tasmania*
greg.oates@utas.edu.au

In this paper, the SOLO taxonomy is used to identify different levels of student understanding of the statistical concepts associated with sampling distribution. This study was part of a research project investigating students' conceptual understanding of concepts of hypothesis testing taught with the support of simulation learning activities. The study involved eight students enrolled in a first-year tertiary introductory statistics unit and the examination of their written responses to three questions about sampling distribution concepts. The SOLO taxonomy categorisations revealed that some students had only pre- and unistructural understanding of sampling distribution, and none providing responses at the extended abstract level.

The teaching of statistics has developed and evolved both in terms of curricula and pedagogical practices, yet statistics is still a difficult subject to teach and learn (Horton, 2015). Over time, researchers have investigated what makes the subject of statistics difficult. For example, Watts (1991) suggested that maybe the terms used in statistics—such as "a mean", "variance", "probability," or "a random variable"—together with the abstract nature of the associated concepts are the crux of the difficulties students encounter. Other factors associated with students' difficulties include changes in curriculum (Duckworth & Stephenson, 2002), understanding ideas like resampling and random distribution (Larwin & Larwin, 2011), debate about probability as a separate discipline from statistics (Zieffler et al., 2018), inability to meaningfully interpret problems in statistics (Reaburn, 2014), understanding statistical inference (Chance et al., 2022; Kula & Koçer, 2020), and the lack of individual experience with data when teaching certain topics like sampling distribution (Sued & Valdora, 2021).

The ideas of population and sample are two key concepts in inferential statistics. Students must first understand that the sample they are using is merely one of a huge number of samples that may be taken from the population, and next, to draw conclusions, that the distribution of the means of these samples must be known or modelled. The "sampling distribution" is a crucial idea in statistical inference; the outcome of repeatedly drawing samples from a population of a fixed size, calculating the sample statistic's value (invariably, the mean) for each sample, and then forming a distribution of those values (see, e.g., Bruce, 2014). However, this concept is often poorly understood (Ozmen & Guven, 2019; Sued & Valdora, 2021; Watkins et al., 2014). This report focuses on the learning of sampling distribution.

Why is the concept of sampling distribution hard to teach and learn? Sampling distribution is a multi-faceted abstract concept compared to elementary statistical ideas (Kula & Koçer, 2020; Watts, 1991). In addition, the sampling distribution is more an hypothetical distribution than it is an experiential distribution (Watkins et al., 2014). That is, when students learn sampling distribution, they cannot experience building a full sampling distribution, but instead must abstract it. As well, students commonly develop misconceptions of the complex terminology that is essential to understanding sampling distribution. These misconceptions involve the sample mean, effect of sample size, variance in sample means, standard deviation (Watkins et al., 2014), standard error of

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 73–80). Newcastle: MERGA.

the mean, and improper use of probability language (Chance et al., 2004). Student misconceptions about sampling distribution include:

- The sampling distribution should look like the population distribution (for $n > 1$),
- Sampling distributions for small samples and large samples have the same variability,
- Sampling distributions for large samples have more variability,
- A sampling distribution is not a distribution of sample statistics,
- One sample (of real data) is confused with all possible samples (in distribution) or potential samples,
- The law of large numbers (larger samples better represent a population) is confused with the central limit theorem (distributions of means from large samples tend to form a normal distribution), and
- The mean of a positive skewed distribution will be greater than the mean of the sampling distribution for large samples taken from this population (Ben-Zvi, 2004).

To understand sampling distribution, it is necessary to conceptualise sampling from a population, sampling variability, effect of sample size, long run frequency, that is, knowing that the process of random selection causes variability in outcomes, but a stable distribution is achieved over a long run (Pfannkuch et al., 2012), and an understanding of the relationship between population parameters and sample means(Ozmen & Guven, 2019).

One teaching strategy employed in tertiary statistics education is the use of simulations as a means of depicting real life situations. Statistical simulations allow multiple retrials of a given sample to generate a larger data size (Blejec, 2003), and are known as a computer simulation when a computer is used as a means of generating virtual data quickly. Simulations are believed to be effective because it takes advantage of the dual relationship between a distribution and a sample from that distribution. Learners can control the simulation, allowing them to discover the effects of changing the sample size by generating various sample sizes (Hesterberg, 2015).

Although, the teaching of sampling distribution has been researched extensively (Garfield et al., 2015; Kula & Koçer, 2020; Ozmen & Guven, 2019; Watkins et al., 2014), the research focus has been mostly on statistical inference. For example, Chance et al. (2022), Makar and Rubin (2018), Morris et al. (2019), Rossman and Chance (2014), Sigal and Chalmers (2016) all explored tertiary student understanding of sampling distribution within the context of statistical inference but with using simulation. The various principles related to sample distribution are what make it complex. Using the SOLO taxonomy (further discussed in the analysis section), Watson (2004) studied pupils at various stages of reasoning about samples. Using the SOLO taxonomy, an analysis of variation—another concept connected to sampling distribution—showed different levels of knowledge development in a primary six class (Watson et al., 2022). The SOLO taxonomy will be used here to analyse students' understanding of sample distribution at various levels as ideas connected to sampling distribution are investigated. Hence, the research question addressed in this paper is:

- What does a SOLO taxonomy categorisation tell us about students' understanding of sampling distribution?

## Research Methods

The research reported in this paper adopted a pragmatist paradigm (Mackenzie & Knipe, 2006) to explore tertiary students' understanding of sampling distribution. The methodology was chosen to capture specifically the students' level of understanding of sampling distribution demonstrated using designated statistical simulations. The research was conducted in a first-year undergraduate statistics unit over three semesters at a regional Australian university. The unit, *Data Handling and Statistics 1,* was taken by students from courses that require a foundational statistics unit and was offered over a period of 12 weeks in both Semester 1 and 2 of the academic year. Students in this study participated in the unit by attending weekly 1-hour lectures and 2-hour tutorial sessions. Assessment of the unit comprised written responses to quiz questions, completion of projects, and a final examination.

Four simulation activities that focused on different statistical topics were implemented as a pedagogical intervention in lectures. The focus of this report is the extension tasks that followed one of these simulation activities, which focussed on sampling distribution. The learning intentions of this activity and the extension tasks were to understand sampling distribution, simulation, sampling bias, and randomness. The first author participated as an observer during the implementation of the activity. The data collected for this study comes from the eight participants who were enrolled in the unit, gave informed consent to participate in the research, and provided written responses to the extension tasks. These were completed in their own time after the lecture that included the simulation activity. Participants also engaged in after-class activities, one of which is reported and discussed in this paper.

### *Task*

Scenarios used in the simulation activity and the questions used in the extension tasks afterwards were sourced from a moderated statistics blog called, *Ask Good Questions: A Blog About Teaching Introductory Statistics* (https://askgoodquestions.blog/). A simulation activity (Gettysburg Address) was implemented in the lecture. The Gettysburg Address is an activity where students were given a speech by Nelson Mandela, were asked to circle any ten words, and then asked to find the average number of letters for each word in their sample. This was first done manually by recording, on paper, the words circled and the number of letters in each word, and then calculating the average number of letters. Students then shared their means and started to manually build a sampling distribution. They were then introduced to a simulation app, which selected ten words from the Gettysburg at random (without showing the actual words), and then calculated the mean automatically. They also produced 20-word samples, determined the means, and produced a distribution of the means. At the end of the lecture, students were provided with an online link to the extension tasks, which were the stimulus for the data collection in this study, and which are described in the results.

### *Data Analysis*

The Structure of the Observed Learning Outcome (SOLO) taxonomy (Biggs & Collis, 2014) was used to analyse the data. SOLO is a qualitative model of assessment, established from existing cognitive models, which seeks to assess students' conceptual knowledge at varying levels of understanding. SOLO helps to separate the respondents from their responses, measuring only their knowledge or level of performance demonstrated at a particular time. This form of assessment has been used successfully to evaluate both summative and formative assessment in statistics education (e.g., Groth & Bergner, 2006; Watson et al., 2022). The SOLO taxonomy has five levels (Biggs & Collis, 2014), namely:

*Prestructural.* Responses at this level could be "I don't know", or involve merely reiterating the question, or have no relation to the question.

*Unistructural.* Responses might use a single cue from the question or relevant domain. There is no interrelationship of ideas.

*Multistructural.* Responses provided or interpreted at this level incorporate two or more aspects relevant to the questions, but they may not be interrelated, and conclusions in the response might be inconsistent.

*Relational.* Responses at this level weave together all connecting aspects in the response and make a coherent whole response to a given question.

*Extended abstract.* Responses at this level involve generalisation, set out principles on which responses are based, and may indicate application to other situations.

The students' responses were initially categorised against the SOLO levels by the first author. They were then categorised independently by the third and fourth authors, and any discrepancies were discussed and resolved.

## Results

Comprehensive understanding of sampling distribution is evidenced by understanding of the sample mean, effect of sample size, variance in sample means, standard deviation (Watkins et al., 2014), standard error of the mean, and correct use of probability language (Chance et al., 2004). Presented below is each scenario with the corresponding questions, followed by a model response (sourced from https://askgoodquestions.blog/), a brief discussion about cogent words or description necessary for answering the question, general description of varying level of participants' responses using SOLO (Biggs & Collis, 2014), two examples of a given response, and a justification for the level of categorisation determined.

### *Scenario 1: Cats*

| |
|---|
| Assume that domestic housecats' body lengths (excluding the tail) have a mean of 18 cm and a standard variation of 3 cm. |

*Question 1*. (Cat tail length longer than 20cm).

> Which is more likely—that a randomly chosen cat's length is longer than 20 cm, or that the average length of a randomly selected sample of 50 cats is more than 20 cm, or are these probabilities equal? Describe your thinking.

*Model response.* Since a length of 20 cm is longer than the mean and average values vary less than individual values, the likelihood of a cat exceeding 20 cm is higher than the likelihood of the sample average exceeding 20 cm.

*Discussion.* The model response implies variation (variation in the sample means, but greater variation in the individual values); when the mean of the whole population is 18 cm, the sample averages will be around this value, and thus likely to be less than 20 cm, whereas a single cat's length is more likely to be longer than 20 cm (correct use of probability language).

Out of eight respondents, most responses used the correct probability language, and most understood variation in sample mean. Three responses were classified as multistructural because they referred to multiple elements, such as the sample average, sample size, variation in mean, standard deviation and the correct use of probability language. One response was considered relational because it used mathematical calculations to back up its reasoning, and the other four were either pre-structural (e.g., gave vague responses, or just reiterated the question), or unistructural (only referred to the variation in mean or standard variation). Of particular interest here are two responses that were difficult to categorise.

> Response A: I think the probability of a sample of fifty cats having its mean fall at 20 cm is more likely than a single cat measuring exactly 20 cm, because 20 cm is only 1 standard deviation away.

Response A could be classified as a unistructural response or a pre-structural response. On one hand, it could be argued that the response was unistructural because it states a statistical claim with reference to one element of the problem (standard deviation). The claim, "because 20 cm is only 1 standard deviation away," shows that the respondent had some understanding of standard deviation, although the concept of standard deviation was not well-used. On the other hand, it could be said the response was pre-structural because the standard deviation is not relevant in this case. This response was ultimately categorised as unistructural because the standard deviation was applied in the justification.

> Response B: The first probability is larger. The main difference is that when working with a sample, we use the standard error of the sample, which is equal to standard error divided by the square root of the sample size. p1 = p (> 20) = 1-NORM.DIST(20, 18, 3, 1) = 0.25; p2 = p(average length > 20; n = 50) = 1-NORM.DIST(20, 18, 3/sqrt 50, 1) = 0.0000012.

Response B was debated among the authors, with a multistructural categorisation initially considered, based on insufficient written explanation to address the instruction that said, "Describe your thinking." However, it was ultimately categorised as relational because the respondent justified the choice of standard error, chose the correct distribution, and then performed the correct calculations to justify the claim that the first probability was larger.

*Question 2*. (Cat tail length between 17–19 cm).

> Which is more likely—that a randomly chosen cat's length falls between 17 and 19 cm, or that an average length of 50 randomly chosen cats falls which this range, or are these probabilities equal? Describe your thinking.

*Model response.* Because the range is centered on the population mean, the probability of length being between 17 and 19cm is greater for a sample average than for an individual cat.

*Discussion.* In the analysis of this question, some responses related the sample mean to the population mean, which showed understanding of variation, sample mean, effect of sample size, and the use of correct probability language. Using the SOLO categorisations, most responses were at a unistructural level, and while two respondents realised that question one and question two were similar, the way this connection between question was stated brought about a question of where their responses should be placed on the SOLO categorisation scale.

> Response A: The same reason.

No other explanation was given, so if we go by the definition of the prestructural SOLO level, then, "The same reason" was considered prestructural. However, it is asking to take the justification for the other response (response given in question one) as the response for question two. If we go by the given response in question one "The second one should be larger generally as a more stable and closing to normal distributing data," which was categorised as unistructural, then we might categorise the response to question two as also being unistructural.

> Response B: Using the same logic as in the last question, the probability that the sample of 50 is between 17 and 19 cm will be higher.

Response B was categorised as multistructural, because the respondent was able to relate the first question to the second question using the phrase "using the same logic in the last question." In the previous question the participant wrote:

> The probability that the length of a randomly selected cat is no longer than 20cm is higher. The standard error in the sample of 50 will be lower than the standard deviation for one, as there is a 1/sqrt (50) multiplier. Thus, the confidence intervals will be smaller for the sample.

This response was also categorised as multistructural with respect to Question 1.

### *Scenario 2: Hospital Problem*

> Suppose that a region has two hospitals. Hospital A has about 10 births per day, and Hospital B has about 50 births per day. About 50% of all babies are boys, but the percentage who are boys varies at each hospital from day to day.

*Question 3*. Hospital problem.

Over the course of a year, which hospital will have more days on which 60% or more of the births are boys—A, B, or negligible difference between A and B? Explain your option.

*Model response.* With a large sample, we would expect the mean proportion of boy births to be around 50%, whereas smaller samples will vary from this mean more often. Hospital A, which has fewer births (i.e., smaller sample size) will have more days with 60% or more of the births being boys because the smaller hospital will have more variation in the percentages of boys born on a day.

*Discussion.* The third question is a well-known statistical task in the field of introductory statistics. Out of eight respondents, most responses used the correct probability language. Three were prestructural because the question was only reworded or the response seemed like memorised theory with no depth to them, and the three unistructural responses only referred to variation in sample mean with no connection to other elements. Two individuals provided multi-structural responses to this question by explaining effect of sample size (see summary of responses in Table 1). Two responses of particular importance are recounted below.

Response A: Hospital A. With less data, a smaller denominator of the standard error, means a larger range.

Response A could be determined to be at a prestructural level or a unistructural level of learning outcome. This is because only one useful piece of information was mentioned (standard error) which shows some indication of reasoning (unistructural) but this singular use of a term without adequate expression might arise as merely a memorised term (prestructural).

Response B: Hospital A will have more variable days because of the low sample size.

Response B was categorised as multistructural as the respondent showed an understanding of variation and sample size, although the given response was not well expressed to show the implied relation between sample size of 10 versus 50. More connectivity between the two may have taken this response to the relational level. Also, the respondent could have been reiterating memorised theory, but it was not possible to confirm that possibility with the data available.

A summary of the responses for the three questions are presented in Table 1. Across the three questions and 24 responses in total, there were five prestructural, nine unistructural, nine multistructural, and one relational level responses.

**Table 1**

*Number of Responses for Each SOLO Level*

| | SOLO Level | | | |
|---|---|---|---|---|
| | Prestructural | Unistructural | Multistructural | Relational |
| Question 1 | 1 | 3 | 3 | 1 |
| Question 2 | 1 | 3 | 4 | 0 |
| Question 3 | 3 | 3 | 2 | 0 |

## Discussion

The analysis of the student responses demonstrated that participants generally understood the correct probability language that applied to the questions posed. Probability is one of the few topics in statistics studied from Years Foundation–12 in Australia (Australian Curriculum, Assessment and Reporting Authority, 2018), so it might be expected the participants have brought this familiarisation with the concept of probability from high school. This suggests that probability as an important idea in sampling distribution (Pfannkuch et al., 2012) could be an effective starting point for the teaching and learning of sampling distribution.

The five responses classified as either unistructural or multistructural support the observation that component elements of sampling distribution like “sample,” “sampling,” “variation,” and “averages” (Watson, 2004; Watson et al., 2022) are vital to the understanding of bigger concepts. Lack of such understanding, however, may underpin the remaining prestructural responses, suggesting the misconception of variability between large and small samples (Ben-Zvi, 2004). Although not confirmed in the data presented here, we further postulate that higher levels of understandings could stem from using simulation as a pedagogical approach in teaching this concept (Chance et al., 2022).

Students’ grasp of the ideas related to sampling distribution varied greatly, and using SOLO taxonomy proved helpful in distinguishing different levels of understanding. Higher order thinking abilities like analysis, synthesis, and assessment were encouraged by the form of the questions answered. However, only four of the SOLO taxonomy’s five levels of learning outcomes were evidenced in the responses given. None of the questions achieved the abstract extended level of learning outcome, which may be because the questions were not expressed specifically to yield answers at that level.

Categorisation of the data using SOLO was at times problematic because responses were brief, non-existent, or ambiguous. This illustrates the problem of presentation or internalisation (Reaburn, 2014), as observed in the ways respondents wrote their responses. Since understanding of the concept can only be inferred based on what is written, lower-level responses could be due to a misunderstanding of the question wording, its presentation, or accompanying instructions. This suggests additional data, such as from an interview, is needed to support the SOLO classifications. Interviews using open-ended questions would provide participants the opportunity to demonstrate understanding at the extended abstract level described by Groth and Bergner (2006). The results in Table 1 shows the majority of responses were unistructural or multistructural, with only one relational response This suggests that the simulation activity was not fully successful in supporting students’ conceptual understanding for the extension tasks, certainly in making the connections between the simulation observations and different contexts.

## Conclusion

The study reported in this paper differs from previous research in that it has looked at the understanding of the fundamental components of sampling distribution, rather than the more common focus on sampling distribution for statistical inference. The evidence presented here serves to shed light on the elements of sampling distribution best understood by students, and the levels of understanding observed in their responses. It highlights the complexity of sampling distribution and suggests that achieving sophisticated understanding of the concept is possible if the fundamental components of the concept are well-understood. The results from this report could potentially be used to inform a methodological approach in analysing students’ understanding of fundamental components of a statistical concept.

## References

Australian Curriculum, Assessment and Reporting Authority. (2018). *The Australian curriculum: Mathematics* (senior secondary). ACARA. https://www.australiancurriculum.edu.au/senior-secondary-curriculum/mathematics/

Ben-Zvi, D. (2004). Reasoning about variability in comparing distributions. *Statistics Education Research Journal*, *3*(2), 42–63. https://doi.org/10.52041/serj.v3i2.547

Biggs, J. B., & Collis, K. F. (2014). *Evaluating the quality of learning: The SOLO taxonomy (structure of the observed learning outcome)*. Academic Press.

Blejec, A. (2003). Teaching statistics by using simulations on the Internet. In *Statistics and the Internet. Proceedings of the IASE/ISI satellite conference*, Berlin, Germany. https://iase-web.org/documents/papers/sat2003/Blejec.pdf?1402524992

Bruce, P. C. (2014). *Introductory statistics and analytics: A resampling perspective*. John Wiley & Sons.

Chance, B., delMas, R., & Garfield, J. (2004). Reasoning about sampling distribitions. In D. Ben-Zvi, & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 295–323). Springer. https://doi.org/https://doi.org/10.1007/1-4020-2278-6_13

Chance, B., Tintle, N., Reynolds, S., Patel, A., Chan, K., & Leader, S. (2022). Student performance in curricula centred on simulation-based inference. *Statistics Education Research Journal, 21*(3), Article 4. https://doi.org/10.52041/serj.v21i3.6

Duckworth, W. M., & Stephenson, W. R. (2002). Beyond traditional statistical methods. *The American Statistician, 56*(3), 230–233. https://doi.org/10.1198/000313002173

Garfield, J., Le, L., Zieffler, A., & Ben-Zvi, D. (2015). Developing students' reasoning about samples and sampling variability as a path to expert statistical thinking. *Educational Studies in Mathematics, 88*(3), 327–342. https://doi.org/10.1007/s10649-014-9541-7

Groth, R. E., & Bergner, J. A. (2006). Preservice elementary teachers' conceptual and procedural knowledge of mean, median, and mode. *Mathematical Thinking and Learning, 8*(1), 37–63. https://doi.org/10.1207/s15327833mtl0801_3

Hesterberg, T. (2015). What teachers should know about the bootstrap: Resampling in the undergraduate statistics curriculum. *The American Statistician, 69*(4) 371–386, http://doi.org/10.1080/00031305.2015.1089789

Horton, N. J. (2015). Challenges and opportunities for statistics and statistical education: Looking back, looking forward. *The American Statistician, 69*(2), 138–145. https://doi.org/10.1080/00031305.2015.1032435

Kula, F., & Koçer, R. G. (2020). Why is it difficult to understand statistical inference? Reflections on the opposing directions of construction and application of inference framework. *Teaching Mathematics and its Applications: An International Journal of the IMA, 39*(4), 248–265. https://doi.org/10.1093/teamat/hrz014

Larwin, K. H., & Larwin, D. A. (2011). Evaluating the use of random distribution theory to introduce statistical inference concepts to business students. *Journal of Education for Business, 86*(1), 1–9. https://doi.org/10.1080/08832321003604920

Mackenzie, N., & Knipe, S. (2006). Research dilemmas: Paradigms, methods and methodology. *Issues in Educational Research, 16*, 193–205. https://www.iier.org.au/iier16/mackenzie.html

Makar, K., & Rubin, A. (2018). Learning about statistical inference. In D. BenZvi, J. Garfield, & K. Makar (Eds.), *International handbook of research in statistics education* (pp. 261–294). Springer International Publishing.

Morris, T. P., White, I. R., & Crowther, M. J. (2019). Using simulation studies to evaluate statistical methods. *Statistics in Medicine, 38*(11), 2074–2102. https://doi.org/10.1002/sim.8086

Ozmen, Z. M., & Guven, B. (2019). Evaluating students' conceptual and procedural understanding of sampling distributions. *International Journal of Mathematical Education in Science and Technology, 50*(1), 25–45.

Pfannkuch, M., Wild, C. J., & Parsonage, R. (2012). A conceptual pathway to confidence intervals. *ZDM-Mathematics Education, 44*(7), 899–911. https://doi.org/10.1007/s11858-012-0446-6

Reaburn, R. (2014). Introductory statistics course tertiary students' understanding of *p*-values. *Statistics Education Research Journal, 13*(1), 53–65. https://doi.org/10.52041/serj.v13i1.298

Rossman, A. J., & Chance, B. L. (2014). Using simulation-based inference for learning introductory statistics. *Wiley Interdisciplinary Reviews: Computational Statistics, 6*(4), 211–221. https://doi.org/10.1002/wics.1302

Sigal, M. J., & Chalmers, R. P. (2016). Play it again: Teaching statistics with Monte Carlo simulation. *Journal of Statistics Education, 24(*3), 136–156. https://doi.org/10.1080/10691898.2016.1246953

Sued, M., & Valdora, M. (2021). Each student with her/his own data: Understanding sampling distributions. *arXiv,* Article 2105.02727. https://doi.org/10.48550/arxiv.2105.02727

Watkins, A. E., Bargagliotti, A., & Franklin, C. (2014). Simulation of the sampling distribution of the mean can mislead. *Journal of Statistics Education, 22*(3). https://doi.org/10.1080/10691898.2014.11889716

Watson, J. M. (2004). Developing reasoning about samples. In D. Ben-Zvi, & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 277–294). Springer.

Watson, J., Wright, S., Fitzallen, N., & Kelly, B. (2022). Consolidating understanding of variation as part of STEM: Experimenting with plant growth. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-022-00421-1

Watts, D. G. (1991). Why is introductory statistics difficult to learn? And what can we do to make it easier? The *American Statistician, 45*(4), 290–291. https://doi.org/10.2307/2684456

Zieffler, A., Garfield, J., & Fry, E. (2018). What is statistics education? In D. BenZvi, J. Garfield, & K. Makar (Eds.), *International handbook of research in statistics education* (pp. 37–70). Springer International Publishing. https://doi.org/10.1007/978-3-319-66

# What Matters With Out-of-field Teaching: A Preliminary Analysis of Middle Years Teachers of Mathematics in South Australia

Amie Albrecht
*University of South Australia*
amie.albrecht@unisa.edu.au

Lisa O'Keeffe
*University of South Australia*
lisa.okeeffe@unisa.edu.au

Anne Morrison
*University of South Australia*
anne.morrison@unisa.edu.au

There has been a lot of media attention about 'out-of-field' teaching, but much of it does not capture the complexities of the term or account for the range of knowledge, experience, and attitudes relevant for teaching mathematics in the middle years. In this paper we report on a survey conducted to better understand the diversity of the profession teaching middle school mathematics (Years 7-10) in South Australia. A preliminary analysis indicates that self-identification as a teacher of mathematics is a key contributing factor to confidence in teaching mathematics. We suggest that looking through an identity lens will better inform planning and support for out-of-field teaching in mathematics.

Described as "education's dirty little secret" (Brodbelt 1990, p. 282), out-of-field teaching occurs when a teacher is assigned to teach one or more subjects for which they are not qualified or adequately trained. This phenomenon is reported in many countries, including Australia, Germany, Indonesia, Ireland, the UK, and the USA (Price et al., 2019), South Korea (Kim 2011), New Zealand (Post Primary Teachers' Association, 2018), and others. Terminology synonymous with out-of-field teaching includes mis-assignment (Sharplin, 2014; Van Overschelde, 2022), teaching across specialisations (Hobbs & Törner, 2014), non-specialist teaching (Goos et al., 2019), and teaching out-of-area (Hobbs & Törner, 2019, p. xi).

The scale of out-of-field teaching in Australia is greater than in other comparable countries (Marginson et al., 2013) and is a concern for every Australian state and territory (Weldon, 2016). The problems experienced globally may be exacerbated by Australian geographical complexities such as rural and remote communities. Using the 2013 Staff in Australia's Schools survey data, Weldon (2016) reported that 17% of mathematics classes in Years 7-10 are being taught by an out-of-field teacher but that the problem is inequitably distributed, with the figure being 26% of classes in remote locations compared to 14% in metropolitan locations. These issues can compound over multiple years of schooling, with modelling by the Australian Mathematical Sciences Institute suggesting that there is a 76% chance of being taught by an out-of-field teacher in Years 7 to 10 and that 'less than one in four Year 7 to 10 students [will] have an in-field maths teacher every year' (Prince & O'Connor, 2018, p. 3).

In this paper we explore the notion of out-of-field (OOF) mathematics teaching in a South Australian context. Like others, we posit that labelling teachers as OOF is complex, particularly when it does not account for the evolving nature of their knowledge (both pedagogical and content), experience and attitudes and, as such, may create deficit perspectives of OOF teaching that are unwarranted. We show data that points to the significance of teacher identity as a key factor that should be considered when planning responses to OOF teaching in mathematics.

## Defining Out-of-Field Teaching

Despite the widespread occurrence of out-of-field teaching, there is no single understanding of the phenomenon. The broad characterisation—'assigning teachers to teach subjects they are not qualified to teach' (Hobbs et al., 2022b, p. 5)—centres on criteria used to qualify teachers.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 81–89). Newcastle: MERGA.

Consequently, 'because of state, national and international differences in teacher registration, approval and certification, there is no single definition of what makes a teacher out-of-field' (Hobbs et al., 2021, p. 126). Hobbs and Porsch (2021) suggest a wider scope for out-of-field teaching, such as 'situations where teachers are learning something new, like a teacher learning to use new technology' (p. 369), while Hobbs et al. (2020, p. 1) note that, in practice, principals and teachers often judge the suitability of a teacher to teach a particular subject to a particular year level based on a range of factors and standards—not just qualification.

A multi-faceted definition of out-of-field teaching, as shown in Figure 1, was devised by Hobbs et al (2022a) based on work for the Victorian Department of Education and Training, and comprises four key categories which are elaborated below.

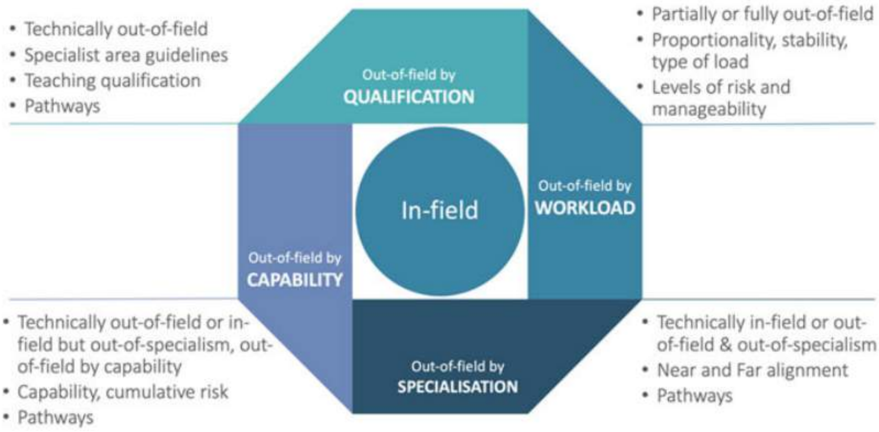



Figure 1. Multifaceted definition of teaching out-of-field from Hobbs et al. (2022a, p. 30).

In this paper we use an overarching definition of OOF aligned to the definition of 'OOF by qualification' by Hobbs et al. (2022a, p. 33), which refers to misalignment 'between the subject required to teach and [a teacher's] qualifications'. We label this QOOF. We follow Hobbs et al. (2022a) in also considering discipline qualifications (i.e., qualified to teach mathematics or not) and school level qualifications (i.e., qualified to teach primary or middle years) as part of QOOF, but separate them out as follows:

- QOOF-T: Technical misalignment between a teacher's discipline qualification and current teaching.
- QOOF-P: Phase misalignment between a teacher's school level qualification and current teaching.
- QOOF-B: Both technical and phase misalignment.

Teachers not meeting any of these criteria can be considered 'in-field by qualification', which we call QIN.

Hobbs et al. (2022a) refer to 'OOF by specialism' which considers whether there is misalignment between a teacher's qualification and the sub-discipline they are teaching. While this classification is clearly defined and understood in a composite subject area like science, it is less useful in mathematics. As such, we have not used a parallel definition in our study. For 'OOF by workload', which Hobbs et al. (2022a, p. 38) used to describe 'the proportion of load that is out-of-field at any one time or across a period of time, the stability of [a teacher's] workload allocation, and the type of load', we have used an adapted definition. We categorise the proportion of workload relative to total workload. However, this study was intended to provide a snapshot of respondents' current experiences and therefore we did not look at stability of teaching load over time.

Hobbs et al. (2022a) defined 'OOF by capability' as their final dimension, which relies on factors beyond qualification and relates to a teacher's 'perceived and/or actual capability'. Rather than make a judgement of teachers' capability, we have chosen to explore self-reported confidence, interest, and identity. In our study we have used these additional dimensions as ways in which to further understand the complexities of out-of-field by qualification.

## Research Design

The aim of this study, which employed a survey-research design, is to better understand the diversity of the profession teaching middle school mathematics (Years 7-10) in South Australia (SA). The survey was designed using the principles of Hobbs et al.'s (2020) classifications of 'out-of-field', AITSL's (2021) report on the SA teacher workforce, and Weldon's (2016) study of the out-of-field issue in Australia. The survey was distributed online, via the South Australian Department for Education, for one month. All middle school teachers were invited to complete the survey, irrespective of what they were currently teaching. We opened the survey to all middle school teachers (not just those teaching mathematics) in order to capture the voices of *all* teachers working within and across these years who could be impacted (either positively or negatively) by this issue. The research questions guiding the aspects of the study reported in this paper are:

- What are the in-field/out-of-field teacher demographics of SA middle school teachers of mathematics?
- What are the teacher attitudes and levels of interest, enjoyment, confidence and commitment in teaching mathematics?

## Findings

A total of 232 participants completed the survey, of which 196 have taught middle school mathematics during their career. Of this cohort of 196 teachers, 23 indicated they were teaching in a primary context (Year 6 and below) and 133 in a secondary school context (Year 7 and above) at the time the survey was conducted. This left 40 respondents who were not teaching mathematics. Each respondent was classified as QIN or QOOF (-T, -P, or -B) using information they provided about their teaching qualifications. Of the 133 secondary teachers, 60% are considered OOF, either by technical or phase misalignment, or both (Figure 2).

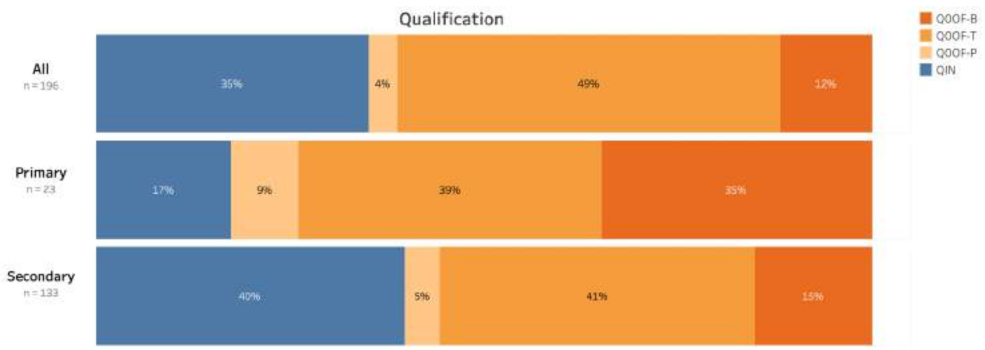



Figure 2: QIN & QOOF classification by phase (n=196).

Further exploration of the data for secondary teachers (n=133) indicates little variance between groupings by gender, and examining age revealed that 74% of respondents aged under 30 years are QOOF (-T, -P and -B), with similar percentages for the 30-39 and 40-49 age groups. In contrast, respondents aged 50+ years are more likely to be in-field by qualification, with 62% of those aged 50+ years and 68% of those 60+ years in the QIN group. Looking at teachers aged under 40 years, those teaching senior secondary mathematics (Years 11-12) are more likely to be in-field by qualification (45%) than those teaching in the middle years (30%).

The longer teachers have been in the profession, the more likely it is that they are in-field by qualification (QIN), regardless of whether they teach Years 7-10 or Years 11-12. A greater proportion of the teachers with 5-10 years' experience are out-of-field (71%) than those with less than 5 years' experience (69%). Given the findings that early career teachers are leaving the profession due to feeling unsupported and overwhelmed by workload (Johnson et al., 2014; Windle et al., 2022), this warrants further investigation.

For this sample of teachers there was little difference in the prevalence of QOOF based on school location across major cities, inner regional, and outer regional (56%, 53%, and 63% respectively) or by school category (between 60% and 66% across school categories 2-7; category 1 was 80% QOOF but with only 5 respondents cannot be considered representative).

To examine OOF by workload, we calculated the proportion of mathematics classes for respondents currently teaching in a secondary school context (n=133), and categorised by Low (0-25%), Low-Medium (25-50%), Medium-High (50-75%), High (75-100%). Figure 3 shows the proportion of workload by QIN and QOOF. There are two main features to note.

- 17% of teachers who are in-field by qualification have less than 50% of their teaching load in mathematics, with 4% having less than 25% allocated to mathematics. In the context of a shortage of teachers of mathematics, it may be worth exploring what 'other' teaching this cohort is doing.
- 50% of teachers who are out-of-field by qualification have more than 50% of their teaching allocated to mathematics.

For this sample of teachers there was little difference in the prevalence of WOOF (our term for OOF by qualification and with more than 50% of their teaching workload in mathematics) by gender. However, as age increases (and similarly, years of experience), the proportion of WOOF teachers increases. The WOOF proportion is relatively consistent across all locations (45% or more in each location), and common across all school categories. Around 50% of permanent teachers who are out-of-field by qualification are also out-of-field by workload.

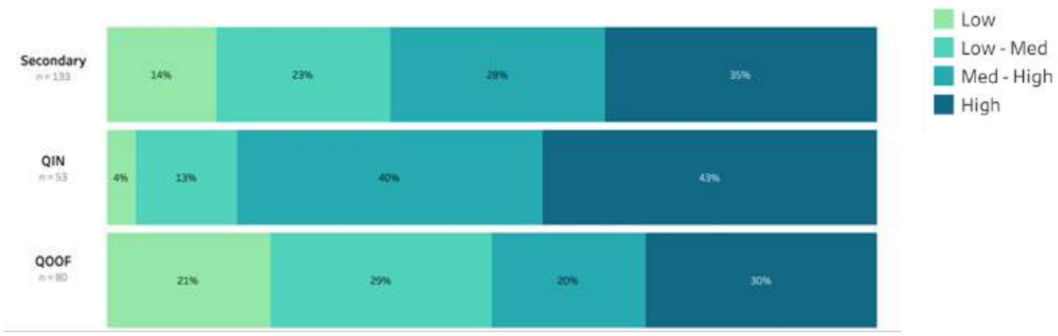



Figure 3. Secondary teachers' workload in mathematics.

## *Teacher Identity*

As well as classifying teachers as QIN or QOOF, we asked two questions about identity. The first was whether or not respondents self-identify as teachers of mathematics. Of the 176 teachers who have taught mathematics and who responded to the question, 76.7% (n=135) self-identify as teachers of mathematics, leaving 23.2% (n=41) who do not. The second question was whether or not they self-identify as out-of-field. Of the 175 teachers who have taught mathematics and responded to this question, 33.1% (n=58) self-identify as OOF, leaving 66.9% (n=117) who do not. For convenience, we refer to the last group as self-identifying as in-field, even though we didn't explicitly frame the question in this way. Using their responses, we can assign each respondent to one of four 'identity' groups. Table 1 summarises the number and percentage of respondents in each group for the total cohort who answered both questions (n=175; columns 2-3), as well as further broken down by whether they are in- or out-of-field according to their teaching qualifications, that is, QIN or QOOF.

Perhaps unsurprisingly, the majority of teachers who are in-field by qualification (QIN) also self-identify as teachers of mathematics and as in-field. Of particular interest are the QOOF teachers. Half of these teachers consider themselves in-field and half out-of-field. Of those who identify as in-field, 79% consider themselves teachers of mathematics. In contrast, of those who identify as out-of-field, only 51% consider themselves as teachers of mathematics.

**Table 1**

*Self-identification as a Teacher of Mathematics and In- or Out-of-field*

| | All (n=175) | | QIN (n=60) | | QOOF (n=115) | |
|---|---|---|---|---|---|---|
| | Self-identifies as teacher of maths (n=134) | Does not identify as teacher of maths (n=41) | Self-identifies as teacher of maths (n=59) | Does not identify as teacher of maths (n=1) | Self-identifies as teacher of maths (n=75) | Does not identify as teacher of maths (n=40) |
| Self-identifies as in-field (n=117) | 59.4% (n=104) | 7.4% (n=13) | 96.6% (n=58) | 1.7% (n=1) | 40.0% (n=46) | 10.4% (n=12) |
| Self-identifies as out-of-field (n=58) | 17.1% (n=30) | 16.0% (n=28) | 1.7% (n=1) | 0% (n=0) | 25.2% (n=29) | 24.3% (n=28) |

### *Teacher Interest, Enjoyment, Confidence, and Commitment*

Teachers were asked to indicate their personal interest in mathematics, their enjoyment in teaching mathematics, confidence in their mathematical content knowledge (CK), confidence in their pedagogical approaches for teaching mathematics (PCK), and their personal commitment to develop their own CK and PCK. Respondents rated their responses on a scale from 0 to 5, with 0 being low and 5 being high. The means are shown in Table 2. We grouped respondents by whether they were in- or out-of-field according to their teaching qualifications (columns 3-4). To explore the impact of identity, we also grouped respondents according to self-identity as a teacher of mathematics (columns 5-6) and out-of-field (columns 7-8).

**Table 2**

*Teacher Self-Reported Interest, Enjoyment, Confidence, and Commitment*

| | All (n=175) | QIN (n=60) | QOOF (n=115) | Self-identifies as teacher of maths (n=134) | Does not identify as teacher of maths (n=41) | Self-identifies as in-field (n=117) | Self-identifies as out-of-field (n=58) |
|---|---|---|---|---|---|---|---|
| Personal interest in mathematics | 3.92 | 4.58 | 3.58 | 4.42 | **2.32** | 4.28 | 3.21 |
| Enjoyment teaching mathematics | 3.88 | 4.37 | 3.63 | 4.36 | **2.34** | 4.14 | 3.37 |
| Confidence in CK | 3.90 | 4.68 | 3.48 | 4.38 | **2.34** | 4.38 | 2.91 |

| | All (n=175) | QIN (n=60) | QOOF (n=115) | Self-identifies as teacher of maths (n=134) | Does not identify as teacher of maths (n=41) | Self-identifies as in-field (n=117) | Self-identifies as out-of-field (n=58) |
|---|---|---|---|---|---|---|---|
| Confidence in PCK | 3.64 | 4.23 | 3.32 | 4.10 | **2.15** | 3.97 | 2.96 |
| Commitment to develop CK | 3.83 | 4.35 | 3.55 | 4.35 | **2.15** | 4.10 | 3.26 |
| Commitment to develop PCK | 3.88 | 4.53 | 3.54 | 4.47 | **2.00** | 4.18 | 3.28 |

Respondents who do not self-identify as teachers of mathematics reported the lowest confidence in all six categories, and comparisons of means using an independent t-test[3] (with equal variances not assumed as per Levene's test) point to statistically significant differences (all with $p < .001$) between the groupings of teachers identified earlier. This accords with the findings by Hobbs (2012, p. 27) who found that how 'a teacher sees themselves in an out-of-field role will influence their interest and ability to engage with professional learning and professional development designed to up-skill teachers'.

We also analysed responses using the groupings introduced in Table 1; these findings are shown in Table 3. Within the QOOF cohort, the data indicates that self-identifying as a teacher of mathematics appears to have the greatest impact on their perceived enjoyment, confidence, and commitment. While we cannot generalise these findings due to the smaller sample sizes resulting from the sub-groupings (e.g. n=11), we believe this warrants further investigation.

**Table 3**

*Teacher Self-Reported Interest, Enjoyment, Confidence, and Commitment by Identity Grouping*

| | QIN | QOOF | | | |
|---|---|---|---|---|---|
| | Self-identifies as ToM | Self-identifies as teacher of mathematics | | Does not identify as teacher of mathematics | |
| | Self-identifies as in-field (n=58) | Self-identifies as in-field (n=46) | Self-identifies as out-of-field (n=28) | Self-identifies as in-field (n=11) | Self-identifies as out-of-field (n=27) |
| Personal interest in mathematics | 4.62 | 4.38 | 4.11 | 2.33 | 2.29 |
| Enjoyment teaching mathematics | 4.41 | 4.42 | 4.18 | 1.92 | 2.54 |
| Confidence in CK | 4.76 | 4.38 | 3.68 | 2.67 | 2.18 |
| Confidence in PCK | 4.29 | 4.13 | 3.71 | 1.83 | 2.25 |
| Commitment to develop CK | 4.43 | 4.40 | 4.14 | 1.67 | 2.39 |
| Commitment to develop PCK | 4.60 | 4.42 | 4.29 | 1.50 | 2.25 |

### *Teacher Confidence and the Curriculum*

Teachers were asked to indicate their level of confidence (low, medium or high) in teaching each strand of the Australian Curriculum: Mathematics (AC:M) and in each year level. To turn this into one summary measure, scores were assigned as follows: low = 0, med = 2.5, high = 5. The average (mean) level of confidence for each strand in each year level was computed and reported and shown in Table 4. Due to the smaller sample sizes of some cohorts, we have not reported by identity sub-groupings.

From Table 4 it is clear that self-identification as a teacher of mathematics is a contributing factor to confidence, more so than self-identification as OOF or actual classification of QOOF. Additionally, as the year level increases, confidence generally decreases. There are three exceptions to this (identified with grey shading in the cell) in which average confidence is higher with Year 8 mathematics than Year 7 for: respondents who self-identify as teachers of mathematics, respondents who self-identify as in-field, and respondents who are classified as QIN. We speculate that this indicates a lack of ease with the Year 7 curriculum by these cohorts, given that Year 7 only recently moved to secondary contexts in SA. The mean differences between each teacher grouping for Years 8, 9 and 10 are statistically different, all with $p<.001$.

**Table 4**

*Teacher Self-reported Confidence in Teaching All Strands of the AC:M, by Year Level*

| | All (n=196) | QIN (n=69) | QOOF (n=127) | Self-identifies as teacher of maths (n=135) | Does not identify as teacher of maths (n=41) | Self-identifies as in-field (n=117) | Self-identifies as out-of-field (n=58) |
|---|---|---|---|---|---|---|---|
| Year 7 | 4.47 (n=161) | 4.57 (n=55) | 4.42 (n=106) | 4.65 (n=124) | 3.86 (n=37) | 4.51 (n=109) | 4.38 (n=52) |
| Year 8 | 4.41 (n=165) | **4.93 (n=57)** | 4.14 (n=108) | **4.74 (n=127)** | 3.32 (n=38) | **4.60 (n=111)** | 4.02 (n=54) |
| Year 9 | 3.97 (n=158) | 4.77 (n=57) | 3.51 (n=101) | 4.38 (n=125) | 2.41 (n=33) | 4.36 (n=109) | 3.11 (n=49) |
| Year 10 | 3.48 (n=158) | 4.53 (n=60) | 2.83 (n=98) | 4.03 (n=123) | 1.55 (n=35) | 4.06 (n=109) | 2.19 (n=49) |

The mean difference between the Year 8 teachers who self-identify as in-field in comparison with the teachers who self-identify as out-of-field is also statistically significant but with a p-value of .009. Year 7, however, is an outlier. The mean differences across all three groupings are not statistically significant suggesting:

- The confidence levels (across all strands of Year 7 mathematics) of those who self-identify as teachers of mathematics and those who don't are not statistically different.
- The confidence levels (across the strands of Year 7 mathematics) of those who self-identify as out-of-field and those who don't are not statistically different.
- The confidence levels (across the strands of Year 7 mathematics) of the teachers are QIN (qualified to teach mathematics) and those who are QOOF (not qualified to teach mathematics) are not statistically different.

## Summary and Conclusion

The approach taken in this paper to examine out-of-field mathematics teaching in South Australia draws on the conceptual framing developed by Hobbs et al. (2022a). Working from their multifaceted definition, we also posit that defining and categorising out-of-field teaching is a complex endeavour. Defining OOF solely by qualifications does not account for the range of experiences, knowledge and attitudes that accumulate throughout a teacher's career. The survey data indicates that looking through multiple lenses can provide a more nuanced view.

We presented a snapshot of workload in mathematics for in- and out-of-field teachers. More analysis is needed to make inferences from the data. For example, a teacher with low maths workload might be either 'just filling in' or 'dipping a toe' into a new learning area. Similarly, a teacher with high maths workload might be completely overwhelmed or, alternatively, have gained the knowledge and confidence to teach in an area they were not initially qualified.

Identity as a teacher of mathematics, in particular, was shown to be a key factor influencing interest, enjoyment, confidence and commitment in teaching mathematics. We suggest that future planning and support for teachers of mathematics would be better informed by carefully examining OOF through an identity lens, including targeting professional learning at needs of particular cohorts of teachers.

## References

AITSL. (2021). *Australian teacher workforce data: National teacher workforce characteristics report: South Australia*. Melbourne, Vic: Australian Institute for Teaching and School Leadership.

Brodbelt, S. (1990). Out-of-field teaching. *The Clearing House, 63*(6), 282–285.

Goos, M., Bennison, A., Quirke, S., O'Meara, N., & Vale, C. (2019). Developing professional knowledge and identities of non-specialist teachers of mathematics. *International handbook of mathematics teacher education* (Vol. 1, pp. 211–240). Brill.

Hobbs, L. (2012). Teaching out-of-field: Factors shaping identities of secondary science and mathematics. *Teaching Science*, *58*(1), 21–29.

Hobbs, L., Campbell, C., Delaney, S., Speldewinde, C., & Lai, J. (2020). *Defining and mapping out-of-field teaching in Victorian government schools*. Report for the Victorian Department of Education and Training.

Hobbs, L., Campbell, C., Delaney, S., Speldewinde, C., & Lai, J. (2022a). Defining teaching out-of-field: An imperative for research, policy and practice. In L. Hobbs, & R. Porsch (Eds.), *Out-of-field teaching across teaching disciplines and contexts* (pp. 23–48). Springer.

Hobbs, L., Delaney, S., Campbell, C., Speldewinde, C. A., & Lai, J. (2021). Developing a multifaceted definition of out-of-field teaching: Distilling the literature and capturing expert insights. In P. J. White, T. Tytler, J. Ferguson, & J. C. Clark (Eds.), *Methodological approaches to STEM education research* (Vol. 2, pp. 126–150). Cambridge Scholars Publishing.

Hobbs, L., Du Plessis, A., Oates, G., Caldis, S., McKnight, L., Vale, C., O'Connor, M., Rochette, E., Watt, H., & Weldon, P. (2022b). *Australian national summit on teaching out-of-field: Synthesis and recommendations for policy, practice and research.* Waurn Ponds, Vic: Deakin University.

Hobbs, L., & Porsch, R. (2021). Teaching out-of-field: Challenges for teacher education. *European Journal of Teacher Education, 44*(5), 601–610.

Hobbs, L., & Törner, G. (Eds.) (2014). *Taking an international perspective on "out-of-field" teaching: Proceedings and agenda for research and action from the 1st Teaching Across Specialisations (TAS) Collective Symposium*. TAS Collective. https://www.uni-due.de/TAS.

Hobbs, L., & Törner, G. (2019). Teaching out-of-field as a phenomenon and research problem. In L. Hobbs, & G. Törner (Eds.), *Examining the phenomenon of 'teaching out-of-field': International perspectives on teaching as a non-specialist* (pp. 3–20) Springer.

Johnson, B., Down, B., Le Cornu, R., Peters, J., Sullivan, A., Pearce, J., & Hunter, J. (2014). Promoting early career teacher resilience: A framework for understanding and acting. *Teachers and Teaching, 20(*5),530–546.

Kim, E.-G. (2011). Out-of-field secondary school teachers in Korea: Their realities and implications. *KEDI Journal of Educational Policy, 8*(1).

Marginson, S., Tytler, R., Freeman, B., & Roberts, K. (2013). STEM: Country comparisons. *Report for the Australian Council of Learned Academies*. http://www.acola.org.au

Post Primary Teachers' Association (2018). *Secondary school staffing survey report 2018*. New Zealand: PPTA.

Price, A., Vale, C., Porsch, R., Rahayu, E., Faulkner, F., Ríordáin, M. N., Crisan, C., & Luft, J. A. (2019). Teaching out-of-field internationally. In L. Hobbs, & G. Törner (Eds.), *Examining the phenomenon of "teaching out-of-field": International perspectives on teaching as a non-specialist* (pp. 53–83). Springer.

Prince, G., & O'Connor, M. (2018). *Crunching the numbers on out-of-field teaching in maths*. AMSI Occasional Paper, 1. Australian Mathematical Sciences Institute.

Sharplin, E. D. (2014). Reconceptualising out-of-field teaching: Experiences of rural teachers in Western Australia. *Educational Research*, *56*(1), 97–110.

Van Overschelde, J. P. (2022). Value-lost: The hidden cost of teacher misassignment. In L. Hobbs, & R. Porsch (Eds.), *Out-of-field teaching across teaching disciplines and contexts* (pp. 49–70). Springer.

Weldon, P. R. (2016). *Out-of-field teaching in Australian secondary schools (Policy Insights, Issue #6*). Camberwell, Vic: Australian Council for Educational Research.

Windle, J., Morrison, A., Sellar, S., Squires, R., Kennedy, J-P., & Murray, C. (2022). *Teachers at breaking point: Why working in South Australian schools is getting tougher (report to the Australian Education Union, South Australian Branch).* University of South Australia.

# Identifying and Assessing Students' Transition Barriers Between Additive and Multiplicative Thinking

Lei Bao
*Deakin University*
baolei@deakin.edu.au

This paper reports on 73 Grades 3 to 6 students' written responses to equal groups, arrays, multiplicative comparison and Cartesian product word problems. It is part of a larger study relating to students' development of multiplicative thinking (MT). The potential of using multiplicative word problems as a diagnostic assessment to reveal students' transition barriers from additive thinking to MT was explored. The development of the assessment and some of the research findings are described here. Findings show that students experience different types of barriers during their transition from additive to MT. Some recommendations are given on how these barriers might be overcome during primary school years.

There has been a real and persistent barrier to many middle and upper primary school students' progress in MT (Hurst & Hurell, 2016). The transition from operating the single units to coordinating two composite units (that is, groups of equal size and the number of groups) simultaneously is a conceptual leap for many students which potentially constitutes an obstacle during the development of MT (Cheeseman et al., 2020). Transition from additive to MT is a process of cognitive change where transition barriers can be identified between developmental stages (Siemon et al., 2019). Many local and international researchers have been using various assessment styles including interviews and pen-and-paper tests to identify students' developmental stages of MT in order to support students' transition from additive to MT.

There has been extensive research (e.g., Downton & Sullivan, 2017; Cheeseman et al., 2020) into early and middle years students' responses towards various multiplicative problems, which mainly explore the complexity of early and middle stages of development in MT. However, there has been limited work exploring later stages of development in MT to reveal students' ability to discern multiplicative relationships between two quantities and their ability to extend to other pairs of quantities. This work is essential to identifying students' transition barriers and common errors based on their strategy choices. Therefore, an assessment is needed for middle and upper primary students to shed light on the following research question:

- To what extent can diagnostic assessment under four key situations—equal groups, arrays, multiplicative comparison and Cartesian product—reveal primary school students' transition barriers between additive and MT?

## Theoretical Framework

Evidence-based Learning Trajectories (LTs) have been used to inform students' progress over time by examining their knowledge and thinking through solution strategies to mathematical problems and students' level of understanding on a developmental continuum (Siemon et al., 2019). In a recent review based on studies of LTs, Siemon et al., (2019) highlighted that transition barriers are boundaries between two developmental stages during students' development. When students first learn the concept of multiplication, their strategies are often concrete and inefficient, relying on drawing equal groups and counting objects one by one to find the total. The transition barrier at this stage is failing to recognise an equal grouping structure (Siemon et al., 2019). Progressing to counting composites (skip counting and repeated addition) or even double count (that is, keeping a running total and keeping the track of the number of groups) (Steffe, 1992) becomes one of the key transition barriers during students' development in MT because of the relationship of many-to-one correspondences (Cheeseman et al., 2020). Once students learn to coordinate composite units, they

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 91–98). Newcastle: MERGA.

might use a mix of additive and multiplicative strategies such as doubling/halving and/or partitioning/splitting strategies to solve problems (Siemon et al., 2006). This will be superseded by using multiplication facts and procedural based methods (Downton & Sullivan, 2017). However, procedural based learning could be a barrier which limits students to recognise multiplicative relationships and apply properties of multiplication (Hurst & Hurell, 2016). When students start using more abstract and efficient strategies such as properties of multiplication (Downton & Sullivan, 2017), some students still experience difficulty in dealing with two-digit by two-digit multiplications (Larsson, 2016).

In addition, LTs have been used for the design of rich assessment tasks to assess students' understanding of the areas of mathematics and their ability to apply their knowledge in unfamiliar situations and explain or justify their reasoning (Siemon et al., 2019). In this study, a framework of LTs in MT based on the literature was used to design 15 test items and coding categories for each item by identifying students' prior knowledge through their solution strategies ranging from counting all strategies to more sophisticated multiplicative strategies. The items ranged from familiar to unfamiliar multiplicative situations and from two single-digit factors to two two-digit factors to adjust the difficulty level of each item, aiming to help identify students' strategy choices including errors that underline students' performance.

## The Design of the Items

Researchers such as Greer (1992) distinguished four classes of multiplicative situations: equal groups, arrays, multiplicative comparison and Cartesian product, indicating that a wide range of contexts are needed to identify students' understanding of multiplicative relationships and transition barriers from additive to MT (e.g., Larsson, 2016; Downton & Sullivan, 2017). The test items in this study are situated in four situations and embedded with real life context.

Understanding a situation mathematically requires students to recognise relationships between given quantities. Each simple multiplicative relationship encloses one multiplication and two division operations involving three quantities where each has a particular role. Any one of the three quantities can be unknown, which could provide three mathematical problems sharing the same relationship. Including both multiplication and division operations offers a window to see how students would respond between the operations (Downton, 2013).

Some items were adopted with slight modification from previous research as shown in Table 1 and others were created based upon theoretical, literature-based analysis and on the interpretation of the performance that students exhibit from additive strategies to sophisticated multiplicative strategies. Since the items are designed for Grades 3 to 6, it is important that the items provide students with multiple entry points to demonstrate their level of thinking.

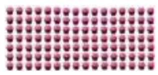
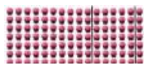
The test items in Table 1 were organised into four tasks: 1) Australian Coins, 2) Michelle's Bakery, 3) Beth's Cupcakes and 4) Sam's Outfits and sequenced based on students' familiarity of multiplicative situations, operations and size of numbers. The test items aim to reveal students' understanding of multiplicative relationships and potential barriers by analysing their solution strategies and tapping common errors in the transition from one stage to the next.

Representing equal groups, items 1a and 1b involve many-to-one correspondences where five 20c coins are matched with \$1. In this situation, the number of groups of five 20c coins stipulates the relation between the number of 20c coins and the dollar equivalent. In item 1b, the size of equal groups (5) is known and the number of groups is unknown. In item 2e, the size of equal groups is unknown and the number of groups (25) is known.

Representing arrays, the equation $6\times4=24$ in item 2b could be challenging for some students since it requires students to recognise the structure of the array as 4 rows of 6 without needing to fill in the unseen space (Downton & Sullivan, 2017). The equation $16\times7=102$ in item 3b is intended to assess students' understanding of partitioning and properties of multiplication. The use of a two-digit factor tests whether the size of numbers influence students' strategy choice across the same situation. Item 3c provides an opportunity to test student's understanding of the associative property of multiplication and to reveal a common transition barrier in moving beyond additive thinking (Squire et al., 2004; Larsson, 2016).

**Table 1**

*Multiplication and Division Word Problems Used in the Study*

| Multiplicative Thinking Assessment Items | Multiplicative Situations/References |
|---|---|
| 1a. The value of five 20c coins is same as one \$1 coin. How many 20c coins are same as \$4?  | Equal Groups/Downton (2013) |
| 1b. If you have 45 20c coins, how many \$1 coins can you make? | Equal Groups/Downton (2013) |
| 2a. Michelle bakes 40 biscuits. She puts them in rows of 8 biscuits on a baking tray. How many rows of biscuits does Michelle bake? | Arrays/Downton (2013) |
| 2b. Michelle puts party pies on a baking tray like in the picture and fills the tray. How many party pies does Michelle bake?  | Arrays/Downton & Sullivan (2017) |
| 2c. Michelle bakes 18 pies. She also bakes 4 times as many sausage rolls as pies. How many sausage rolls does Michelle bake? | Multiplicative Comparison/Downton (2013) |
| 2d. Michelle sold 15 pies on Friday and 60 pies on Saturday. How many times as many pies were sold on Saturday? | Multiplicative Comparison/Larsson (2016) |
| 2e. Michelle needs 25 boxes to pack her 200 party pies. How many party pies are in each box? | Equal Groups/Hurst & Hurrell (2016) |
| 3a. Beth put cupcakes on the bench like in the picture. Can you work out the total number of cupcakes that Beth made?  | Arrays/Hurst & Huntley (2020) |
| 3b. In order to work out the total number of cupcakes, Beth divided the cupcakes into 3 sections like in the picture. How did Beth work out the total number of cupcakes?  | Arrays/Hurst & Huntley (2020) |
| 3c. Beth baked 12 rows of cookies with 15 cookies in each row. Three of her children tried to work out the total number of cookies. Sam did $12\times15=10\times17=170$. Tom did $12\times15=10\times10+2\times5=110$. Emily did $12\times15=6\times30=180$. Who do you think is correct? Why? | Arrays/Larsson (2016) |
| 4a. Sam has 4 jumpers and 3 shorts. If Sam chose a blue jumper, what might be Sam's choice of outfits? | Cartesian Product/Wright (2011) |
| 4b. Sam has 4 jumpers and 3 shorts. How many different outfits are there in total? | Cartesian Product/Wright (2011) |
| 4c. After Christmas, Sam has 5 jumpers and 30 outfits. How many shorts does Sam have? | Cartesian Product/Wright (2011) |
| 4d. Sam's Dad has 18 jumpers and 13 shorts so he has 234 different outfits. Sam's younger brother has 13 jumpers and 18 shorts. How many different outfits does Sam's younger brother have? | Cartesian Product/Squire et al. (2004) |
| 4e. Sam's older brother has 13 jumpers and 19 shorts. How many different outfits does Sam's older brother have? | Cartesian Product/Larsson (2016) |

Representing multiplicative comparison, item 2c tests students' understanding of "times as many". Item 2d involves reversibility of item 2c by asking students to find the unknown multiplier, aiming to see whether students' responses reflect barriers reported by Squire et al., (2004) and Larsson (2016) where "times as many" was confused with adding additional groups.

Cartesian product situations present another transition barrier for students (Wright, 2011; Downton & Sullivan, 2017) since repeated equal sets are not obviously presented. Item 4b aims to see whether students can construct repeated equal sets of different outfits by either using the jumpers or shorts. Item 4c involves reversibility of item 4b which requires students to share the total number of 30 outfits into 5 equal sets or to pair with 5 jumpers. Item 4d and 4e not only assess students' understanding of Cartesian product situation but also aim to identify students' understanding of the commutative and distributive properties in multiplication. In item 4e, students need to recognise the multiplicand or multiplier increasing by 1 so they can use derived facts (e.g.,$13\times19=234+13=247$) or distributive property to solve the problem efficiently. Item 4e also aims to identify another transition barrier in moving away from additive thinking (e.g., $13\times18=234$ so $13\times19=234+1=235$) as claimed by Squire et al., (2004) and Larsson (2016).

## Method

This study is part of a doctoral research study investigating the development of MT among primary school students, for which all necessary ethics approvals have been attained. As a part of iterative cycles of design-based research, prior to this study, the size of numbers, the use of language, format and the sequence of items within the task were evaluated during the focus group discussions with a panel of experts in the field. The use of language, the context and instructions of the test items were also discussed with a group of expert teachers through semi-structured interviews. The feedback from both groups was taken into account in the process of revising and refining the tasks shown in Table 1.

In August 2022, 27 Grade 3 students, 18 Grade 4 students, 18 Grade 5 students and 10 Grade 6 students in two different schools completed the assessment within 45 minutes during their regular maths class time. Students were asked to try a sample question first to ensure that they understood how to respond to the tasks in the assessment and individual students were provided with reading assistance if necessary. The first version of the test items was given to 48 students from Grades 3 to 6 at one primary school. Two weeks later, a slightly varied sequence of the same tasks was given to 25 students in a combined Grade 3/4 at a different school with the aim of seeking more responses on Cartesian product situations.

Students' solution strategies were coded according to the extent of multiplicative thinking shown in Table 2, drawing on the findings of the literature (Siemon et al., 2006; Downton & Sullivan, 2017), as well as necessary modifications based on students' actual answers from the study. The coding categories were flexible and accommodative of different possible strategies that students could use to solve a given test item. An example of the coding category is provided in Table 2 for item 2d. Two-digit codes were used to separate correct and incorrect responses with specific strategies. Students' correct responses were coded from a deficient response 10 through intervening stages to an optimal response 15. No response or irrelevant response or responses with no indication of strategies were coded 70. Incorrect responses resulting from adding or subtracting the two giving numbers were coded 71 as superficial strategies. Responses indicating additive strategies with errors were coded 72 and responses indicating a mix of additive and multiplicative strategies with errors were coded 73. Responses indicating multiplicative strategies with calculation errors were coded 74. Students' responses reflecting an error based on inappropriate generalisation of additive thinking were coded 75. Students' responses based on the coding categories were entered into a spreadsheet for analysis across the Grade levels, multiplicative situations, size of numbers and operations.

Two-digit coding (10, 11, 12, 13, 14, 15, 70, 71, 72, 73, 74, 75) as strategies listed in Table 2 were used to categorise the strategies students used to solve the problems. Based on the analysis of data, there are differences in students' strategy choices across four situations, size of numbers and operations. Findings suggest that students experience different types of transition barriers and overreliance on additive strategies seems hinder their transition from additive to MT. More students struggled with items involving two two-digit factors and multiplicative comparison and Cartesian product situations.

**Table 2**

*Coding and Solution Strategies for Item 2d in Michelle's Bakery Task*

| Coding | Strategies | Responses |
|---|---|---|
| 10 | Counting All | Relies on drawing to count by 1s. |
| 11 | Skip Counting | Shows skip counting by 15s 4 times. |
| 12 | Repeated Addition | Shows 15+15+15+15= 60 or 15(1), 30(2), 45 (3), 60 (4). |
| 13 | A Mix of Additive and Multiplicative Strategies | Uses a mix of additive and multiplicative strategies, e.g., using split strategies or showing doubling or halving strategies. |
| 14 | Multiplicative Operation | Uses 15×4=60 or 60÷15=4. |
| 15 | Holistic Thinking | Shows 10×4+5×4=60 or 30÷15=2 so 60÷15=4 |
| 70 | No Strategies | Shows no response or an irrelevant response. |
| 71 | Superficial Strategies | Incorrect response resulting from adding or subtracting two given numbers, e.g. 60-15=45. |
| 72 | Additive Strategies | Uses additive strategies with errors or an incomplete response. |
| 73 | A Mix of Additive and Multiplicative Strategies | Uses a mix of additive and multiplicative strategies with errors or an incomplete response. |
| 74 | Multiplicative Strategies | Uses multiplicative strategies with errors or an incomplete response, e.g., showing 60÷15. |
| 75 | Inappropriate Strategies | Shows overreliance on additive thinking, e.g., 4-1=3. |

## Results and Discussion

According to students' strategy choices based on the Grade levels in Table 3, the use of counting all strategy (10) in Grade 3 is 15%, the highest among the four grades, and gradually declining by grade level. The barrier for these students is failing to recognise an equal grouping structure (e.g., Steffe, 1992; Siemon et al., 2019). 22% of students (using strategies 11, 12 & 72) recognised the size of equal groups and used skip counting or repeated addition to solve the problems. Some used double count strategy, indicating a shift from operating a single unit to composite units but some responses (72) indicate students failed to recognise the number of equal groups (e.g., Cheeseman et al., 2020). Reliance on additive strategies (10, 11 & 12) to reach a correct solution remains almost unchanged (approximately 26%) across all grades.

The fact that students in Grades 5 and 6 fail to see many-to-one correspondence and still rely on additive strategies, which points to an important barrier. While 15% of responses show that students can use procedural based methods such as vertical multiplication or lattice method to get correct answers, for example, to 13×19, only 1% of students solved problems such as item 4e by applying the distributive property of multiplication to find 13×19 when the value of 13×18 =234 is given. Lack of knowledge of the distributive property of multiplication clearly remains a barrier for upper primary school students.

**Table 3**

*The Mean Frequency of Students' Strategy Choices Based on the Grade Levels (N=73)*

| | Strategies | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Year Levels | 10 | 11 | 12 | 13 | 14 | 15 | 70 | 71 | 72 | 73 | 74 | 75 |
| Grade 3 (n=27) | 15% | 7% | 4% | 6% | 7% | 0% | 32% | 15% | 7% | 2% | 2% | 3% |
| Grade 4 (n=18) | 10% | 11% | 4% | 8% | 9% | 0% | 28% | 12% | 10% | 2% | 3% | 4% |
| Grade 5 (n=18) | 9% | 9% | 8% | 6% | 20% | 1% | 25% | 9% | 6% | 1% | 3% | 3% |
| Grade 6 (n=10) | 5% | 8% | 11% | 3% | 38% | 1% | 19% | 5% | 5% | 0% | 4% | 2% |
| Mean (n=73) | 11% | 9% | 6% | 6% | 15% | 1% | 27% | 11% | 7% | 2% | 2% | 3% |

In Table 4, items involving equal groups situations provoked the highest frequency of using the additive strategies (10, 11, 12 & 72) of 53%. It seems that the overuse of equal groups situations may not assist students transition from additive to MT, a concern raised by Larsson (2016) and Cheeseman et al., (2020). On the other hand, array items produced a high frequency of using multiplicative strategies of 27% and the lowest of frequency of superficial strategies of 4%. Array items, unlike equal groups, provided students with a visual image encouraging students to recognise the composite units in the array structure (Hurst & Huntley, 2020).

**Table 4**

*The Mean Frequency of Students' Strategy Choices Based on Multiplicative Situations (N=73)*

| | Strategies | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Multiplicative Situations | 10 | 11 | 12 | 13 | 14 | 15 | 70 | 71 | 72 | 73 | 74 | 75 |
| Equal Groups | 18% | 16% | 10% | 6% | 18% | 0% | 12% | 7% | 9% | 2% | 1% | 0% |
| Arrays | 10% | 13% | 4% | 7% | 19% | 1% | 23% | 4% | 8% | 2% | 4% | 5% |
| Multiplicative Comparison | 3% | 8% | 10% | 13% | 15% | 0% | 8% | 27% | 5% | 3% | 3% | 3% |
| Cartesian Product | 10% | 1% | 3% | 3% | 9% | 0% | 50% | 14% | 5% | 0% | 2% | 3% |

Multiplicative comparison situations as shown in items 2c and 2d (Table 1) also require students to apply multiplicative relationships. In Table 4, 21% of students used additive strategies (10, 11 & 12) to reach a correct answer whereas 28% of students used effective multiplicative strategies (13 & 14) to solve the problems. However, 27% of students used superficial strategies by adding or subtracting two given numbers, indicating no understanding of the notion of "times as many". Clearly, multiplicative comparison situations are a barrier for nearly half of students and need attention.

In this study, Cartesian product situations showed the clearest barrier, producing the highest frequency 64% either showing no response (70) or using superficial strategies (71) by adding or subtracting two given numbers. Clearly, many students are unfamiliar with this situation since repeated equal sets are not clearly presented which constitutes a barrier for students to construct set of ordered pairs from two sets in Task 4. Wright (2011) and Downton and Sullivan (2017) also confirmed that Cartesian product situations present a barrier for many students.

Based on the results in Table 5, items involving two two-digit factors produced the highest frequency 67%–either showing no response (70) or using superficial strategies (71) by adding or subtracting two given numbers and the highest frequency12% using inappropriate strategies, indicating two-digit by two-digit multiplication is a barrier for students' development in MT. Some responses reflect errors of inappropriate generalisation of additive thinking (Squires et al., 2004; Larsson, 2016). For example, in item 3c some students wrote 12×15=10×17 by moving 2 to 15 or 12×15=10×10+2×5 by splitting 12 into 10 and 2 and splitting 15 into 10 and 5. These students use place value partitioning in addition for multiplication problems. However, some of these students successfully solved the item 2c, 18×4=72 by partitioning 18 into 10 and 8 so 10×8+4×8=72. This shows that these students do not fully understand the distributive property in two-digit by two-digit multiplication. In item 4e, some students wrote 13×19=234+1=235 because in item 4d 13×18=234 and 19 is 1 more than 18. It shows how some students attempted but failed to understand the multiplicative relationship between 13×18 and 13×19. In addition, items involving two single-digit factors provoked the highest frequency of 47% using additive strategies (10, 11, 12 & 72), indicating the need to move beyond single-digit multiplication and division problems.

**Table 5**

*The Mean Frequency of Students' Strategy Choices Based on the Size of Numbers (N=73)*

| | Strategies | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Size of Numbers | 10 | 11 | 12 | 13 | 14 | 15 | 70 | 71 | 72 | 73 | 74 | 75 |
| Two Single-digit Factors | 21% | 11% | 8% | 3% | 18% | 0% | 22% | 9% | 7% | 1% | 1% | 0% |
| Single & Two-digit Factors | 2% | 5% | 4% | 6% | 8% | 2% | 15% | 15% | 12% | 10% | 11% | 9% |
| Two two-digit Factors | 0% | 1% | 0% | 6% | 6% | 1% | 58% | 9% | 4% | 0% | 2% | 12% |

According to students' strategy choices based on operations in Table 6, the use of additive strategies (10, 11, 12 & 72) such as counting all, skip counting and repeated addition strategies in items involving division operations (37%) is higher than in items involving multiplication operations (29%). It is evident that there are more students using additive strategies in division than multiplication problems. For example, in items 1b and 2a, a typical strategy was to successively separate out groups of the specified size until the total number was reached and then to count the number of groups. Even though the group of equal size was created in the division problems, the calculation procedure does not reflect on the use of equal grouping structure which is the main barrier for the development of MT. There are significantly more students using superficial strategies in division items (19%) than multiplication items (7%) which indicates that students experienced more difficulty in understanding multiplicative relationships involving division operations than multiplication operations. This contrasts with the conjecture claimed by Downton (2013). Therefore, identifying an equal grouping structure in division items appears to be a barrier for students.

**Table 6**

*The Mean Frequency of Students' Strategy Choices Based on Operations (N=73)*

| | Strategies | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Operations | 10 | 11 | 12 | 13 | 14 | 15 | 70 | 71 | 72 | 73 | 74 | 75 |
| Multiplication | 9% | 8% | 4% | 6% | 15% | 1% | 32% | 7% | 8% | 2% | 3% | 4% |
| Division | 13% | 10% | 9% | 6% | 16% | 0% | 18% | 19% | 5% | 1% | 1% | 1% |

## Implications and Conclusion

This study investigated the extent to which diagnostic assessment under four key multiplicative situations can reveal primary school students' transition barriers between additive and MT. The data discussed above go a long way to answering this key research question. For example, in dealing with equal groups, many students appeared to not recognise and coordinate the size of equal groups and the number of equal groups, and so relied on additive strategies such as counting all, skip counting and repeated addition to answer the questions. This suggests that the overuse of equal groups situations may not assist students' transition from additive to MT. Instruction where multiplication is introduced as repeated addition or equal groups could be a contributor that leads students to rely on additive thinking. The use of arrays and multiplicative comparison situations showed clearer evidence of multiplicative strategies underlining the importance of using these kinds of tasks. However, understanding the notion of "times as many" in multiplicative comparison situations was a barrier for students who appeared to confuse "times as many" with "times more". This confusion clearly requires attention. Finally, Cartesian product situations presented the biggest barrier for many students because they could not identify repeated equal sets. Introducing two-digit by two-digit multiplication problems in arrays and Cartesian product situations also revealed that many students did not understand or use the distributive and associative properties of multiplication. Overcoming this barrier is an important step in the development of MT. This study highlights the need to move beyond single-digit multiplication and division problems for supporting students' transition from additive to MT. The use of procedural based methods, such as vertical multiplication and lattice methods, may limit students' ability to see the multiplicative relationships including the distributive and associative properties of multiplication between the pairs of quantities involved in the operation.

## References

Cheeseman, J., Downton, A., Ferguson, S., & Roche, A. (2020). Meeting multiplicative thinking through thought-provoking tasks. *Mathematics Education Research Journal*, 1–32.

Downton, A. (2013). Making connections between multiplication and division. In V. Steinle, L. Ball, and C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow. Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia* (pp. 242–249). Melbourne, Victoria: MERGA.

Downton, A., & Sullivan, P. (2017). Posing complex problems requiring multiplicative thinking prompts students to use sophisticated strategies and build mathematical connections. *Educational Studies in Mathematics*, *95*, 303–328.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). New York: Macmillan.

Hurst, C., & Huntley, R. (2020). Distributivity, Partitioning, and the Multiplication Algorithm. *Journal of Research and Advances in Mathematics Education*, 5(3), 231–246.

Hurst, C., & Hurrell, D. (2016). Investigating children's multiplicative thinking: Implications for teaching. *European Journal of STEM Education*, *56*, 1–11.

Larsson, K. (2016). *Students' understandings of multiplication* [Doctoral dissertation, Stockholm University]. Stockholm.

Siemon, D., Barkatsas, T., & Seah, R. (2019). *Researching and using progressions (trajectories) in mathematics education. Brill.*

Siemon, D., Breed, M., Dole, S., Izard, J. and Virgona, J. (2006). *Scaffolding Numeracy in the Middle Years—Project Findings, Materials, and Resources*, Final Report submitted to Victorian Department of Education and Training and the Tasmanian Department of Education.

Squire, S., Davies, C., & Bryant, P. (2004). Does the cue help? Children's understanding of multiplicative concepts in different problem contexts. *British Journal of Educational Psychology*, *74*, 515–532.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259–309.

Wright, V. J. (2011). *The development of multiplicative thinking and proportional reasoning: Models of conceptual learning and transfer* [Doctoral dissertation, University of Waikato]. Waikato.

# Increasing Participation of Students from Disadvantaged Backgrounds in Challenging Mathematics Subjects

Anne Bennison
*University of the Sunshine Coast*
abenniso@usc.edu.au

Calls for high levels of participation and success in Science, Technology, Engineering and Mathematics (STEM) by all Australians exacerbates the need for research into improving participation in mathematics by students from disadvantaged backgrounds. Rogoff's three planes of analysis is used to foreground the institutional plane in a low socioeconomic status school with sustained increased enrolments in Mathematics B (now replaced by Mathematics Methods). Processes in this plane are examined by drawing on semi-structured interviews with a Deputy Principal and Head of Mathematics. Contributing factors appear to be the way that the Mathematics Program and Pastoral Care Program are delivered at the school.

Science, Technology, Engineering and Mathematics (STEM) disciplines are crucial in a rapidly changing world in which advances in technology effect almost every aspect of daily life (Department of Education, 2022). A long-term strategic view of STEM to ensure Australia's economic prosperity was outlined almost 10 years ago (Office of the Chief Scientist, 2014). Australia's competitiveness (characterised by innovation, links between industry and research, and a flexible workforce), education and training, research, and international engagement were seen as crucial for building a competitive economy. One of the education and training goals was for "high levels of participation and success in STEM for all Australians, including women, Indigenous students, and students from disadvantaged and marginalised backgrounds" (p. 20). A barrier to participation and success in STEM for students from disadvantaged backgrounds is that they are over-represented among those who do not meet national and international benchmarks in mathematics (e.g., Thomson et al., 2020). Consequently, students from such backgrounds are less likely than their peers from more privileged backgrounds to study challenging mathematics subjects, such as Mathematical Methods (a subject taken in the final two years of school that includes the study of calculus, probability, and statistics), that underpin STEM disciplines. Murphy (2018), for example, found that the proportion of students enrolled in Mathematics Methods in Year 12 in Victoria averaged across 2014 to 2016 was 0.064 in the highest socioeconomic status (SES) schools compared to 0.039 in the lowest SES schools. Mathematical Methods is not only required for STEM disciplines but also provides the necessary foundation for an extremely broad range of professions including economics and the social sciences. In a recent review of Australasian research on equitable, socially just, and ethical mathematics education, Vale et al. (2020) called for more research on how to improve participation and achievement in mathematics in schools in low SES communities. This study seeks to contribute to addressing this gap in literature.

The aim of this study was to identify and understand factors that contribute to promoting sustained engagement with mathematics in early secondary years (Year 7, Year 8, and Year 9) among students from disadvantaged backgrounds. The study extends earlier research (Bennison et al., 2018), which focussed on students in the later years (Year 10 and Year 11), by shifting the focus to younger students who are yet to make critical subject choices for the senior phase of schooling. The earlier research identified some schools located in low SES areas in Queensland that have achieved sustained increased enrolments in Mathematics B (replaced by Mathematics Methods in 2019). The present study explores factors that may influence students' mathematical aspirations and engagement in the early years of secondary schooling in one of these schools. The goal is to identify factors that may motivate students in this school to choose to study Mathematics B in their final two years of school. Using Rogoff's (1995) three planes of analysis, overlapping personal, social, and institutional factors were explored to identify and understand possible reasons for students' sustained

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 99–106). Newcastle: MERGA.

engagement in mathematics in this school. The focus of this paper is at the institutional level. The following research question is addressed:

- What institutional factors appear to be effective in promoting sustained engagement with mathematics at Cassowary Secondary College?

## Theoretical Framework

There are many competing and interwoven factors that influence students' selection of mathematics subjects for their final two years of schooling. For this reason, a sociocultural approach that takes into consideration interconnected personal, social, and institutional factors is warranted. Rogoff's (1995) three planes of analysis provides a way of examining the *person-in-context* that allows the focus of analysis to foreground the individual, their interactions with others, or institutional processes while keeping in mind that activities in each of these planes take place in the context of the other planes. Rogoff regards these planes of focus "not as separate or as hierarchical, but as simply involving different grains of focus with the whole sociocultural activity. To understand each requires the involvement of the others" (p. 141). Thus, any analysis using this theoretical framework must begin by developing an understanding of each of the planes. Developing students' aspirations to study more challenging mathematics subjects in their final two years of schooling was seen as the sociocultural activity of interest in this study. The personal plane *zooms in* on the individual and encompasses factors such as a student's beliefs about mathematics and their future career plans. The social plane *zooms out* to view how interactions the individual has with peers, teachers, and parents shape their aspirations. *Zooming out* further brings the institutional plane into focus. This plane allows attention to be placed on practices within the school that provide the context in which personal and interpersonal processes take place. This plane is foregrounded in this paper.

Rogoff (1995) used the metaphors of *apprenticeship*, *guided participation*, and *participatory appropriation* to explain developmental processes taking place in the institutional, social, and personal planes, respectively. She used the metaphor of apprenticeship for the institutional plane to illustrate how attention in this plane is on the role of the individual and others as well as the "cultural/institutional practices and goals of the activities to which they contribute" (p. 143). Individual development in this plane takes place through participation in activities with others in a community of practice (Wenger, 1998). Thus, the institutional plane is foregrounded while recognising the contribution of processes taking place in the personal and social planes. In this study, the institutional plane encompasses how the school arranges activities (i.e., how mathematics is offered) and supports students in the junior secondary years that may contribute to sustained engagement in mathematics. The outcome of this sustained engagement is students choosing to study Mathematics B in their final two years of schooling.

## Research Design

This project was conducted in 2019 and is developing a case study (Stake, 2003) of a school located in a low socioeconomic status (SES) area in Queensland where there was a sustained increase in enrolment in Mathematics B (the predecessor of Mathematics Methods in Queensland) over 2013-2017 (the most recent 5-year period for which relevant data was available when the project was conducted). Ethics approval (A191208) was granted by the university involved in the project, and participants and parents/caregivers gave informed consent. Permission to conduct the research at the school was given by the jurisdictional authority and the principal. The name of the school and names of participants are pseudonyms.

### *School Selection and Context*

The project employed purposeful sampling using publicly available Queensland Curriculum and Assessment Authority (QCAA) enrolment data to determine possible sites. A sustained increase in enrolment in a subject was considered to have occurred if the ratio of enrolment in Mathematics B in 2017 in Year 12 to that in 2013 was greater than one and the ratio of the school population in 2017 to that in 2013 was close to one (i.e., the school population was stable). A further constraint was that subject enrolment for both years was greater than 10. For schools which met these criteria, schools with an Index of Community Socio-Educational Advantage (ICSEA) (ACARA, 2015) of less than 1000 were identified. Final school selection was determined by convenience sampling based on school location and willingness of the principal to participate in the study. Cassowary State Secondary College met the criteria and was chosen for the study.

### *Participants*

Cassowary State Secondary College's senior administration consisted of a principal supported by three deputy principals, including *Chris* who was responsible for mathematics, science, and numeracy. The Mathematics Department was led by *Elizabeth*, and included *Karen*, who taught a Year 7 Extension class; *Susan* and *Janelle*, who each taught a Year 8 Extension class; *Melissa*, who taught a Year 9 Extension class; and *Will* and *Paul* who were responsible for numeracy in the first and second half of 2019, respectively. All named school staff participated in the study. Twenty-one students from the extension classes (seven from Year 7, nine from Year 8 and five from Year 9) also participated in the study though focus group interviews. Data collection took place during school visits in June and November. As attention in this paper is on factors at the institutional level, data drawn on is the interviews with Chris and Elizabeth.

### *Data Collection*

Chris and Elizabeth were interviewed in June and a follow-up interview was conducted with Elizabeth in November. Semi-structured interviews were used to determine what institutional factors might have an impact on students' aspirations to study Mathematics B in their final two years of schooling. Questions sought information on the participant's background, how mathematics was delivered at the school, and why participants thought there had been an increase in enrolments in Mathematics B in the nominated 5-year period. Interviews lasted between 34 and 43 minutes, were audio recorded, and transcribed.

### *Data Analysis*

An inductive content analysis (Silverman, 2013) of interview transcripts was employed to identify factors at the institutional level that may contribute to engagement in mathematics at the Cassowary Secondary College. All transcripts were initially read to identify potential themes. These tentative themes were used to code the transcript of the interview with Chris (Deputy Principal), then applied to the two interviews with Elizabeth (Head of Mathematics). Categories were refined or expanded to accommodate any different responses. Finally, the names of the themes were refined to reflect the underlying *institutional practice*.

## Institutional Practices at Cassowary Secondary College

Cassowary Secondary College is a mid-sized (875 students in 2019) co-educational government school in an outer metropolitan area of Brisbane. The school ICSEA value in 2019 was 955, with 82% of students from backgrounds below the Australian median for Socio-Educational Advantage (including 49% in the bottom quarter). Nine percent of students were from Indigenous backgrounds and 10% were from a language background other than English (https://www.myschool.edu.au/). Student enrolments in Mathematics B in Year 12 were 28 in 2013 and 36 in 2017 (ratio: 1.29) while the school population increased from 910 to 955 (ratio: 1.05). Thus, there had been an increase in enrolment in Mathematics B in the context of a relatively stable student population.

Six institutional practices that may contribute to sustained increase in enrolments in Mathematics B were identified and grouped under two programs operating in the school: the Mathematics

Program and the Pastoral Care Program. Four of the institutional practices were related to the Mathematics Program and the two remaining institutional practices were part of the Pastoral Care Program (see Table 1).

**Table 1**

*Institutional Practices Contributing to Sustained Engagement in Mathematics*

| School program | Institutional practice |
|---|---|
| Mathematics Program | Staffing of extension classes |
| | Selecting students for extension classes |
| | Supporting students' mathematical learning |
| | Providing access to resources |
| Pastoral Care Program | Building relationships |
| | Providing support networks |

## *Mathematics Program*

The Mathematics Program at Cassowary Secondary College employs a version of streaming that is characterised by providing students with opportunities to achieve their full potential: "a lot of our kids that go to extension classes would not be in traditional extension classes" at other schools (Craig, June 2019). The school usually has one Year 7 Extension class and two Extension classes in Year 8 and Year 9. In addition to extension and Core classes, there are two Horizon classes in Year 7 and one in Year 8. These classes are for students who have "too many gaps in mathematics" (Elizabeth, June 2019) and the goal is for these students to be able to transition into Year 9 or Year 10 Core classes. There appear to be four features of the Mathematics Program that potentially contribute to developing students' aspirations for mathematics: *staffing of extension classes*, *selection of students for extension classes*, *supporting students' mathematical learning*, and *providing access to resources*.

*Staffing of extension classes*. Craig and Elizabeth ensure that all the extension classes are taught by experienced and qualified mathematics teachers to ensure that students are well-prepared to study Mathematics B and possibly Mathematics C (a subject similar to Specialist Mathematics that can be taken alongside Mathematics B). According to Craig, the school has "good teachers … We do put them on our extension classes and our senior classes … that's very deliberate … it's very hard to extend kids if you don't know where you are extending them to" (Craig, June 2019). Elizabeth provided further explanation of the rationale for this approach:

> We put very experienced teachers in those [extension] classes. Teachers who are teaching senior maths like Maths B, Maths C. We give them these extension classes because they know where they have to lead these kids to, what skills are very important, say like algebra skills, the first thing for Maths B, Maths C, problem solving. By the time the kids are in 10 extension—so we have a good group who is going to Maths B class—they feel confident. (Elizabeth, June 2019)

This approach is not at the expense of providing quality mathematics teaching for all students: "We have other teachers who are exceptional at working with students who need support" (Craig, June 2019). There are some teachers in the school who were trained as primary teachers and moved to Cassowary Secondary College when Year 7 moved from primary to secondary schooling in Queensland. One of these teachers takes the Year 7 Extension class for both mathematics and science. Teachers taking the Year 8 and Year 9 Extension classes are "maths teachers who are trained to be maths teachers" (Elizabeth, June 2019). Students in Extension classes have the same teacher in Year 8 and Year 9.

*Selecting students for extension classes.* When students enter Cassowary Secondary College, they may be allocated to a Year 7 Extension Mathematics class based on their results in Year 5, Year 6, and numeracy in the National Assessment Plan—Literacy and Numeracy (NAPLAN). These students "might not be very extended or super genius in maths, but they're good kids, they're hard-working kids" (Elizabeth, June 2019). Students with potential are identified during Year 7 and a second Extension class is added for Year 8. The two Extension classes continue in Year 9 and, if warranted, Year 10: for Year 9 in 2019, "we've very deliberately kept two extension classes. So even though the kids aren't quite at that level, we push them, extend them" (Craig, June 2019). The make-up of the extension classes is not static with a review at least at the end of every term:

> You're never pigeonholed in … they can move. If they're doing well in Core, we will move them up. If we find they're really struggling in maths, then we'll move them down. If it's just a lack of work ethic, we try to push them harder. (Craig, June 2019)

Craig believes that the groundwork laid through the extension program is effective: by the time students get to Year 11, "we've done a lot of work with them, and they get there and they're pretty successful at getting through" (Craig, June 2019).

*For entry into Mathematics B and Mathematics C in Year 11, students need to obtain an A or B in Year 10 Extension mathematics*. The school runs a Summer School in the last two weeks at the end of Year 10 to give students every opportunity to demonstrate the required knowledge and skills if they did not meet this prerequisite.

*Supporting students' mathematical learning*. The school has a Homework Club that operates twice a week for one hour after school: "Lots of students go there. Lots of teachers. Nearly every maths teacher is there" (Elizabeth, June 2019). Homework Club provides an opportunity for students to get assistance with concepts but also provides an opportunity to engage in mathematics beyond what students do in their usual mathematics classes: "We've had extension groups and instead of being normal Homework Club, we would sit them in a room and give them a bit of food and just do some really hard problem solving" (Craig, June 2019).

*Providing access to resources*. Lack of access to textbooks and technological resources to support mathematics learning presents a challenge that the school is slowly addressing. Resources are being built up over time. Class sets of textbooks have been purchased by Elizabeth since she came to the school four years ago: "I was looking after junior maths and every year I was buying a set or two class sets for extension classes. Now, all extension classes have enough textbooks in the library so they can borrow" (November 2019). Most students now have access to graphics calculators: "one year we saved money, and we bought two sets of 25 calculators for the library, and last year I have organised the borrowing system so students can borrow a calculator for a year" (Elizabeth, November 2019). While the issue of access to textbooks and graphics calculators is being addressed, access to computers is still problematic: "We are very under-resourced with computers. We have three computer rooms, four computer rooms and three of them are permanently booked so it's literally impossible to get into a computer room" (Elizabeth, November 2019).

## *Pastoral Care Program*

The Pastoral Care Program at Cassowary State College includes a vertical house structure that focusses on *building relationships* between students and teachers and *providing support networks* such as Breakfast Club, and access to a wide range of professional support services. This program means that "school is a happy and safe place for many kids" (Elizabeth, June 2019).

*Building relationships.* The house system has four houses, each with four house leaders who are heads of department or experienced senior teachers. House leaders oversee three care classes which may have "20 students—there will be Year 7 students, Year 8 students, some Year 9, some Year 10 and some Year 11, 12" (Elizabeth, June 2019). Thus, house leaders are responsible for about 60 students who represent all year levels. A teacher conducts the day-to-day activities of each Care class and is supported the house leader: "I visit them every morning, wishing them good morning, asking how they been. If there are any issues, concerns, so I have to address it" (Elizabeth, June 2019). Students stay in the same Care class from the time they enter the school in Year 7 until they leave the school, thereby creating a close-knit group: "Every year three, four new students come and it's like little families. That's very good" (Elizabeth, June 2019). The major benefit of this approach are the relationships that are built: "When they're having those conversations [about future subject choices] so there is some buy in because of the relationships that are formed. As well as that, there is buy in from the parents" (Craig, June 2019).

Providing support networks. The school is acutely aware of the need to support students' physical and emotional wellbeing:

> "Some of what happens to our students outside school is horrendous, absolutely horrendous, what they live through. Without that support, they just wouldn't be able to come in and engage but with that support, thankfully they can get to school, and they can engage and be a little bit successful" (Craig, June 2019).

Although Craig acknowledged that it was unlikely that students he was referring to in this instance would be in Extension classes, the concern for these students was indicative of the culture of the school. In addition to professional support provided by a nurse and guidance counsellor, the schools pays

> for an extra guidance officer. Youth support coordinator, we pay a fairly large proportion of them, like maybe 50% … We've got two chaplains that we work hard to keep. So yeah, we are aware of the support needs but the big thing is our focus on learning … The other stuff, having good behaviour management or having lots of support networks is about supporting them to learn (Craig, June 2019)

Another way students are supported is through the provision of food. Breakfast is provided four days a week and students can get lunch if needed: "For some of our kids that's their meal for the day … We also provide lunch so if they rock up and haven't got food, staffrooms will have cheese toasties" (Craig, June 2019).

## Discussion

The person-in-context perspective provided by Rogoff's (1995) personal, social, and institutional planes of analysis has allowed a focus on institutional processes within Cassowary State College. These processes provide the context for the development of students' aspirations to study Mathematics B (now replaced by Mathematics Methods) in their final two years of schooling. The Mathematics Program and Pastoral Care Program were identified as potentially making a positive contribution to these aspirations.

The structure and staffing of the Mathematics Program coupled with opportunities for students to participate in extension classes, support for students' mathematical learning and the provision of resources provide a positive mathematics experience for students. There are many similarities between the Mathematics Program and the one at Marigold State High School—a school in a low SES area that had also shown a sustained increase in enrolments in mathematics B (see Bennison, et al., 2018). Both programs began with a single Year 7 Extension class with a second Extension class added for Year 8, Year 9, and Year 10. In both cases, therefore, as many students as possible were given the opportunity to follow a pathway to Mathematics B. There were also opportunities for students to move from Core classes to Extension classes, although this flexibility was spelt out much more clearly at Cassowary State College. Staffing of Extension classes in both schools was strategic with fully qualified mathematics teachers taking Extension classes as well as senior classes. The rationale for staffing choices was similar—to provide an accessible pathway to Mathematics B for students. Unsurprisingly, access to textbooks and technology (e.g., computers and graphics calculators) was an issue in both schools but the Heads of Department had worked hard at overcoming this issue. Interestingly, Marigold State High School had established a STEM program several years ago to encourage students to study mathematics and science whereas Cassowary State College had decided to do STEM-related activities within normal classes. One factor identified at Marigold State High School, not reported on here for Cassowary State College, was the positive culture of the Mathematics Department which possibly extends into the classroom to build a supportive environment. The culture of the Mathematics Department at Cassowary State College may contribute to building a supportive environment, but this would be in the context of the much broader school culture.

While the school's goal is to support student learning, there is recognition that this cannot happen if students' physical and emotional wellbeing is not addressed. Relationship building and access support networks is at the core of the Pastoral Care Program. The vertical house structure means that students can build relationships with their house leader and their Care teacher throughout their time at the school. It also provides an opportunity for older students to be role models for their younger peers. Recognition of the importance of support networks is evidenced by the school contributing financially to provision of extra services and ensuring that students whose access to basic necessities is limited are provided with food.

While the focus in this paper is at the *institutional* level, there is evidence of elements of Rogoff's (1995) *interpersonal* plane at work, particularly in the Pastoral Care Program. Analysis of interviews with teachers of the extension classes will shed further light on this plane. Additionally, focus group interviews with students will contribute to understanding the *social*, and *personal* plane. One of the limitations of this study is its size and scope. Data were collected in a single school at two points in time over a 6-month period. While the findings are promising, further research is needed.

## Concluding Remarks

Calls for high levels of participation and success in Science, Technology, Engineering and Mathematics (STEM) by all Australians (Office of the Chief Scientist, 2014) exacerbates the need for research into improving participation and achievement in mathematics by students from disadvantaged backgrounds (Vale et al., 2020). Lower participation rates in more challenging mathematics subjects, such as Mathematics Methods, in the senior years of schooling (e.g., Murphy, 2018), means that students from disadvantaged backgrounds have narrower post-school options than their peers from more affluent backgrounds. It is crucial, therefore, to identify factors that appear to contribute to promoting sustained interest and engagement in mathematics among students from disadvantaged backgrounds. One approach, as taken in this study, is to identify schools located in low SES areas that have shown sustained increase in enrolments in these subjects and investigate personal, social, and institutional processes that may influence students' subject choices.

The case study presented in this paper focusses on institutional processes at Cassowary State College and suggests that the Mathematics Program and Pastoral Care Program have contributed to a sustained increase in enrolments in Mathematics B in this school. The findings are largely consistent with earlier research (Bennison et al, 2018) and point to possible ways that institutional practices in schools in low SES areas can promote positive aspirations for mathematics. Further research is needed to explore practices in other schools in low SES areas that have demonstrated similar sustained increases in enrolments in Mathematics Methods (which has replaced Mathematics B), including through longitudinal studies that follow students from their entry into Year 7 until these students make subject choices for the senior phase of schooling.

## Acknowledgements

This study was supported by an Early Career Researcher Grant from the Faculty of Science, Humanities, Engineering, and Education. The author thanks participants for their contributions to the study.

# References

Australian Curriculum, Assessment and Reporting Authority. (2015). *What does the ICSEA value mean?* ACARA. https://docs.acara.edu.au/resources/About_icsea_2014.pdf

Bennison, A., Goos, M., & Bielinski, D. (2018). Identifying practices that promote engagement with mathematics among students from disadvantaged backgrounds. In J. Hunter, P. Perger, P., & L. Darragh (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 154–161). MERGA.

Department of Education (2022). *Why is STEM important?* https://www.education.gov.au/australian-curriculum/national-stem-education-resources-toolkit/introductory-material-what-stem/why-stem-important

Education Council (2015). *National School STEM Education Strategy 2016–2026*. https://www.education.gov.au/education-ministers-meeting/resources/national-stem-school-education-strategy

Murphy, S. (2018). School location and socio-economic status and patterns of participation and achievement in Year 12 enabling mathematics. In J. Hunter, P. Perger, & L. Darragh (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 583–590). Auckland: MERGA.

Office of the Chief Scientist (2014). *Science, technology, engineering and mathematics: Australia's future*. Australian Government. http://www.chiefscientist.gov.au/wp-content/uploads/STEM_AustraliasFuture_Sept2014_Web.pdf

Rogoff, B. (1995). Observing sociocultural activities on three planes: Participatory appropriation, guided appropriation and apprenticeship. In J. V. Wertsch, P. Del Rio, & A. Alverez (Eds.), *Sociocultural studies of mind* (pp. 139–164). Cambridge University Press.

Silverman, D. (2013). *Doing qualitative research* (4th ed.). Sage.

Stake, R. (2003). Case Studies. In N. Denzin, & Y. Lincoln (Eds.), *Strategies of qualitative inquiry* (2nd ed., pp. 134–164). Sage.

Thomson, S., De Bortoli, L., Underwood, C., & Schmid, M. (2020). *PISA 2018: Reporting Australia's results. Volume II: Student and school characteristics*. Melbourne: Australian Council for Educational Research (ACER). https://research.acer.edu.au/ozpisa/49

Vale, C., Averill, R., Hall, J., Forgasz, H., & Leder, G. (2020). Equity, social justice, and ethics. In J. Way, C. Attard, J. Anderson, J. Bobis, H. McMaster, & K. Cartwright (Eds.), *Research in mathematics education in Australasia 2016–2019* (pp. 177–208). Springer.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge University Press.

# Instructional Clarity, Classroom Disorder, and Student Achievement in Mathematics: An Exploratory Analysis of TIMSS 2019

Nathan Berger
*Western Sydney University*
n.berger@westernsydney.edu.au

Kathryn Holmes
*Western Sydney University*
k.holmes@westernsydney.edu.au

Erin Mackenzie
*Western Sydney University*
e.mackenzie@westernsydney.edu.au

It is generally understood that both clear teacher instruction and orderly classroom climates support student achievement in mathematics. However, to what extent does instructional clarity "compensate" for classroom disorder? In this exploratory study, we analyse data from 8,864 Year 8 students sampled by TIMSS 2019 to investigate the relationship between instructional clarity, classroom disorder, and mathematics achievement. The findings demonstrate the deleterious effects of classroom disorder for mathematics achievement, even in the presence of high instructional clarity. The findings contribute to an emerging international body of work and reinforce the importance of teachers having an optimal combination of classroom skills to support student learning.

A key competency for any teacher is to be able to clearly communicate with students to facilitate learning in the classroom. Instructional clarity can be defined as a teachers' capacity to deliver classroom instruction clearly and concisely (Maulana et al., 2016). In mathematics, instructional clarity is of key importance, as many students develop negative attitudes to the subject as they progress through school (Brown et al., 2008) and attitudes of this type are associated with lower mathematics achievement (Namkung et al., 2019). Instructional clarity has been found to enhance student self-efficacy and interest (Maulana et al., 2016) and is associated with higher achievement in mathematics (Thomson et al., 2021)

Disorder in secondary classrooms is a significant concern for teachers and students (Duesund & Ødegård, 2018), with significant negative impacts on student achievement, school belonging, and motivation (Hurd et al., 2018), as well as teacher stress and job satisfaction (Nash et al., 2016). Disorderly behaviour detracts from student learning and contributes to poor classroom climates in which social and emotional needs are unmet (Duesund & Ødegård, 2018). There are significant gender differences in teachers' perceptions of student misbehaviour, with male students viewed as being more likely to be disruptive and more difficult to control in the classroom (Glock & Keen, 2017).

Previous research has demonstrated that one precursor to disorderly classroom behaviour is poor instructional clarity. That is, if students do not understand the content and skills being taught, they may disengage and misbehave (Cothran et al., 2009). On the other hand, there are many other causes for student misbehaviour (Nash et al., 2016), and if this misbehaviour escalates to distract other students in the classroom, it is likely to detract from the impact of instructional clarity. Previous studies using Trends in International Mathematics and Science Study (TIMSS) 2019 data have found an association between instructional clarity, classroom management, and positive attitudes to mathematics in the US (Chen, 2022) and in the UK, but not in Hong Kong (Chen & Lu, 2022), revealing important contextual differences between countries.

In reporting Australian results for TIMSS 2019, Thomson and colleagues (2021) found that 40% of Year 8 students reported high clarity of mathematics instruction (compared to the 46% international average) and observed that these students had significantly higher mathematics scores

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 107–114). Newcastle: MERGA.

than those who reported low clarity instruction. For classroom climate, some 65% of Australian Year 8 students reported disorderly behaviour occurred in *some* lessons and 24% in *most* lessons (Thomson et al., 2021). Higher mathematics achievement was found to be strongly associated with lower levels of disorderly behaviour. However, a significant gap in the literature is the extent to which instructional clarity compensates for the impact of classroom disorder on academic achievement in mathematics in Australian classrooms. In this study, we explore the relationship between students' perceptions of instructional clarity, disorderly behaviour in mathematics classes, and mathematics achievement using Australian data from TIMSS 2019.

# Method

## *Dataset and Sample*

Data are from TIMSS 2019 administered by the International Association for the Evaluation of Educational Achievement (IEA). In Australia, a representative sample of 9,060 Year 8 students from 284 secondary schools participated in standardised mathematics achievement tests and completed context questionnaires about their mathematics classrooms (Thomson et al., 2020). The sample used in this paper is 8,864 due to missing data on key variables.

## *Variables*

*Instructional clarity during mathematics lessons.* Students' perceptions of instructional clarity in their mathematics lessons were measured using a seven-item scale. The question was *how much do you agree with these statements about your mathematics lessons* and sample items included *my teacher is easy to understand* and *my teacher does a variety of things to help us learn.* Students responded to each item using a four-point scale from *disagree a lot* to *agree a lot*. Scale cut scores were calculated by the IEA to indicate Low Clarity (disagree a lot or a little), Medium Clarity (agree a little), and High Clarity (agree a lot). The scale had excellent reliability for the Australian sample ($\alpha = .92$).

*Disorderly behaviour during mathematics lessons.* Students' perceptions of classroom disorder in their mathematics lessons were measured using a six-item scale. Sample items included: *My teacher has to wait a long time for students to quiet down* and *it is too disorderly for students to work well*. Students responded to each item using a four-point scale from *never* to *every or almost every lesson*. Scale cut scores were calculated by the IEA to indicate Low Disorder (disruption during few or no lessons), Medium Disorder (disruption in some lessons), and High Disorder (disruption in most lessons). This scale was highly reliable for the Australian sample ($\alpha = .92$).

*Classroom climate.* To explore the relationship between instructional clarity and disorderly behaviour, the two TIMSS variables described above were combined to create the following nine groups characterising the classroom climate from the students' perspective: 1) Low Clarity/Low Disorder (LC/LD), 2) Low Clarity/Medium Disorder (LC/MD), 3) Low Clarity/High Disorder (LC/HD), 4) Medium Clarity/Low Disorder (MC/LD), 5) Medium Clarity/Medium Disorder (MC/MD), 6) Medium Clarity/High Disorder (MC/HD), 7) High Clarity/Low Disorder (HC/LD), 8) High Clarity/Medium Disorder (HC/MD), and 9) High Clarity/High Disorder (HC/DC).

*Mathematics achievement.* TIMSS assessed student achievement across content and cognitive domains and item response theory was used to generate a single achievement score where higher scores indicated better achievement in the subject (Fishbein et al., 2021). Due to TIMSS's complex sampling method, each participant's mathematics achievement score was recorded as a series of five plausible values. These require special handling during analyses to prevent biased estimates of mathematics achievement (Berger et al., 2020).

### *Analysis*

Analyses were conducted using IBM SPSS Statistics 27. Syntax for analyses involving plausible values were created using the IEA IDB Analyser software, allowing accurate estimates of achievement scores and standard errors and statistical weighting so the sampled students represent the total population of Australian Year 8 students (Berger et al., 2020). The association between mathematics achievement, classroom climate, and gender was primarily investigated using linear regression (Fishbein et al., 2021). Dummy coding was used to enter categorical variables into linear regressions as independent variables.

## Results

Table 1 shows descriptive statistics for the classroom climate groups. The largest groups were Medium Clarity/Medium Disorder (28%) and High Clarity/Medium Disorder (28%) while the smallest group was Low Clarity/Low Disorder (1%). Mathematics achievement was highest in the High Clarity/Low Disorder group (581.33) and lowest in the Low Clarity/Medium Disorder group (485.87). While overall there were equal numbers of girls and boys in the sample, the proportions differed in each of the classroom climate groups. To highlight the highest deviations from the sample split, there were more girls than boys in the Low Clarity/Medium Disorder group, while there were more boys than girls in the High Clarity/Medium Disorder and High Clarity/High Disorder groups.

**Table 1**

*Descriptive Statistics for Classroom Climate Groups*

| | Entire sample | | Girls | | Boys | |
|---|---|---|---|---|---|---|
| Group | n (%)* | *M* (SD) | *n* (%)† | *M* (SD) | *n* (%)† | *M* (SD) |
| LC/LD | 98 (1%) | 490.03 (100.45) | 52 (53%) | 488.08 (92.15) | 46 (47%) | 491.69 (106.96) |
| LC/MD | 852 (9%) | 485.87 (79.36) | 463 (54%) | 491.27 (80.20) | 389 (46%) | 479.73 (77.91) |
| LC/HD | 488 (5%) | 489.67 (77.69) | 258 (53%) | 494.09 (67.59) | 230 (47%) | 484.61 (87.45) |
| MC/LD | 314 (4%) | 555.05 (88.37) | 165 (53%) | 548.30 (81.46) | 149 (47%) | 561.95 (94.31) |
| MC/MD | 2508 (28%) | 514.88 (86.71) | 1278 (51%) | 517.05 (82.97) | 1230 (49%) | 512.67 (90.30) |
| MC/HD | 807 (9%) | 488.67 (86.42) | 418 (52%) | 484.77 (81.70) | 389 (48%) | 492.79 (90.39) |
| HC/LD | 726 (8%) | 581.33 (85.83) | 368 (51%) | 574.69 (80.33) | 358 (49%) | 587.64 (90.26) |
| HC/MD | 2529 (28%) | 538.53 (86.85) | 1195 (43%) | 529.65 (80.29) | 1334 (57%) | 546.61 (91.66) |
| HC/HD | 542 (6%) | 504.14 (80.43) | 231 (43%) | 502.63 (71.55) | 311 (57%) | 505.32 (86.74) |
| | 8864 (100%) | 518.40 (89.19) | 4428 (50%) | 515.83 (83.40) | 4436 (50%) | 520.94 (94.50) |

*Percentages as proportion of entire sample.
†Percentages as proportion within group. Percentages may not add to 100% due to rounding.

### *Classroom Climate and Mathematics Achievement*

The first analysis explored whether there was an association between mathematics achievement and classroom climate. Figure 1 shows mathematics achievement in each of the classroom climate groups.

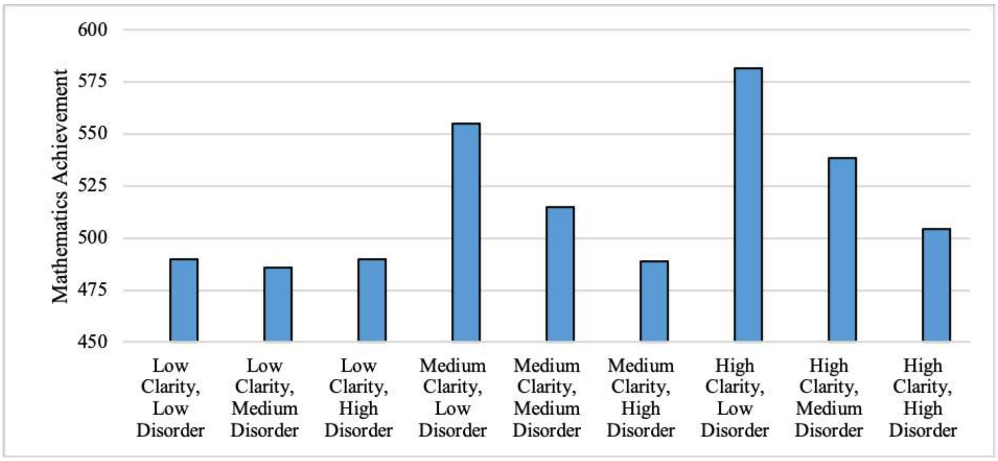



Figure 1. Mathematics achievement in classroom climate groups.

Table 2 shows the results of the linear regression analysis with mathematics achievement as the dependent variable and classroom climate group as the dummy-coded categorical independent variable. Low Clarity/Low Disorder was the reference category and was held constant in the regression. Mathematics achievement in the Medium Clarity/Low Disorder, High Clarity/Low Disorder, and High Clarity/Medium Disorder groups was statistically significantly higher than in the reference category. As such, mathematics achievement in medium clarity instructional environments was no different to low clarity instructional environments when classroom disorder also was medium or high. Furthermore, while high clarity instructional environments appeared to compensate for low and medium levels of classroom disorder, high levels of classroom disorder nullified the effect of high clarity. The mathematics achievement of students in High Clarity/High Disorder classrooms was not statistically different to that of their peers in Low Clarity/Low Disorder classrooms.

**Table 2**

*Regression Coefficients for Mathematics Achievement in Clarity/Disorder Groups*

| Group | B | SE | t |
|---|---|---|---|
| (Constant) | 490.03 | 16.81 | 29.15 |
| LC/MD | −4.15 | 16.77 | −0.25 |
| LC/HD | −0.36 | 18.7 | −0.02 |
| MC/LD | 65.03* | 15.82 | 4.11 |
| MC/MD | 24.86 | 16.02 | 1.55 |
| MC/HD | −1.35 | 17.11 | −0.08 |
| HC/LD | 91.31* | 16.12 | 5.66 |
| HC/MD | 48.50* | 16.30 | 2.98 |
| HC/HD | 14.11 | 16.90 | 0.83 |

*Note.* Reference group (constant) is LC/LD.
*Statistically significant difference from the reference group at $p < 0.05$.

*Gender Differences*

The second analysis explored whether there were gender differences in the association between classroom climate and mathematics achievement. Figure 2 shows mathematics achievement in each of the classroom climate groups split by gender.

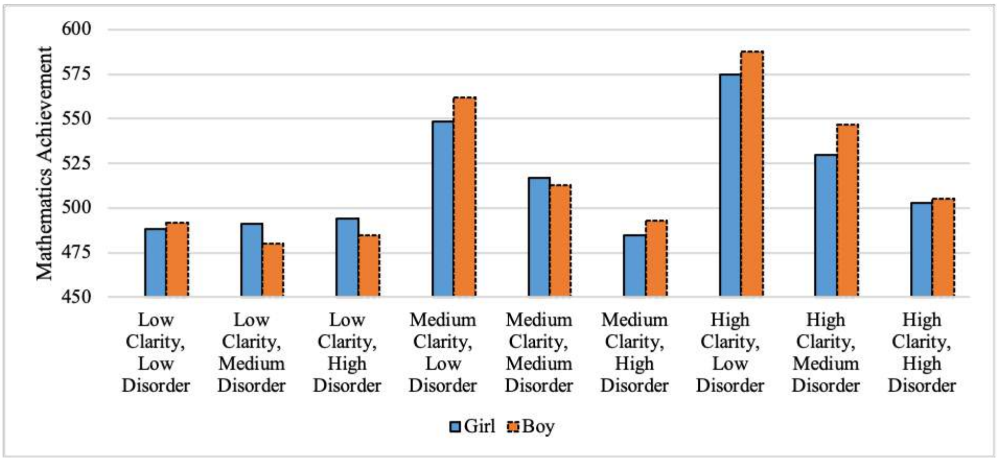



Figure 2. Mathematics achievement by gender in classroom climate groups.

Table 3 shows the results of linear regression analyses with mathematics achievement as the dependent variable and gender as the dummy-coded categorical independent variable, with separate regressions for each classroom climate group. Girls were the reference category which was held constant in each regression. The only statistically significant difference between genders was in High Clarity/Medium Disorder classrooms where boys scored higher on mathematics achievement than girls. As such, the observed association between classroom climate and mathematics achievement largely does not differ between boys and girls.

**Table 3**

*Regression Coefficients for Maths Achievement by Gender in Classroom Climate Groups*

| Group | Gender | B | SE | t |
|---|---|---|---|---|
| LC/LD | (Constant) | 488.08 | 18.92 | 25.80 |
| | Boys | 3.61 | 31.42 | 0.11 |
| LC/MD | (Constant) | 491.27 | 6.55 | 75.01 |
| | Boys | −11.54 | 7.63 | −1.51 |
| LC/HD | (Constant) | 494.09 | 7.17 | 68.96 |
| | Boys | −9.48 | 11.83 | −0.80 |
| MC/LD | (Constant) | 548.30 | 12.62 | 43.45 |
| | Boys | 13.65 | 18.76 | 0.73 |
| MC/MD | (Constant) | 517.05 | 4.15 | 124.55 |
| | Boys | −4.39 | 5.68 | −0.77 |
| MC/HD | (Constant) | 484.77 | 6.31 | 76.87 |
| | Boys | 8.03 | 9.07 | 0.89 |
| HC/LD | (Constant) | 574.69 | 6.95 | 82.71 |
| | Boys | 12.95 | 13.43 | 0.96 |

| Group | Gender | B | SE | t |
|---|---|---|---|---|
| HC/MD | (Constant) | 529.65 | 4.84 | 109.54 |
| | Boys | 16.96* | 7.50 | 2.26 |
| HC/HD | (Constant) | 502.63 | 6.26 | 80.24 |
| | Boys | 2.69 | 9.55 | 0.28 |

*Note.* Reference category (constant) is Girls.
*Statistically significant difference from the reference group at $p < 0.05$.

## Discussion

Australian mathematics classrooms are characterised by sustained disorderly behaviour at levels above international averages (Thomson et al., 2021). Previous research into instructional clarity and classroom disorder has demonstrated the impact of these phenomena on mathematics achievement (Namkung et al., 2019; Thomson et al., 2021). While poor instructional clarity can lead to greater classroom disorder (Cothran et al., 2009), the two concepts are not indivisible. Indeed, our analysis showed the extent to which students separately may perceive the phenomena operating in their mathematics classrooms. Some students perceived high instructional clarity in otherwise highly disordered classrooms. In this context, relatively little is known about the extent to which instructional clarity could ‘compensate’ for the sustained levels of disorderly behaviour seen in Australian mathematics classrooms. Recent international research has found associations between instructional clarity, classroom disorder, and achievement differ by country (Chen, 2022; Chen & Lu, 2022). So how might these factors play out in Australian classrooms?

The findings from our exploratory analysis of a representative sample of Australian adolescents reveal important links between instructional clarity, classroom disorder, and student achievement in mathematics. These findings reinforce concerns about the impact of classroom disorder on achievement (Duesund & Ødegård, 2018; Hurd et al., 2018). Specifically, we found that while higher student perceptions of instructional clarity were generally associated with higher achievement, that achievement was significantly lower in the presence of medium or high levels of disorder. If instructional clarity was low, however, the level of classroom disorder did not appear to be related to mathematics achievement levels. Orderly classrooms, therefore, appeared to be an enabling condition for teacher instruction to have the desired effect on student achievement in mathematics.

To highlight a particular finding from our analysis, the mathematics achievement of students in high clarity/high disorder classrooms was *not statistically different* to that in low clarity/low disorder classrooms. As such, instructional clarity was not able to compensate for disorderly behaviour. A practical implication is that an effective teacher must be able to manage their classroom *as well as* provide clear mathematical instruction to students. It is important to note that student misbehaviour is a complex issue with many potential antecedents. Teachers are not solely responsible for managing student disorder in classrooms and must be supported through school and system policies to achieve optimal learning environments.

In terms of gender, we found no differences in the achievement levels of Year 8 boys and girls across the nine categories, except in the case of the high instructional clarity/medium disorder category where boys achieved significantly higher than girls. These results largely accord with those of Thomson et al. (2021) who found no gender differences in mathematics achievement between boys and girls in aggregate. However, the finding begs the question of why boys achieved better than girls in conditions of medium disorder? Research suggests that boys are more likely to be disruptive in the classroom (Glock & Keen, 2017), so in such circumstances are boys more likely than girls to benefit from greater instructional clarity? Further research is required to fully understand this anomalous finding.

### *Limitations and Future Research*

The exploratory analysis presented in this paper used established TIMSS categories (low, medium, and high) to examine the relationships between student perceptions of instructional clarity, classroom disorder, and their mathematics achievement. Future research could draw on person-centred analyses to develop student profiles based on statistical modelling of students with similar characteristics. In this way, a more nuanced understanding of how student perceptions of instructional clarity and classroom disorder are clustered may be revealed, allowing for a more sophisticated analysis of how these variables are related to student achievement. It should also be noted that we examined students' perceptions of classroom disorder. Future research is needed to determine whether teachers' perceptions of classroom disorder align with the views of their students and are similarly related to student achievement.

## Conclusion

There is substantial evidence that instructional clarity and classroom disorder affect student achievement in a variety of subjects, including mathematics (Maulana et al., 2016; Thomson et al., 2021). However, less is known about the combinatorial effects of clarity and disorder, with extant studies having context-dependent outcomes (Chen, 2022; Chen & Lu, 2022). Our exploratory analysis of Australian TIMSS 2019 data provides further evidence of this. In the Australian context, higher mathematics achievement was associated with high instructional clarity but only when levels of classroom disorder were low. The findings point to the critical importance of Australian mathematics teachers developing clear instructional skills *and* being adequately supported to effectively respond to student misbehaviour.

## References


Berger, N., Mackenzie, E., & Holmes, K. (2020). Positive attitudes towards mathematics and science are mutually beneficial for student achievement: A latent profile analysis of TIMSS 2015. *The Australian Educational Researcher, 4*(3), 409–444. https://doi.org/10.1007/s13384-020-00379-8

Brown, M., Brown, P., & Bibby, T. (2008). "I would rather die": Reasons given by 16-year-olds for not continuing their study of mathematics. *Research in Mathematics Education*, *10*(1), 3–18. https://doi.org/10.1080/14794800801915814

Chen, X. (2023). Instructional clarity and classroom management are linked to attitudes towards mathematics: A combination of student and teacher ratings. *British Journal of Educational Psychology*, *93*(2), 591-607. https://doi.org/10.1111/bjep.12580

Chen, X., & Lu, L. (2022). How classroom management and instructional clarity relate to students' academic emotions in Hong Kong and England: A multi-group analysis based on the control-value theory. *Learning and Individual Differences*, *98*, 102183. https://doi.org/10.1016/j.lindif.2022.102183

Cothran, D. J., Kulinna, P. H., & Garrahy, D. A. (2009). Attributions for and consequences of student misbehavior. *Physical Education and Sport Pedagogy, 14*(2), 155–167. https://doi.org/10.1080/17408980701712148

Duesund, L., & Ødegård, M. (2018). Students' perception of reactions towards disruptive behaviour in Norwegian and American schools. *Emotional and Behavioural Difficulties, 23*(4), 410–423.

Fishbein, B., Foy, P., & Yin, L. (2021). *TIMSS 2019 user guide for the international database* (2nd Edn.) International Association for the Evaluation of Educational Achievement.

Glock, S., & Kleen, H. (2017). Gender and student misbehavior: Evidence from implicit and explicit measures. *Teaching and Teacher Education, 67*, 93–103. https://doi.org/10.1016/j.tate.2017.05.015

Hurd, N. M., Hussain, S., & Bradshaw, C. P. (2018). School disorder, school connectedness, and psychosocial outcomes: Moderation by a supportive figure in the school. *Youth & Society, 50*(3), 328–350. https://doi.org/10.1177/0044118X15598029

Maulana, R., Opdenakker, M.-C., & Bosker, R. (2016). Teachers' instructional behaviors as important predictors of academic motivation: Changes and links across the school year. *Learning and Individual Differences*, *50*, 147–156. https://doi.org/10.1016/j.lindif.2016.07.019

Namkung, J. M., Peng, P., & Lin, X. (2019). The relation between mathematics anxiety and mathematics performance among school-aged students: A meta-analysis. *Review of Educational Research*, *89*(3), Article 3. https://doi.org/10.3102/0034654319843494

Nash, P., Schlösser, A., & Scarr, T. (2016). Teachers' perceptions of disruptive behaviour in schools: A psychological perspective. *Emotional and Behavioural Difficulties, 21*(2), 167–180. https://doi.org/10.1080/13632752.2015.1054670

Thomson, S., Wernert, N., Rodrigues, S., & O'Grady, E. (2020). *TIMSS 2019 Australia. Volume I: Student performance.* Australian Council for Educational Research. https://doi.org/10.37517/978-1-74286-614-7

Thomson, S., Wernert, N., Buckley, S., Rodrigues, S., O'Grady, E., & Schmid, M. (2021). *TIMSS 2019 Australia. Volume II: School and classroom contexts for learning.* Melbourne: Australian Council for Educational Research (ACER). https://doi.org/10.37517/978-1-74286-615-4

# Mapping Teacher Moves when Facilitating Mathematical Modelling

Jill P Brown
*Deakin University*
jill.brown@deakin.edu.au

Gloria A Stillman
*Retired*
stillmanga@gmail.com

This paper explores use of a set of diagrammatic tools for representation and analysis of the moves a teacher makes implementing a mathematical modelling task. The focus here is on identifying what the teacher did so we can subsequently interrogate this, as to the why. Data include pre and post lesson teacher interviews and transcripts of a video and audio-recorded task implementation. The analytical tools developed, with one teacher and one task early in a three-year project were particularly useful in ascertaining what the teacher moves were as we subsequently sought to determine reasons for these.

Our focus is on why teachers do what they do, particularly when implementing mathematical modelling tasks with upper secondary students. The first step in answering the 'why' question is to ascertain 'what' they do. The research reported here is part of the Victorian component of the Australian Research Council *Discovery Project* (DP17010555), *Using Mathematics to Solve Real-World Problems: The Role of Enablers Project*. This two-state research project, Queensland and Victoria, included a focus on identifying teaching approaches supportive of Year 10 and 11 students' modelling. We focused on identifying enablers of student success during mathematical modelling (Geiger et al. 2018). The two states have distinct characteristics with regard to expectations for mathematical modelling (Brown, 2020).

The Victorian Curriculum and Assessment Authority (VCAA) Study Design: Mathematics in the Victorian Certificate of Education (VCE) describes essential mathematical activities as including "conjecturing, hypothesising and *problem posing*; estimating, calculating, and computing; abstracting, proving, refuting and inferring; applying, investigating, *modelling and problem solving*" (emphasis added, VCAA, 2015, p. 6). Our interest is firmly centred on mathematical modelling and the associated problem posing and problem solving. Mathematical modelling is currently an expected part of teaching, learning, and assessment in the Victorian upper secondary curricula (i.e., the VCE). With regards to assessment, modelling tasks are expected to be of 2-3 hours duration. In Year 10, modelling is described within the *Learning in Mathematics* statement as part of the problem-solving proficiency, with students expected to consider both unfamiliar and meaningful problems and in doing so "make choices, interpret, formulate, model and investigate problem situations, select and use technological functions and communicate solutions effectively" (VCAA, 2016). Adding to the complexity of the classroom, but also the complexity and authenticity of accessible real-world contexts, students are also expected to make use of mathematical digital technology. In VCE Mathematics, one of the three aims is for students to "use technology effectively as a tool for working mathematically" (VCAA, 2015, p. 6) with students engaging with contexts and problems from "well defined and familiar to open ended and unfamiliar" (VCAA, 2016, p. 6). This can increase student engagement (Stillman et al., 2010), be used throughout the modelling process (Stillman et al. 2015), but further research is needed (Stillman & Brown, 2014).

However, Wedelin and Adawi (2015, p. 26) suggest that school students are generally "taught to *solve well-defined problems by using given methods*". Thus, students too often have scant experiences with the early stages of problem solving which is unavoidable in modelling where students need to make sense of the messy real-world situation and subsequently formulate a problem (i.e., mathematise) that can be solved mathematically. For example, 'best' will need to be defined by the task solvers in many modelling tasks. Attempts to solve modelling tasks typically involves struggle—the task can be challenging to understand as well as challenging to solve. Different solvers may make a different determination of best which can be a novel situation for learners who have mainly experienced one way to solve a problem according to Wedelin and Adawi (p. 30).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 115–122). Newcastle: MERGA.

The nature of the inherent complexity and hence challenge in solving modelling tasks generally sees students working together to solve tasks. This aligns with the argument of Stillman et al. (2020) that the practices in school mathematics classrooms should mirror those of professional mathematical modellers. In particular, the practices of making meaning, anticipating or engaging in mathematical foresight, arguing mathematically, and validating or justifying. These practices are inherent in mathematical modelling at school level when students work in co-present groups (Hager & Becket, 2019). Classroom discourse has the productive effect of "making thinking visible" (Collins et al., 1991, p. 38). When students articulate their thinking, not only is the group more likely to make progress, but also teachers might be more likely to withhold or delay in-the-moment scaffolding as they are more aware of student reasoning. Teachers are the key to developing students as successful modellers, including in articulating their thinking within their group, and to the class. Once modelling is included in curriculum documents, *whether* it is implemented and *how* it is implemented depends on the teacher (Chapman, 2007). Our focus, within a context explicitly including mathematical modelling, is, what is the thinking of teachers who include modelling in their teaching and what pedagogical strategies do they use that facilitate modelling?

## Background to the Study

To illustrate the complexity involved in mathematical modelling we include the modelling cycle diagram (see Figure 1) (itself a model) used by researchers, teachers, and students. This modelling cycle diagram has been particularly useful both in developing a shared understanding between researchers and teachers of the complexities involved in modelling but also as a useful scaffold by teachers and students during modelling. Seven stages, A-G, are shown beginning in the messy real world. During modelling, modellers move back and forward between stages, rather than as simplified in the diagram. Between each stage, represented by an arrow, are sets of activities required to transition from one stage to another. Examples of relevant activities are part of the diagram. Between the stages, are double headed arrows representing metacognitive activity, an essential part of successful modelling.

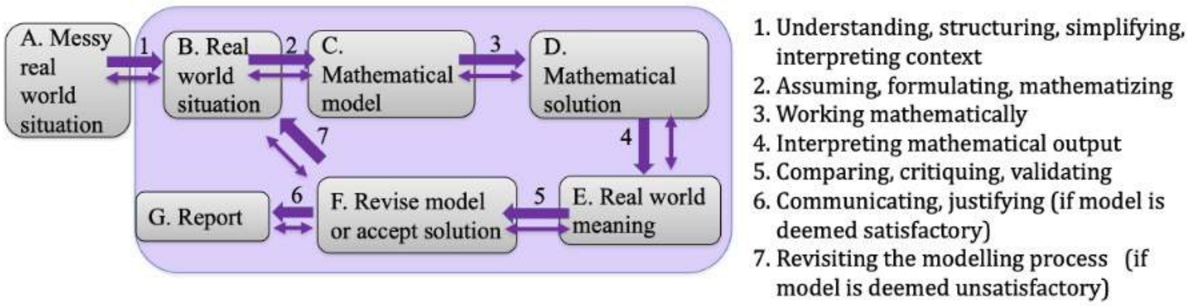



Figure 1. Representation of modelling cycle diagram used in the project (Stillman, 2011).

Elsewhere we have focused on student reasoning in data-rich modelling tasks (Stillman & Brown, 2023) and teacher tactics (Brown & Stillman, 2023). In this paper we focus on the actions of the teacher. We are particularly interested in ascertaining *why teachers do what they do*, but this must be preceded by determining *what teachers do*. In any mathematics classroom, this can be a challenge. It is certainly a challenge in upper secondary classrooms with students working in groups using digital technology and working on challenging modelling tasks.

## Research Design and Methodology

One teacher is the focus of the analysis here, selected based on her participation in year 1 of the project. As an experienced teacher of mathematics, including mathematical modelling, we expected she would be able to articulate the reasons for her actions as well as if these were planned or based on in-the-moment decisions. The school was a regional senior secondary college with co-educational classes. The Year 11 year-long mathematics subject included expectations of problem solving, investigations, and modelling (VCAA, 2015). Due to administrative challenges in the research project, the class was only able to work on one modelling task, *Bush Walking with Hilary* (BWwH). In seeking to determine what the teacher was doing and saying, as she interacted with the co-present groups (Hager & Becket, 2019)—our unit of analysis—we employed multiple data sources and several analytical techniques. Although this paper focusses on one teacher, our close analysis of the data was intended to apply to other components in the three-year project (different teachers, classes, tasks, and over time). Data included in the analysis reported here includes semi-structured pre- and post-lesson implementation teacher interviews (Flick, 2006), video and audio recordings, and field notes.

Using video and audio recordings, student scripts, and researcher field notes, we created a case record (Miles et al., 2014) for individual student groups. The case record included the transcript, which documented the discourse and actions of the students and teacher. These were supplemented with stills from the video recordings, at times annotated to show gestures of interest. Subsequently a case record for the teacher was constructed. What occurred as the lesson unfolded may not be so obvious to the researcher. For example, a teacher may spend more time interacting with one group than others. This may have been pre-planned. For example, a group included a student new to the class, so the teacher wanted to ascertain the group collaboration was proceeding smoothly (i.e., teacher intends to interact closely with Group 1). The reason may be more general with the intention focused on working with any group apparently struggling to get started (i.e., teacher interacts with Group 2 who seem to be unfamiliar with the context) (see Stillman & Brown, 2023). The interactions, intended or realised, may be of increased frequency or of longer duration. However, as we had multiple data sources, including pre and post teacher interviews, we had rich data to interrogate.

### *Teacher Moves*

Diagrams are particularly useful analytical tools (Miles et al., 2014). Diagrams show what is hidden within transcripts and case records. Diagrams also allow us to focus simultaneously rather than sequentially on patterns in the data (p. 108). One question of interest related to the way in which teacher interactions with students unfolded during task implementations. To analyse activity in the classroom as students collaborated on a modelling task, we coded the case record into four categories of interactions. We refer to *Teacher Group Interactions* (TGIs) as occurring when the teacher interacted with a student group. The TGI could be initiated by the teacher or a student. Unsurprisingly, at times an interaction was preceded by the teacher listening to what a particular group was saying or doing. At times the teacher engaged in this listening but did not immediately follow this with a TGI. We described such interactions as *Teacher Group Listening* (TGL). The student group may or may not have been aware of a TGL. Where the TGL was immediately followed by an interaction, this was coded as part of the TGI. On some occasions, this also occurred at the conclusion of an interaction. *Whole Class Interactions* (WCI) occurred when the teacher was interacting with the whole class. We note that as the students are upper secondary students, at times they chose to ignore the teacher's intentions. We coded *Group Interactions* (GI) where the students were working on the task, independently of the teacher. At times this involved some or all group members in discourse with each other, sharing screens from their technological devices, and working together to record ideas and outputs from technological devices. At other times, individuals in the group were working more independently, albeit mainly toward a common goal.

### *Sequence Diagrams*

In looking at how the teacher moved around the room, we asked how the interactions with each group unfolded. There were seven student groups working on the *BWwH task.* Figure 2 shows that she interacted with each group, before a second interaction with any group occurred, with the exception of Group 6. The first interaction was with Group 7, followed by Group 6 and Group 4. The shaded rectangle indicates a TGL. This analysis and subsequent diagram told some, but not all, of the story of how the teacher interacted with the students.

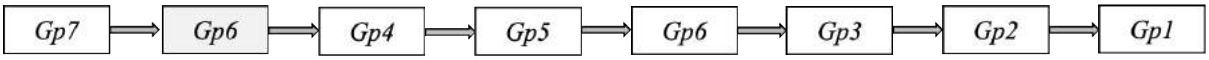



Figure 2. Sequence whereby TB1 interacts with all groups.

## Mud Maps

The layout of the room may also have influenced the order of the TGIs so we created a mud map of the classroom layout, as shown in Figure 3. A mud map is a rough sketch (later digitised) created by the researchers at the beginning of the lesson. Each rectangle represents the table and approximate location of the named group. Group 1 is located at the front of the room adjacent to the teacher table, whereas Group 6 was located at the back to the room near the door. Several video and audio recorders were located in the room to capture as much of the rich data as possible, but these are not shown on in Figure 3. On the righthand side of Figure 3, the teacher's initial set of interactions with the student groups staring from the yellow dot is shown. The mud map represents the same information as Figure 1, but this two-dimensional view provides a greater sense of how these interactions unfolded in the actual classroom.

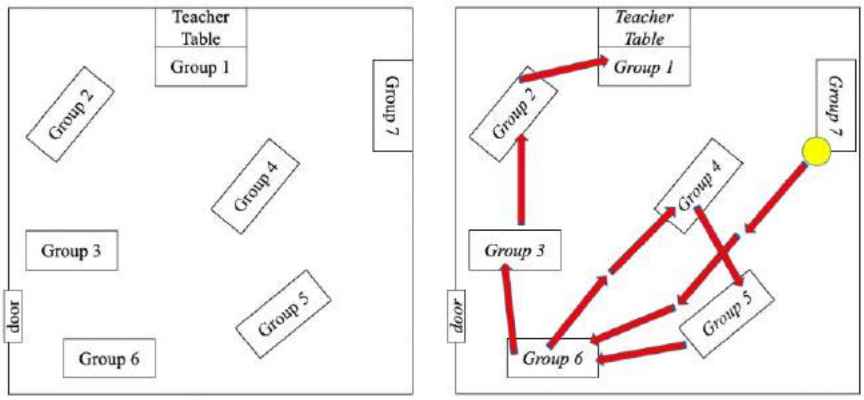



Figure 3. Mud map of the classroom setup by the teacher (left) and the initial teacher group interactions.

The layout of the room was considered to determine if this might have influenced the sequence of interactions. In some lessons in the project, the teacher deliberatively set up the room, in others, the furniture was used as left by the previous class. How the groups were determined, and located in the room, also varied (teacher or student selection or a combination) and these were considered in our analysis. In this class, the teacher selected the groups of two to three students and indicated where they should sit during the modelling lesson.

## Timeline Diagrams

Noting that the sequence diagram showing teacher student interactions highlighted some aspects of what occurred, we also created timeline diagrams. The timeline is shown in 6 second segments and we acknowledge times are estimates, due to rounding and also using multiple recording devices that began recording at different times. Our careful re-listening and re-watching the recordings has however resulted in an accurate portrayal of what occurred in the classroom. The timeline diagram has three sets of columns. Column 1 is segmented in 30 second sections, white for the first 30 seconds, then grey for the second thirty seconds of each minute. Each of these are further partitioned into five 6 second sections in column 2. Column 3 indicates the type of interactions. A section of a timeline diagram is presented in Figure 4.

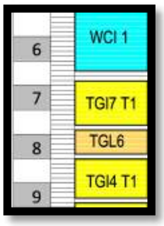



Figure 4. A snippet of a timeline diagram for the class.

Initially, a WCI is in progress as the teacher introduced the task and initiates a discussion about the context. This is followed by 12 seconds where the teacher is not interacting with the students. From 6:12-6:18 mins the first TGI occurs as the teacher initiated an interaction with Group 7. The coding T1 indicates this is the first teacher-initiated interaction with this group. This is followed by the teacher listening to, but not interacting with Group 6. Next, we see the first interaction between the teacher and Group 4, also initiated by the teacher. The white rectangles in column 3 are indicative of the times when the teacher might be moving from one group to another, or watching the class more globally, or preparing to bring the class together for a WCI. The case record would be examined where this detail was needed.

## Analysis and Findings

Our analysis, although presented here for the *Bush Walking with Hilary* (BWwH) implementation, is focused on what we can learn about 'why teachers do what they do' more broadly. The sequence of teacher-student interactions over the course of the lesson is presented in Figure 5 with the mud map of these in Figure 6 (with arrows coloured red, orange, green then black). Each new row shows where the teacher began a new mini sequence of interactions. In rows 1 and 2, a TGI occurred with each group, indicating interaction with every group. However, in row 3 only five TGIs occurred before returning to Group 6. This part of the lesson saw no interactions with Groups 4 and 5. This sequence suggests the teacher's intention was to interact with each group and then repeat this as the lesson unfolded. The location of WCI2-4 in the sequence is indicated by black arrows with WCI1 and WCI5 bookending the sequence.

The timeline diagram (Figure 7) indicates that in this modelling lesson implementation, the teacher began with an extended *Whole Class Interaction* (WCI1, 6 mins 13 secs) and most of the lesson saw the students working in their groups. Three shorter WCIs occurred in the final stages of the lesson and a final WCI occurred as the teacher wrapped up the lesson. The timeline diagram also shows the teacher interacting with various groups for almost all of the lesson. The number of TGIs per group varied in number (3-5) and duration (13 sec—5:13 mins). The shortest TGI (TGI6 T1) came shortly after the only TGL. The total TGI time per group varied from 1:30 minutes (Group 1) to 9:29 mins (Group 6).

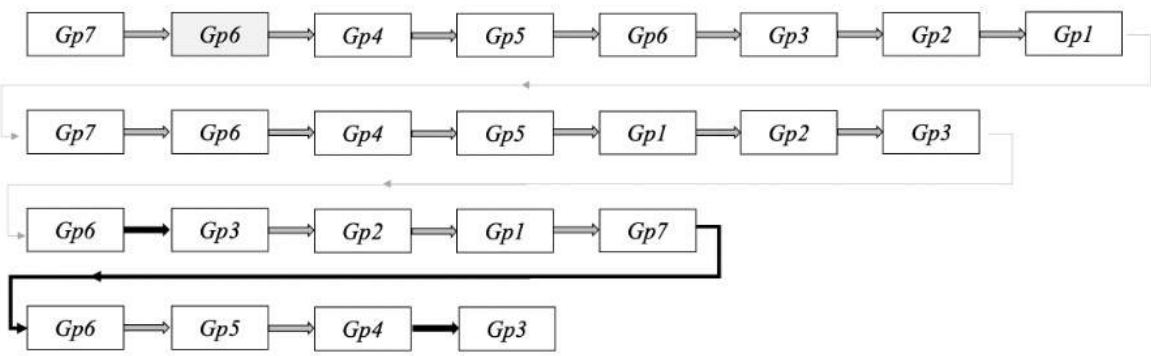



Figure 5. Sequence of teacher-student interactions during task implementation.

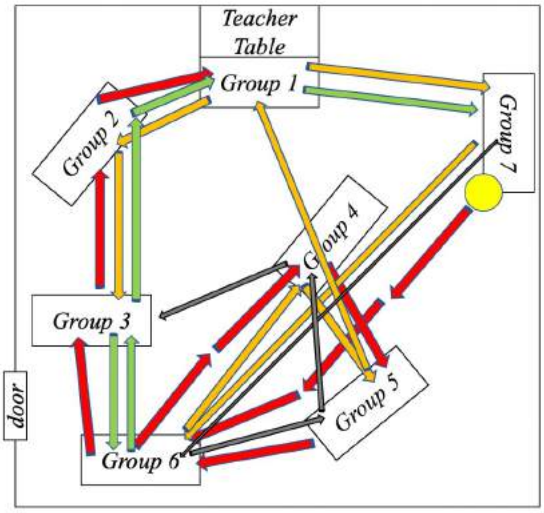



Figure 6. The complete mud map with TGIs and TGL for the class undertaking the *BWwH task*.

## Discussion and Concluding Remarks

While our ultimate focus in on *why teachers do what they do,* this paper focuses on ways of ascertaining *what they do*. The diagrams proved particularly useful in guiding further analysis of the complexity of the thinking (Hager & Beckett, 2019) both mathematical and pedagogical. Discourse analysis was very much a focus of our broader analysis (e.g., Stillman & Brown, 2023) during various types of interaction and the visual displays created and presented here were particularly productive. Here we report our broad categories of discourse in this task implementation as the teacher interacted with the co-present groups. For this teacher, we generalise three distinct categories of interactions, based on what the teacher did and said. These correspond with Row1, 2, and the combined rows 3 and 4 presented in Figure 5 and coloured paths in Figure 6. In the first set of TGIs, the teacher asked each group to respond to her question: *what is the problem about?* In her second mini sequence of teacher student interactions, she again asked, *what is the problem?* but added, *what is your approach*? In the third mini sequence, her focus was: *can you verify your model? can you convince me it works?* During the WCIs, she encouraged students to *write down where you are at*.

In her pre-lesson implementation interview, the teacher explained she thought the students would be "be very slow to get started. This particular class like rules, formulae, here's how you do it, go and apply these rules. We don't do a lot of open-ended questions." She elaborated further, "slow to start and then they are very talkative, … they'll follow each other a little bit".

She wanted to see "if they adapt and change [their] planning if it's not working … keep the good bit …. syphoning out what bit is the wrong bit, that's not helping and throwing that away but keeping the part" that is on track. After explaining how she would introduce the task, she described her approach of supporting students, "I won't say do this, do that, I'll give them hints to think about" noting she would give more structured hints if "there's a group that's not really able to get started" [Pre-lesson implementation interview].

As the lesson unfolded, it was evident that one purpose of the teacher's interactions was to encourage collaboration and communication even if she did not always, in the moment, follow what they were doing. "They were a little bit hesitant to say what they thought they were doing. They didn't communicate it very well" [Post-lesson implementation interview]. "When they were telling me what they were doing, sometimes I knew what they were doing and made out I didn't, so they'd say it a bit better" thus helping them communicate their thinking. At other times, "I really didn't know what they were saying, they didn't make sense."

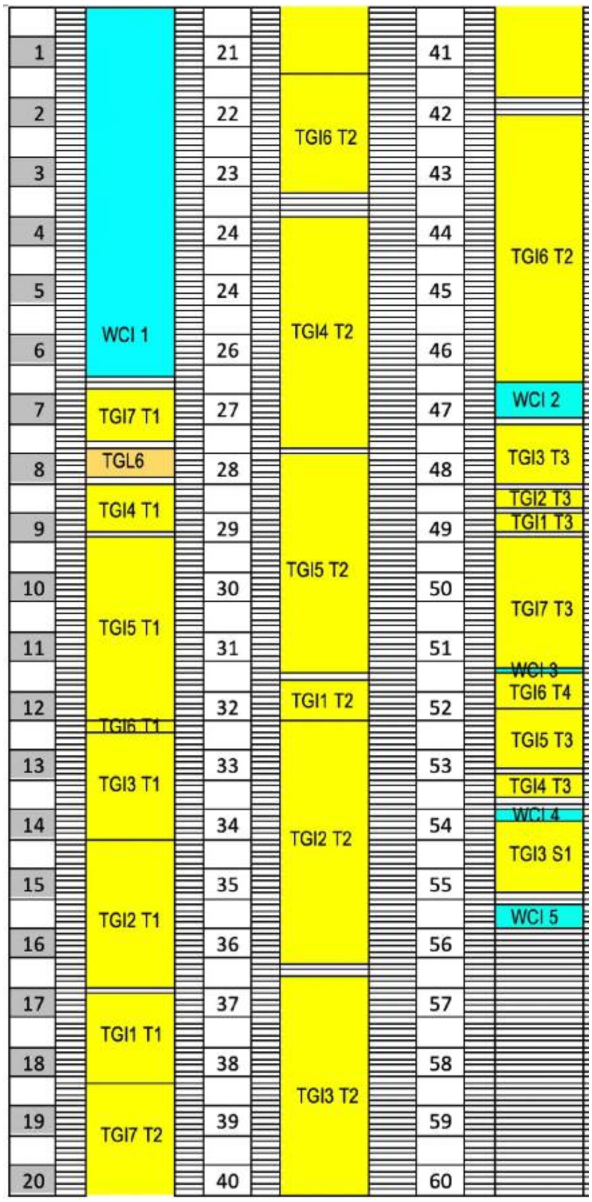



Figure 7. The complete timeline diagram for the class undertaking the *BWwH task*.

The analysis presented here, particularly the various diagrams, was particularly useful in helping us determine what the teacher did during task implementation. This subsequently guided the focus of our analysis as to why she acted in particular ways. The methods used here were then applied to other classrooms in our study, looking for commonalities and variation within and across classes. Each type of diagram, sequence, mud map, and timeline, contributed both individually and collectively to our understanding of *what the teacher did*. Each provides different insight as to what the teacher does. They provide different lenses for us as we also focus on the discourse to further consider *why teachers did what they did*. Our diagrams are handcrafted, and researcher generated, but powerful in allowing us to 'see' what the actors did. We concur with Miles et al. (2014, p. 108), that visual displays presenting information systematically allow "credible and trustworthy analysis" to answer our research questions. Noticing continues to be crucial to successful modelling (Galbraith et al. 2017).

## Acknowledgments

DP17010555 Chief Investigators were V. Geiger, G. Stillman, J. Brown, and P. Galbraith. M. Niss was a Partner Investigator. The views expressed herein are those of the authors.

# References

Brown, J. P. (2020). Expectations for challenge in modelling and its assessment In G. A. Stillman, G. Kaiser, & C. E. Lampen (Eds.), *Mathematical modelling education and sense-making* (pp. 371–382). Springer. https://doi.org/10.1007/978-3-030-37673-4_32

Brown, J., & Stillman, G. (2023). *Teaching Tactics to Manage Mathematisation during Mathematical Modelling,* [Manuscript submitted for publication.]

Chapman, O. (2007). Mathematical modelling in high school mathematics: Teachers' thinking and practice. In W. Blum, P. L., Galbraith, H. W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education*. Springer. https://doi.org/10.1007/978-0-387-29822-1_34

Collins, A., Brown, J. S., & Holum, A. (1991). Cognitive apprenticeship: Making thinking visible. *American Educator, 6*(11), 38–46.

Flick, U. (2006). *An introduction to qualitative research* (3rd ed.). Sage.

Galbraith, P., Stillman, G., & Brown, J. (2017). The primary of 'noticing': A key to successful modelling. In G. Stillman, W. Blum, & G. Kaiser (Eds.), *Mathematical modelling and application* (pp. 83–94). Springer.

Geiger, V., Stillman, G., Brown, J., Galbraith, P., & Niss, M. (2018). Using mathematics to solve real world problems: The role of enablers*.* *Mathematics Education Research Journal*, *30*(1), 7–19.

Hager, P., & Beckett, D. (2019). *The emergence of complexity: Rethinking education as a social science*. Springer.

Miles, M., Huberman, M., & Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Sage.

Stillman, G. (2011). Applying metacognitive knowledge and strategies in applications and modelling tasks at secondary school. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 165–180). Springer.

Stillman, G., & Brown, J. (2014). Evidence of "implemented anticipation" in mathematising by beginning modellers. *Mathematics Education Research Journal 26*(4), 763–789.

Stillman, G. A., & Brown, J. P. (2023). Modeling the phenomenon versus modeling the data set. *Mathematical Thinking and Learning*, *25*(3). https://doi.org/10.1080/10986065.2021.2013144

Stillman, G. A., Brown, J. P., & Czocher, J. (2020). Yes, mathematicians do X so students should do X, but it's not the X you think. *ZDM*, *52*(6), 1211–1222. https://doi.org/10.1007/s11858-020-01183-5

Stillman, G., Brown, J., & Galbraith, P. (2010). Identifying challenges within transition phases of mathematical modeling activities at year 9. In R Lesh, P. Galbraith, C. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 385–398). Springer.

Stillman, G., Brown, J., & Geiger, V. (2015). Facilitating mathematisation in modelling by beginning modellers in secondary schools. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modeling in education, research and practice: Cultural, social, and cognitive influences* (pp. 93–103). Springer.

VCAA. (2015). *Victorian Certificate of Education: Mathematics study design* (2016–2020). VCAA.

VCAA. (2016, online). *Victorian Curriculum Mathematics: Foundation-10*. VCAA.

Wedelin, D., & Adawi, T. (2015). Warming up for PBL: A course in mathematical modelling and problem solving for engineers. *Högre utbildning, 5*(1), 23–34.

# The Deeply Engrained Behaviourist Assessment Ideologies Constraining School Mathematics

Rebecca Burtenshaw
*University of the Sunshine Coast*
rebecca.burtenshaw@research.usc.edu.au

Behaviourism proposes successful learning to be dependent on the performance of conditioned behaviours that are distinctly observable and objectively measurable. Over the past 100 years, various behaviourist concepts have been superseded by sociocultural and cognitive learning theories, but the entwined areas of assessment in mathematics education have not received the same focus. Outdated beliefs and unquestioned norms surrounding behaviourist assessment ideologies continue to dominate schools' decision-making around mathematics education. This paper includes the theoretical foundations for a PhD that aims to surface the tension of what is said to be valued in mathematics as "success" and what is systematically labelled as "success".

While behaviourism is theoretically viewed as obsolete, this learning theory is still inadvertently tethered to mathematics education by outdated yet deeply engrained beliefs around assessment (Shepard, 2000). The perpetuating endurance of these beliefs is a major contributor suppressing more current understandings of good mathematical learning, especially as assessment and learning should be deeply intertwined (Shepard, 2000). Musial (2021) writes how the word "assess" derives from the Latin verb *assidere*, which means to "sit beside". To "sit beside" conjures mental images of two people sitting alongside each other, determining what a learner does know or does not know yet (Musial, 2021). This is a stark contrast to assessment practices observed in some mathematics classrooms. For example, rows of individual students quietly circling answers to sets of multiple-choice questions posed to validate the attainment of normalised benchmarks (Watt, 2005). While Musial (2021) explains how assessment is commonly linked to other concepts around reporting, grades, and performance, "they are by no means the same" (p. 50). However, these concepts can all be quite synonymous with behaviourism. This paper questions if unwillingness to let go of behaviourist assessment ideologies is restraining the advancement of school mathematics education and narrowing beliefs around "success". While research into mathematics education can rarely be linked to any one theory, theoretical frameworks can act as "virtual reality systems that help practitioners connect to day-to-day realities" (Stoilescu, 2016, p.140). Therefore, this paper will explore behaviourism and how its appropriation into educational practices influenced learning, teaching and, most of all, assessment. This paper will then propose—despite time passed and research espousing sociocultural practices—that behaviourist assessment ideologies are still deeply engrained in beliefs concerning assessment and, therefore, drives behaviourist approaches in mathematics classrooms.

## The Emergence of Behaviourism

Stemming from psychology, the theory of behaviourism proposes that human behaviour can be systemically conditioned to respond to specific stimuli and, if trained correctly, these behaviours are expected to be objectively observed through performance (Eisenberg, 1975; Hatfield, 2003; Stoilescu, 2016). Behaviourists suggest conditioned behaviours are adjusted through repetition, reinforcement and continuous feedback on which stimulus responses are correct and which are incorrect (Skinner, 1985; Stoilescu, 2016). Hatfield (2003) explains how the rise of behaviourism was not initially due to its own theoretical merit but instead driven by scepticism towards introspection and the knowing mind. Inspired by earlier philosophers, such as Pavlov, founding behaviourist theorists like psychologist John Watson were driven to establish that human function is measured solely by external behaviours, rejecting any influence of internal states of mind,

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 123–130). Newcastle: MERGA.

emotions, and psychological conditions (Hatfield, 2003). Watson initially experimented with theories on rats, rabbits, and monkeys before applying these same experimental theories to the study of humans, which he shared in the prominent *Behaviorist Manifesto* (1913). Other behaviourist theorists at least acknowledged the presence of emotional factors, introspection, and the impact of external factors but then ultimately disregarded these in favour of externalised and measured performance (Hatfield, 2003).

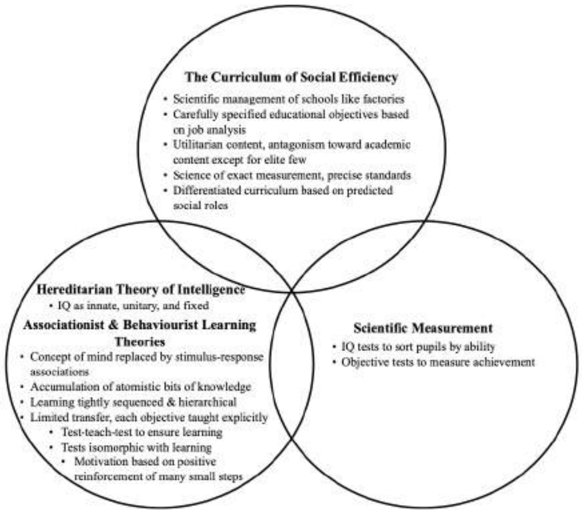



Figure 1. The interlocking paradigms that dominated the 20th Century, as adapted from Shepard (2000).

At the turn of the 20th Century, around the same time that psychology became interested in behaviour, systemic and structural developments were required to cope with population growth, movement, and revised social agendas (Hatfield, 2003; Schneider & Hutt, 2013). As seen in Figure 1, Shepard (2000) demonstrates the interlocking paradigms that dominated the 20th Century and how the overarching social efficiency of this time was closely linked to hereditarian theories of intelligence, commonly associated with behaviourism and scientific measurement. She writes how "the social efficiency movement grew out of the belief that science could be used to solve the problems of industrialisation and urbanisation" (Shepard, 2000, p. 4). Most of the social efficiency movement involved some form of depersonalisation and standardisation (Shepard, 2000), which educational institutions began to rely upon to maximise school efficiency (Schneider & Hutt, 2013). For example, Schneider and Hutt (2013) write how "administrators refashioned themselves as professional managers whose job was to manage burgeoning systems in the most efficient way possible" (p. 7). Reporting systems that relied upon uniform standards and easily communicated grades were modified from other industries, such as cattle and timber (Schneider & Hutt, 2013). Another example of an incipient measurement scheme was the Intelligence Quotient (IQ), which attempts to measure someone's intellectual potential or lack of (Gipps & Stobart, 2009). Edward Thorndike, credited as the "father of scientific measurement" (Shepard, 2000) and a founder of educational psychology, was a strong supporter of the emerging IQ assessment. Alongside his studies in human conditioning, Thorndike (1910) believed an individual's original nature—intellectually, characteristically, and morally—is inherently bound by ancestry, and individuals should be educated according to their innate genetic capabilities. Though, he also believed a "man's original nature" could be improved and that this conditioning would contribute to the "success in controlling human nature and changing it to the advantage of the common weal" (Thorndike, 1910, p. 8). Hilgard (1996) writes how Thorndike and John Dewey were colleagues for many years but held different views about education. While Dewey was a progressive reformer, Thorndike "valued data above all else" (Hilgard, 1996, p. 423) and believed school improvement was achieved through quality control rather than innovation. The education system favoured and embraced Thorndike's theories as it accommodated other beliefs at this time. Instigated by various societal factors, behaviourism learning theory emerged and was applied to various school subject areas, such as mathematics education.

## Behaviourism in Mathematics Education

Behaviourism learning theory commonly characterises learning and instruction as the transmission of carefully sequenced knowledge passed down from one to another and typically accomplished through the mimicking and repetition of universal, static procedures and instructed strategies (Askew et al., 1997; Richardson, 1996; Stoilescu, 2016). Stoilescu (2016) explains how behaviourism is commonly intertwined with "the belief that mathematics can be transmitted by inoculating the right knowledge and discourse at the right time" via "already established theories, knowledge paths and experiences" (p.141). Behaviourism, the oldest and most well-known theoretical orientation, aligns closely with a transmission model (Richardson, 2001; Stoilescu, 2016). Traditionally, behaviourism learning theory centres teachers as the arbiter of knowledge (Richardson, 2001). However, more modern representations of this learning theory include technology programs platforms playing a role in instruction and the transmission of knowledge (Aydin, 2005). With each mathematical concept sequenced within a wider hierarchy, students are situated as the passive recipients of knowledge previously discovered and owned by masters centuries ago (Bada & Olusegun, 2015; Handal, 2009). This style of learning is often completed through explicit procedural teaching, drill-like practice, and repetitive rote learning (Askew et al., 1997; Handal, 2009). Such an approach also directly aligns with behaviourist beliefs around conditioning behaviour. In fact, within both a transmission model and behaviourism learning theory, each lesson will often have pre-determined goals with pre-determined outcomes, requiring achievement in pre-determined ways (Shepard, 2000). Therefore, "success"—for both the teacher and the student—is judged (or "assessed") on how well a student can repeat and perform the desired behaviour (Jacob et al., 2017).

Behaviourism learning theory views assessment of learning as students' displaying quantifiable changes in behaviours that are distinctly observable and objectively measurable (Eisenberg, 1975; Stoilescu, 2016). This objectivity is repetitively noted as a superior trait that should drive the adoption of behaviourist ideologies (Eisenberg, 1975; Stoilescu, 2016). An example in mathematics education is short answer exams that examine the accuracy of followed procedures to arrive at an answer. The number of correct responses is scored against an overall total and is often then related to a grade. This objective 'assessment' is frequently accompanied by themes of scientific measurement, such as norms, ranking and grading, and terms like 'evidence', 'standards' and 'outcome-based education' (Eisenberg, 1975; Maxwell, 2009; Shepard, 2000; Stoilescu, 2016). It is proposed these "precise standards of measurement" are required to validate that learning has occurred and that each skill is "mastered at the desired level" (Shepard, 2000, p. 4). As Thorndike (1910) claims, if education is about change, then there should be units of measurement to observe such change. While this may be true in some applications, like most things in education, balance is needed when considering any initiative or educational practice.

Some behaviourist and hereditarian theories of intelligence have since been disputed in education, as well as psychology. For example, statements made by Thorndike (1910) have been empirically challenged, such as the accuracy of IQ measurement (Gipps & Stobart, 2009), the varying intellectual capacity of different "races", and the superior mental function of males compared to females. Furthermore, the revolution of cognitive research, social learning theory and Postmodernism dispositions confronted behaviourist theories (Richardson, 2001). Research that supports the ongoing impact of introspection (Desautel, 2009; Munns & Woodward, 2006), the influence of a growth mindset (Dweck, 2014) and the ongoing developments founded through cognitive science. A wide range of research also claims the harms of behaviourist ideologies. For example, Eisenberg's (1975) paper, *Behaviourism: The Bane of School Mathematics*, shared numerous examples of the damaging impact behaviourism can have on mathematical understanding. Piaget and Vygotsky's theories around cognitive development have inspired many more sociocultural learning theories (Goos, 2004; Stoilescu, 2016). In teacher education, behaviourism is commonly noted as a theory of the past in favour of more current learning orientation, such as constructivism (Richardson, 1996). Constructivism orients learning as an active process of meaning-making developed by and adapted from interactions with others, experiences (prior and new) and their subjective states of mind (Bada & Olusegun, 2015; Handal, 2009; Stoilescu, 2016). Teachers as facilitators provide adaptive and experiential learning opportunities as students create their own 'personal mental model' of understanding (Bada & Olusegun, 2015) and "look for similarities and differences against their own cognitive schemata" (Handal, 2009, p. 5).

There are many examples of literature (Bada & Olusegun, 2015; James, 2006; Richardson, 1996, 2001) that discusses both behaviourism and constructivism, typically beginning with a denunciation of behaviourism followed by an exploration of constructivism. This article does not intend to compare or pit behaviourism against constructivism. In fact, Aylward and Cronjé (2022) contend that behaviourism and constructivism should not be viewed as linear opposites as traditionally posed but complementary. However, this behaviourism-constructivism flow of discussion could unintentionally lead the reader to interpret that one has replaced the other. A careful distinction needs to be made about how learning orientations (like behaviourism and constructivism) continue to exist side by side. While one learning orientation may have a greater preference or be more influential in particular subject areas, from its inception, learning orientations exist in parallel (Stoilescu, 2016). It is important to acknowledge that school learning and teaching practices may have evolved to embed cognitive learning ideologies, constructivist frameworks and other sociocultural theories espoused by mathematics education research (James, 2006; Shepard, 2000). However, there is still a strong presence of behaviourist ideologies in mathematics assessment practices. Shepard (2000) writes how "dominant theories of the past continue to operate as the default framework affecting and driving current practices and perspectives" (p.4). These out-of-date ideas can still shape the belief systems of teachers, students, parents, and policymakers (Shepard, 2000), though no more prominently than within assessment.

There is a deep obligation to the behaviourist's notion of objectivity within mathematics education assessment, even if this attachment to psychometric measurement comes at the detriment of student learning and other important assessment design principles (Shepard, 2000; Watt, 2005). Watt (2005) investigated Sydney mathematics teachers' attitudes towards alternative assessment methods in mathematics and found an astounding overreliance on written tests, despite participants also indicating how "traditional mathematics tests cannot be used to infer more general mathematical ability" (p. 23). Participants across various years of teaching experience indicated that alternative assessments—even quite a conservative shortlist, such as student self-assessment, oral tasks and practical tasks—were perceived as too subjective and, in a lesser sense, unsuitable for mathematics (Watt, 2005). The perceived "threat to the objectivity" (Watt, 2005, p. 39) may override the rationality to collate a well-rounded, holistic picture of students' mathematical understanding, which can not fairly or ethically achieved from one assessment modality (Clarke, 1997). Objectivity is also highly valued when assessment is exclusively connected to grading (Watt, 2005). However, Clarke (1997) writes, "Assessment is a process. Grading can be one product of that process. The two should not be confused with each other" (p. 21). Viewing grading and assessment as indistinguishable misrepresents and underestimates the value of assessment and its broader application in classrooms (Clarke, 1997; Munns & Woodward, 2006). After all, "no one has ever gotten taller just by being measured" (Clarke, 1997, p. 4).

That is not to say objectivity is unimportant; however, other rigorous assessment design principles, such as validity, clarity, fidelity, and fairness, are disproportionately considered in the pursuit of objectivity (Musial, 2021; Zane, 2009). Watt (2005) observes, "To date, reliance on the traditional mathematics test has been justified on the grounds of maximising reliability and ensuring comparability, but this has often been at the expense of validity" (p. 24). Validity refers to the extent evidence and design measure what was intended to be measured (Rawlins et al., 2005). Gipps and Stobart (2009) also believe fairness and equity should be embedded within validity arguments by examining sociocultural dispositions, such as the type of knowledge valued, assessment preparation, how assessment is used and the ramifications of such use. They believe the discussion around validity should move from "a *fixed property* of an assessment" to a "*process* that investigates an assessment in terms of both the construct being assessed… and, crucially, the inferences and actions based on the results" (p. 109-110). Assessment that optimises students' expression of their learning requires purposeful consideration of a variety of assessment design principles, including transparency about the measures of success, how this judgment is reached and follow-up actions that continue to support student learning (Clarke, 1997; Rawlins et al., 2005). Without this, the single-minded obligation to objectivity may lead to invalid judgments, restrictive opportunities, and the risk of misinterpreting student learning (Clarke, 1997; Rawlins et al., 2005).

There are many other examples of behaviourist assessment ideologies that are widely implemented and deeply engrained in mathematics classrooms, such as:

- The unceasing teach-test-teach-test culture which requires students to demonstrate their atomised and, possibly, shallow understanding of a specific concept after short-term memorisation (Eisenberg; 1975, Shepard, 2000), potentially resulting in the same concepts being retaught year after year.
- The ongoing consequences of NAPLAN and the ill-conceived notion that high-stakes standardised testing is more reliable than the judgement of education professionals (Klenowski & Wyatt-Smith, 2012). This includes schooling systems using NAPLAN scores to reward or reprimand schools (Thompson & Cook, 2014) and, as Watt (2005) describes, the "backwash" of such requirements and pressure.
- Common practices of collecting "data for data's sake", with often limited opportunity to enact upon or give feedback to students (Black & Wiliam, 2004; Liljedahl, 2021). This also includes poor implementation of this data and the use of data to label or rank students by numerical or abstract grading bands (Clarke, 1997; Watt, 2005).
- The streaming of students and classes (Tieso, 2003).
- Inflexible learning objectives attached to transmission models of mathematics teaching, including pushing through the curriculum and limited differentiation (Shepard, 2000).
- The frequent use of the term "standards", despite the real possibility that completely incomparable stances exist (Maxwell, 2009). Maxwell (2009) explains how standards may differ depending on the type of standard, the focus (facet, unit, scope), the construct and the purpose. For example, the different constructs between determining learning and performance; or the distinct implementations of content standards, performance standards and development standards.
- Schooling systems commending theories regarding measurement models, such as Hattie's (1992) effect sizes. This can reinforce the importance of measurement models in schools and certain pedagogical decision-making (Serow et al., 2016).
- Marginalised representations of what "knowledge" is and which areas of knowledge are deemed as more important. (Gipps & Stobart, 2009; Seth, 2007).

In some of these examples, the limitations and constraints may be acknowledged by schools and educators but ultimately disregarded due to normalisation (Klenowski & Wyatt-Smith, 2012; Serow et al., 2016). However, in other cases, the saturation of practices may be so systemic that the influences of behaviourist assessment ideologies are overtly unrecognisable.

While research has questioned what we think we know about teaching practices and student learning, it is important to ask similar questions about assessment ideologies. If we do not do this, the developments of mathematics education will be in vain as the prevailing dominance of behaviourist assessment ideologies will continue to constrain the holistic forward movement of mathematics education. For example, if decision-making around mathematics education practices is made to match the behaviourist assessment norms and requirements, then there may be hesitation in adopting more current researched practices. This could include teachers' unease in exploring the role of metacognition in mathematical thinking (Desautel, 2009) or providing an environment for students to engage together in creative problem-solving play (Liljedahl, 2021) for fear of the time taken away from outcome-based objectives. Furthermore, if mathematics learning and teaching decisions are only made to match out-of-date assessment norms, then similarly out-of-date practices will always be perceived as more robust and fitting (Shepard, 2000). If assessment only requires students to repeat the thinking of others, then teaching will primarily focus on students learning *what* to think, as opposed to *how* to think. "Assessment should model the mathematical activity we value", writes Clarke (1997, p. 8).

## Broader Implications

Questioning the depth of behaviourist assessment ideologies has wider implications beyond assessment, including belief systems for students and teachers, approaches to mathematics education and the motivations for engaging with other mathematics education research (Beswick, 2012; Ekmekci et al., 2015; Stoilescu, 2016). Shepard (2000) explains that if there is to be "any attempt to change the form and purpose of classroom assessment", including making it a more interconnected part of learning, then we "must acknowledge the power of these enduring and hidden beliefs" (p. 6). For example, Ernest (1989) describes three categories of beliefs influencing mathematics: Instrumentalist, Platonist and Problem Solving. There is a tangible congruence between behaviourism and traditional Instrumentalist philosophies of mathematics that could contribute to the continuation of conservative beliefs regarding the nature of mathematics (Ernest, 1989; Handal, 2009; Hatfield, 2003). Mendick (2005) alleges that this view of mathematical knowledge "as absolute and unquestionable" creates the status of "the ultimate intelligence test". This can contribute to the sustained elitism, irrelevance and disaffection towards mathematics, which Nardi and Steward (2003) describe. Richardson (2001) explains how the late 1960s and early 1970s saw research on teaching emerge as its own field of study with an emphasis on process-product research. Process-product research focused on finding correlations between teacher behaviours and improved student achievement (such as standardised test scores) to gauge the difference between more or less 'effective teaching' (Richardson, 2001). Firmly established studies, such as direct instruction in mathematics, were conducted under the belief that teachers were the transmitters of knowledge and were influenced by behaviourist ideologies (Richardson, 2001). Though the notions of an "effective teacher" shifted from teacher behaviours to "teacher cognition, beliefs and knowledge" (Richardson, 2001, p. 282), these initial ideas of 'effective teaching' constructed a strong image of what it means 'to teach'. Jacob et al. (2017) also demonstrate how students' performance on tests and assessment are still deeply interconnected with teachers' beliefs around the success of their own teaching and what it means to be "a good teacher". Students' perceived and realistic successes and failures emotionally drive teachers' own emotions, interactions, and pedagogical decisions (Jacob et al., 2017). Therefore, beliefs intertwined with behaviourist assessment ideologies influence beyond assessment.

Back in 1975, Eisenberg spoke about the inevitable damage and obscured "gravity of espousing an extreme behaviouristic format" and how—even then—"acting under the guise of accountability, [has] forced classroom teachers and curriculum developers via legislation and funding policies to adopt a behaviouristic framework" (p. 163). Further investigation is required to understand the extent, scope and repercussions of how engrained behaviourist assessment ideologies are, including how assessment could challenge or reinforce the broader beliefs of mathematics. By valuing process equally to or more than product (Handal, 2009; Stoilescu, 2016; Watt, 2005), the "assessment of success" in mathematics could shift perceptions, break down limitations and unlock the subject of mathematics—be it in school, careers, or daily life. However, first, the cyclical perpetuation of behaviourism ideologies stemming from and sustained through assessment will need to be surfaced and questioned. These are the hopes and objectives which my PhD aims to address. There is an ambiguity between what is said to be valued in mathematics education and labelled as "success", and what is systemically demonstrated as valued or judged as "success". Surfacing this tension will reveal beliefs around "success" in mathematics education and begin to broaden success metrics beyond traditional behaviourist structures ingrained in Australian education.

## References


Askew, M., Rhodes, R., Brown, M., Wiliam, D., & Johnson, D. (1997). *Effective teachers of numeracy*. King's College London. http://musicmathsmagic.com/page2/files/EffectiveTeachersofNumeracy.pdf

Aydin, E. (2005). The use of computers in mathematics education: A paradigm shift from "computer assisted instruction" towards "student programming". *Turkish Online Journal of Educational Technology*, *4*(2), 4.

Aylward, R. C., & Cronjé, J. C. (2022). Paradigms extended: How to integrate behaviorism, constructivism, knowledge domain, and learner mastery in instructional design. *Educational Technology Research and Development*, *70*(2), 503–529. https://doi.org/10.1007/s11423-022-10089-w

Bada, S. O., & Olusegun, S. (2015). Constructivism learning theory: A paradigm for teaching and learning. *Journal of Research & Method in Education*, *5*(6), 66–70.

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics*, *79*(1), 127–147. https://doi.org/10.1007/s10649-011-9333-2

Black, P., & Wiliam, D. (2004). The formative purpose: Assessment must first promote learning. *Yearbook of the National Society for the Study of Education*, *103*(2), 20–50. https://doi.org/10.1177/016146810410601402

Clarke, D. (1997). *Constructive assessment in mathematics: Practical steps for classroom teachers*. Key Curriculum Press.

Desautel, D. (2009). Becoming a thinking thinker: Metacognition, self-reflection, and classroom practice. *Teachers College Record*, *111*(8), 1997–2020. https://doi.org/10.1177/016146810911100803

Dweck, C. (2014). Teachers' mindsets: "Every student has something to teach me." Feeling overwhelmed? Where did your natural teaching talent go? Try pairing a growth mindset with reasonable goals, patience, and reflection instead. It's time to get gritty and be a better teacher. *Educational Horizons*, *93*(2), 10–15. https://doi.org/10.1177/0013175X14561420

Ekmekci, A., Corkin, D., & Papakonstantinou, A. (2015). The relationship between teacher related factors and mathematics teachers' educational beliefs about mathematics. In S. M. Che, & K. A. Adolphson (Eds.), *Proceedings of the 42nd annual meeting of the Research Council on Mathematics Learning* (pp. 140–148).

Eisenberg, T. A. (1975). Behaviourism: The bane of school mathematics. *International Journal of Mathematical Education in Science and Technology*, *6*(2), 163–171. https://doi.org/10.1080/0020739750060204

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249–253). Falmer.

Gipps, C., & Stobart, G. (2009). Fairness in assessment. In C. Wyatt-Smith, & J. J. Cummings (Eds.), *Educational assessment in the 21st century: Connecting theory and practice* (pp. 105–118.) Springer Science + Business Media. https://doi.org/10.1007/978-1-4040-9964-9_6

Handal, B. (2009). Philosophies and pedagogies of mathematics. *Ilköğretim Online*, *8*(1), 1–6.

Hatfield, G. (2003). Behaviourism and psychology. In T., Baldwin. *The Cambridge history of philosophy* 1870–1945. Cambridge University Press. https://doi.org/10.1017/CHOL9780521591041

Hattie, J. (1992). Measuring the effects of schooling. *Australian Journal of Education, 36*(1), 5–13.

Jacob, B., Frenzel, A. C., & Stephens, E. J. (2017). Good teaching feels good—but what is "good teaching"? Exploring teachers' definitions of teaching success in mathematics. *ZDM Mathematics Education*, *49*(3), 461–473. https://doi.org/10.1007/s11858-017-0848-6

James, M. (2006). Assessment, teaching and theories of learning. In J. Gardner (Ed.), *Assessment and learning*, (pp. 47–60). Athenaeum Press.

Klenowski, V., & Wyatt-Smith, C. (2012). The impact of high stakes testing: The Australian story. *Assessment in Education: Principles, Policy & Practice*, *19*(1), 65–79. https://doi.org/10.1080/0969594X.2011.592972

Liljedahl, P. (2021). Build*ing thinking classrooms in mathematics: 14 teaching practices for enhancing learning: Grades K–12*. Corwin.

Maxwell, G. S. (2009). Defining standards for the 21st century. In C. Wyatt-Smith, & J. Cumming (Eds.), *Educational assessment in the 21st century* (pp. 263–286). Springer.

Mendick, H. (2005). Mathematical stories: Why do more boys than girls choose to study mathematics at AS-level in England? *British Journal of Sociology of Education*, *26*(2), 235–251.

Munns, G., & Woodward, H. (2006). Student engagement and student self-assessment: The real framework. *Assessment in Education: Principles, Policy & Practice*, *13*(2), 193–213.

Musial, D. L. (2021). Meaningful assessment. In L. Wellner, & K. Pierce-Friedman (Eds.), *Supporting early career teachers with research-based practices* (pp. 50–70). IGI Global.

Nardi, E., & Steward, S. (2003). Is mathematics TIRED? A profile of quiet disaffection in the secondary mathematics classroom. *British Educational Research Journal*, *29*(3), 345–367. https://doi.org/10.1080/01411920301852

Rawlins, P., Brandon, J., Chapman, J., Leach, L., Neutze, G., Scott, A., & Zepke, N. (2005). Standards-based assessment in the senior secondary school: A review of the literature. *New Zealand Journal of Teachers' Work*, *2*(2), 107–115.

Richardson, V. (1996). From behaviorism to constructivism. *Teacher Education and Special Education*, *19(*3), 263–271.

Richardson, V. (2001). Constructivist mathematics instruction and current trends in research on teaching. In T. Wood, B. Scorr Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 275–293). Routledge.

Schneider, J., & Hutt, E. (2014). Making the grade: A history of the a–f marking scheme. *Journal of Curriculum Studies*, *46*(2), 201–224. https://doi.org/10.1080/00220272.2013.790480

Seth, S. (2007). Changing the subject: Western knowledge and the question of difference. *Comparative Studies in Society and History*, *49*(3), 666–688. https://doi.org/10.1017/S0010417507000667

Serow, P., Callingham, R., & Tout, D. (2016). Assessment of mathematics learning: What are we doing? In K. Makar, S. Dole, J. Visnovska, M. Goos, A. Bennison, & K. Fry (Eds.), *Research in mathematics education in Australasia 2012–2015* (pp. 235–254). Springer.

Shepard, L. A. (2000). The role of assessment in a learning culture. *Educational Researcher*, *29*(7), 4–14. https://doi.org/10.3102/0013189X029007004

Skinner, B. F. (1985). Cognitive science and behaviourism. *British Journal of Psychology*, *76*(3), 291–301. https://doi.org/10.1111/j.2044-8295.1985.tb01953.x

Stoilescu, D. (2016). Aspects of theories, frameworks and paradigms in mathematics education research. *European Journal of Science and Mathematics Education*, *4(*2), 140–154.

Tieso, C. L. (2003). Ability grouping is not just tracking anymore. *Roeper Review*, *26(*1), 29–36. https://doi.org/10.1080/02783190309554236

Thompson, G., & Cook, I. (2014). Manipulating the data: Teaching and NAPLAN in the control society. *Discourse: Studies in the Cultural Politics of Education, 35*(1), 129–142.

Thorndike, E. L. (1910). The contribution of psychology to education. *Journal of Educational Psychology*, *1*(1), 5–12.

Watson, J. B. (1913). Psychology as the behaviorist views it. *Psychological Review*, *20*(2), 158–177. https://doi.org/10.1037/0033-295X.101.2.248

Watt, H. M. (2005). Attitudes to the use of alternative assessment methods in mathematics: A study with secondary mathematics teachers in Sydney, Australia. *Educational Studies in Mathematics*, *58*(1), 21–44.

Zane, T. W. (2009). Performance assessment design principles gleaned from constructivist learning theory (Part 2). *TechTrends*, *53*(3), 86. https://doi.org/10.1007/s11528-009-0287-5

# Changes in Year 11 Students' Self-Reported Experiences of Emotions Related to CAS and Pen-and-paper

Scott Cameron
*MGSE, the University of Melbourne*
scameron@unimelb.edu.au

Lynda Ball
*MGSE, the University of Melbourne*
lball@unimelb.edu.au

Vicki Steinle
*MGSE, the University of Melbourne*
v.steinle@unimelb.edu.au

This paper reports three emotions (i.e., anxiety, confidence, & enjoyment) related to pen-and-paper (P&P) and a Computer Algebra System (CAS), that were experienced (or not) by twelve Year 11 Mathematical Methods students in a classroom where CAS was allowed. Some students had experienced the same emotions at the start and end of the study, hence experience with CAS did not always appear to impact students' emotions. Comparison of emotions related to CAS and P&P show students more frequently experienced anxiety related to CAS than P&P, and more frequently experienced confidence and enjoyment with P&P than CAS.

Students' attitudes towards CAS can impact their willingness to use CAS, and persistence in overcoming technical difficulties (Pierce & Stacey, 2004). Attitude can be defined as including beliefs and emotions (Di Martino & Zan, 2010), and both beliefs and emotions can impact student CAS use. Focussing on emotions, W. L. Ng et al. (2003) suggested students must be confident with, and enjoy using, CAS before they can effectively use it, and that anxiety may negatively impact CAS use. The potential impact of emotions (i.e., anxiety, confidence, & enjoyment) results from individuals favouring (or avoiding) stimuli that result in a positive (or negative) emotion (Shuman & Scherer, 2014). Meagher (2012) noted a lack of research exploring students' affective engagement with CAS. The literature review below identified few articles which investigated emotions (an aspect of affect) related to CAS. This study builds on literature which has reported students' emotions related to CAS at a single timepoint (e.g., Kissane et al, 2015; W. L. Ng, 2003) or changes resulting from an intervention designed to support attitudes towards CAS (e.g., W. L. Ng et al., 2005), and further contributes by contrasting emotions related to CAS with emotions related to P&P. The research question for this paper is:

- Which emotions (i.e., anxiety, confidence, & enjoyment) related to P&P and CAS were experienced (or not) by twelve Year 11 novice CAS users in a classroom where CAS was allowed, and were there changes over 8 months?

## Literature Review

There are several definitions for emotions, but Hannula (2020) notes agreement in the literature that emotions involve three processes. Hannula (2006) summarised these as (i) changes in "physiological processes that regulate the body" (p. 219) (e.g., heart rate, body posture), (ii) a "subjective experience" (p. 219) which impacts decisions (e.g., frustration when encountering a CAS syntax error) and (iii) an "expressive process" (p. 219) which results in an observable behaviour (e.g., changing from CAS to P&P)). Negative emotions are experienced when progress towards a goal is interrupted (Hannula, 2020); this may result in selecting a different approach, such as moving from CAS to P&P. Positive emotions are experienced when progress towards a goal is uninterrupted.

Three emotions were identified in CAS literature, namely (i) Anxiety (e.g., W. L. Ng et al., 2005), (ii) Confidence (e.g., W. L. Ng et al., 2005), and (iii) Enjoyment (e.g., Kissane et al., 2015).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 131–138). Newcastle: MERGA.

These emotions are discussed below, including literature relating to P&P, to enable a contrast with CAS.

### *Emotion 1: Anxiety*

Anxiety is an “unpleasant emotion of fear, which is usually directed towards an unexpected outcome in the future” (Hannula, 2020, p. 32). Such situations may include use of CAS in unfamiliar contexts or consideration of the role of CAS and P&P calculations in mathematics (Meagher, 2012). Both Alkhateeb (2002) and W. L. Ng et al. (2005) compared the mean anxiety reported by students with different levels of experience and reported that overall levels of anxiety decrease as students gain experience. More generally, mathematics anxiety involves feeling anxiety in response to problem solving situations (Ashcroft, 2002). Such anxiety has been found to impact a “considerable group” (p. 573) of the 294 secondary students reported in L. K. Ng’s (2012) study. Inability to solve problems and the application of symbols and formulae in algebra topics were identified as prompting mathematics anxiety.

### *Emotion 2: Confidence*

Confidence relates to expectations of future success (Graham & Taylor, 2014), and students must have confidence that they will be able to use CAS successfully to integrate it into their mathematical routines (W. L. Ng, 2003). Ball (2015) reported a teacher who believed that an outcome of students lacking confidence in mathematics could be that they privilege P&P over CAS to demonstrate understanding. Kissane et al.’s (2015) survey of 522 Year 11 and 12 mathematics students found that approximately two-thirds of students were confident with CAS, consistent with W. L. Ng et al.’s (2005) study of 32 students, who also reported confidence increases with experience. Orellana (2016) surveyed 367 Year 11 students about attitudes to technology in mathematics, with technology encompassing a range of technologies. A positive relationship was found between students’ self-reported technology confidence and CAS use, with more experienced students reporting higher levels of technology confidence. Orellana’s use of the Mathematics Technology Attitude Scale (Pierce et al., 2007) enabled the comparison of technology confidence and mathematics confidence. Orellana’s results suggest students had higher levels of technology confidence than mathematics confidence, the latter defined as “students’ perceptions of their ability to obtain good results and handle difficulties in mathematics” (p. 18).

### *Emotion 3: Enjoyment*

Kissane et al. (2015) reported that almost three-quarters of students enjoyed working with CAS, while W. L. Ng (2003) reported that more than half the students enjoyed using CAS and learning mathematics with CAS. CAS enjoyment increases with experience (W. L. Ng et al., 2005). Although we were unable to identify literature explicitly discussing enjoyment with P&P, Hine’s (2023) analysis of 1633 senior secondary students reported a lack of enjoyment as a key factor in students not enrolling in senior mathematics subjects. Students cited senior mathematics subjects as being “...too complex, impractical, unnecessary...” (p. 8) but did not attribute enjoyment to either P&P or technology.

Few studies about emotions related to CAS used a longitudinal research design across a school year and in situations that did not involve a deliberate intervention to support CAS use; hence there is little discussion of how emotions change as students gain experience with CAS under normal classroom conditions, or how emotions related to P&P compared to those related to CAS. This study addresses this gap.

## Research Design

The twelve students reported here were from one Year 11 Mathematical Methods class in a co-educational government school in Victoria, Australia. Technology was expected for teaching, learning and assessment (VCAA, 2015). Although a specific technology was not specified in the curriculum, the functionalities of technology expected to be used by students were contained within a CAS; in addition, CAS is expected in Year 12 mathematics examinations and hence was an integral part of learning mathematics. Students were novice CAS users, so gained experience with CAS across the study. Eight students had not used CAS before Year 11, while four reported limited experience with CAS.

The data used in this study was a subset of data used in the PhD study of the first named author. The data was from a questionnaire which focussed on students' beliefs and emotions; emotions being reported in this paper. A questionnaire was chosen as the research instrument to allow for the identification of emotions experienced by an individual through their agreement or disagreement with individual statements (Ajzen, 2005). A literature review was conducted to identify potential questionnaires (e.g., W. L. Ng et al., 2005; Pierce et al., 2007) for the larger study. However, we were unable to identify an existing questionnaire that met criteria of addressing all beliefs and emotions that had been identified in the literature review. Consequently, a new questionnaire was developed with four sections. The questionnaire was administered twice in one school year: once in April (after studying Linear, Quadratic & Cubic functions) and again in November (after studying Calculus). In completing the questionnaire, students were instructed to consider their use of symbolic functionalities of CAS for completing the algebra required in these topics, as algebra was a focus of the larger study and used across all four topics.

### *Questionnaire Development*

Cohen et al.'s (2018) method guided the development of questionnaire items. Statements (i.e., questionnaire items or student comments) where an emotion was stated or could be inferred, were identified from the literature review. For instance, Ball and Stacey (2005) reported interviews with five Year 12 students, three of whom enjoyed working with CAS. Based on this finding, the first-named researcher rephrased each emotion statement as a questionnaire item. For example, two items about enjoyment related to CAS were "I enjoy using CAS to solve algebra problems" (Item 39) and "I like solving algebra problems with CAS" (Item 41). Emotions refer to algebra as this was the context of the larger study. Some refer to P&P (e.g., "I like solving algebra problems with P&P"; Item 35) due to an expectation that emotions about P&P could influence CAS use (e.g., choosing P&P over CAS due to enjoyment). Co-authors reviewed all items, focusing on rewording items for clarity and removing unnecessary items. The questionnaire was then piloted using Cohen et al.'s (2018) process whereby subject matter experts are asked to provide feedback on content and format. Overall, 15 emotion items were generated relating to anxiety (4 items), confidence (6 items) and enjoyment (5 items). Tables 1–3 provide questionnaire items.

Data were collected using a five-point Likert scale (Strongly Disagree, Disagree, Unsure, Agree, Strongly Agree) despite seven-point scales being more sensitive and reliable (Cohen et al., 2018). As we grouped responses of Agree and Strongly Agree to describe an emotion being experienced and the opposite for responses of Unsure, Disagree or Strongly Disagree, the additional sensitivity of a seven-point scale was not required. The collapsing of the five-point scale to a two-point scale was done due to the small sample size and the inability to statistically validate the questionnaire. Without statistical validation of the reliability and validity of the instrument, the reliability of an analysis which focussed on changes of Agree to Strongly Agree or similar was not deemed appropriate for this sample at this point of our research.

### *Questionnaire Analysis*

Each student reported that they either experienced (E) or did not experience (E′) an emotion in April (i.e., start of the study) and November (i.e., end of the study). A response pattern of EE indicated the student reported experiencing the emotion at both the start and the end, hence their experience of the emotion was the same. A response pattern of E′E′ suggested the emotion was not experienced across the year. Response patterns of EE′ and E′E indicated students whose emotional experiences had changed.

The conclusions reported here are limited by the small number of students in the study. The questionnaire was not statistically validated, and future validation and refinement of the questionnaire would strengthen the reliability and validity of this instrument.

## Results and Discussion

The study aimed to investigate changes in students' emotions (i.e., anxiety, confidence, & enjoyment) related to CAS and P&P as they gained experience with CAS. Results for each emotion are provided below.

### *Anxiety*

Table 1 provides the results for the four anxiety items, grouped into anxiety related to P&P and then with CAS. A total of eight students experienced anxiety when working with CAS (i.e., Item 36) at the start of the study (i.e., EE or EE′), and five of these students also reported experiencing anxiety related to CAS at the end of the study (i.e., EE). This contrasts with anxiety related to P&P, where no students experienced anxiety and both the start and end. Alkhateeb (2002) found that the average level of anxiety related to CAS reported by 100 students reduced as they gained experience with CAS. Similarly, in a study of 32 students, W. L. Ng et al. (2005) reported that average level of anxiety related to CAS increased as students learnt to use CAS but decreased after six months.

**Table 1**

*Students' Experiences of Anxiety*

| Emotion (Item) | Same | | | Change | | |
|---|---|---|---|---|---|---|
| | EE | E′E′ | Total | EE′ | E′E | Total |
| Anxiety related to pen-and-paper | | | | | | |
| I feel anxious when I'm solving algebra problems with pen-and-paper (29) | 0 | **8** | 8 | 3 | 1 | 4 |
| I am more comfortable working with CAS than with pen-and-paper (37) | 0 | **10** | 10 | 1 | 1 | 2 |
| Anxiety related to CAS | | | | | | |
| I feel anxious when I am using CAS to solve algebra problems (36) | **5** | 3 | 8 | 3 | 1 | 4 |
| I am more comfortable working with pen-and-paper than with CAS (30) | 3 | **5** | 8 | 2 | 2 | 4 |

*Note.* A student could experience (E) or not experience (E′) an emotion. EE, E′E′, etc., indicate a response at the start and the end of the study. **Bold** indicates the modal response pattern.

Consequently, we might expect some students who experienced anxiety related to CAS at the start would not at the end (i.e., EE′). Such a change was evident for three students (item 29), less than half of those who experienced anxiety when first working with CAS. In contrast, one student experienced anxiety at the end, but not the start (i.e., E′E). However, the data reported here only indicates whether students experienced anxiety and does not provide insight into how frequently students experienced anxiety or how strong the emotion was. Hence, we can conclude that five students experienced anxiety at the start and end of the study, not whether their level of anxiety had decreased. Meagher (2012) reported that students experienced anxiety when using CAS in unfamiliar contexts; it is possible that students encountered a range of unfamiliar contexts throughout the year as they learnt how to use CAS in a range of different topics, so using CAS in a new topic may be a stimuli that prompts anxiety. This study was conducted under normal classroom conditions without intervention (cf. W. L. Ng et al., 2005), so teachers may need to consider anxiety when working with CAS and have strategies to support students to overcome this emotion.

Items 30 and 37 provided a comparison between CAS and P&P, so a reversed pair was used to determine differences. Responses of E′E′ to Item 37 show that 10 students did not feel less anxious when working with CAS compared to P&P. We expected these 10 would respond EE to Item 30, but only three did, so there were inconsistencies in students' responses to these items. This could occur if a student did not feel comfortable working with either P&P or CAS.

### *Confidence*

Table 2 provides the results for the six confidence items, grouped into confidence in ability to solve problems with P&P and then with CAS. Confidence related to P&P essentially stayed the same across the year (i.e., 29 same cf. 7 different); for students who have been working in P&P classrooms for more than ten years their confidence may have been established over many years. Responses to Items 38 and 41 suggest that very few students were confident in using CAS to solve algebraic problems, however the responses to Item 40 suggest that this lack of confidence might not extend to other types of problems for some students. If students consider algebra to be largely algebraic manipulation which requires complex syntax compared to other procedures (e.g., differentiation), then they may feel less confident in solving algebra problems with CAS than other problems.

**Table 2**

*Students' Experiences of Confidence*

| Emotion (Item) | Same | | | Change | | |
|---|---|---|---|---|---|---|
| | EE | E′E′ | Total | EE′ | E′E | Total |
| Confidence in ability to solve problems with P&P | | | | | | |
| I am good at using pen-and-paper to solve algebra problems (31) | **5** | **5** | 10 | 1 | 1 | 2 |
| I can use pen-and-paper to solve many different problems (33) | **7** | 4 | 11 | 1 | 0 | 1 |
| I feel confident using pen-and-paper to solve algebra problems (34) | 3 | **5** | 8 | 3 | 1 | 4 |
| Confidence in ability to solve problems with CAS | | | | | | |
| I am good at using CAS to solve algebra problems (38) | 2 | **6** | 8 | 0 | 4 | 4 |
| I can use CAS to solve many different problems (40) | **6** | 2 | 8 | 2 | 2 | 4 |
| I feel confident using CAS to solve algebra problems (41) | 1 | **5** | 6 | 2 | 4 | 6 |

*Note.* A student could experience (E) or not experience (E′) an emotion. EE, E′E′, etc., indicate a response at the start and the end of the study. **Bold** indicates the modal response pattern.

Confidence related to CAS did change (i.e., E′E or EE′) for some students, particularly for items related to confidence in solving algebra problems (i.e., 38 & 41). These changes suggest that confidence with CAS was less entrenched than that for P&P. Orellana (2016) found that the average technology confidence for a group of 367 students increased with experience, so we might expect that some students in a cohort would experience confidence and even greater confidence over time, while others would move from lack of confidence to experiencing some confidence with CAS. In

Mathematical Methods algebra is expected to be embedded in all topic areas and revisited across the year (VCAA, 2015), hence the structure of the subject may have supported students' development of CAS facility for solving algebra problems, and hence resulted in a greater number of students experiencing confidence with CAS.

Although comparison of EE response patterns to P&P and CAS items suggests that more students experienced confidence with P&P (15 P&P items cf. 9 for CAS), incorporating students with changed experiences of confidence shows that the occurrence of P&P confidence was similar to CAS confidence (37 instances of E across all response patterns for Items 31, 33 & 34 cf. 34 for Items 38, 40, & 41). Changes in confidence occurred more frequently with relation to CAS than P&P, so it may be that over time, most students would experience confidence in solving problems with both P&P and CAS. Victorian Year 12 mathematics students need to complete both technology-free and technology-active examinations, so it may be important for teachers to consider how to continually develop both CAS confidence and P&P confidence as students progress through Year 11 and 12.

### *Enjoyment*

Table 3 provides the results for the five enjoyment items, grouped into enjoyment when working with P&P and then with CAS. All students provided the same responses to Items 32 and 35, with eight indicating they enjoyed solving algebra problems with P&P at both the start and the end (i.e., EE). In contrast, only 1 (and 3) student(s) reported enjoying solving algebra problems with CAS for Items 39 (42) at both the start and end. Hence, students were more likely to enjoy solving algebra problems with P&P than CAS over the period of the study.

**Table 3**

*Students' Experiences of Enjoyment*

| Emotion (Item) | Same | | | Change | | |
|---|---|---|---|---|---|---|
| | EE | E′E′ | Total | EE′ | E′E | Total |
| Enjoyment when working with P&P | | | | | | |
| I enjoy using pen-and-paper to solve algebra problems (32) | **8** | 1 | 9 | 1 | 2 | 3 |
| I like solving algebra problems with pen-and-paper (35) | **8** | 1 | 9 | 1 | 2 | 3 |
| I enjoy solving problems more when I am using pen-and-paper than when using CAS (43) | **6** | 1 | 7 | 4 | 1 | 5 |
| Enjoyment when working with CAS | | | | | | |
| I enjoy using CAS to solve algebra problems (39) | 1 | **6** | 7 | 1 | 4 | 5 |
| I like solving algebra problems with CAS (42) | 3 | **7** | 10 | 0 | 2 | 2 |

*Note.* A student could experience (E) or not experience (E′) an emotion. EE, E′E′, etc., indicate a response at the start and the end of the study. **Bold** indicates the modal response pattern.

Responses to Item 43 also support this finding, however it is important to note that a student may or may not enjoy solving algebra problems with CAS even though they indicate they enjoy solving problems more with P&P than CAS. Average enjoyment when working with CAS increases as students gain experience (W. L. Ng et al., 2005), so we expect some students who did not enjoy working with CAS at the start would at the end. One-third of students responded E′E to Item 39, which demonstrates this increase. In contrast, half the class did report enjoying CAS at the start or end (i.e., E′E′). Students have explained that they enjoy using CAS (Ball & Stacey, 2005), and enjoy learning mathematics with CAS (W. L. Ng et al., 2005). Given the potential for enjoyment to influence CAS use, it is important for teachers to support students to enjoy working with CAS.

## Conclusions

This study identified how students' emotions related to CAS and pen-and-paper (P&P) changed as they gained experience. We demonstrated that some students experience the same emotion at different points in time, even though Shuman and Scherer (2014) indicate that emotions are short lived. Emotions occur in response to a stimulus, so either the recurrence of the stimulus prompted the same emotion or students' emotions were persistent over time. These students used CAS regularly, so if the stimulus was a particular aspect of CAS use (i.e., solving algebra problems as indicated by confidence items), then students may need additional support to overcome negative emotions (e.g., anxiety) and experience positive emotions (e.g., confidence & enjoyment). It was outside the scope of this study to identify the stimuli that prompted these emotions, but understanding these stimuli may have implications for how teachers introduce CAS into the classroom. Further research could identify relevant stimuli which could then inform interventions supporting students to effectively incorporate CAS into their mathematical routines. For example, students experiencing anxiety related to CAS may avoid using it, so teachers would need to provide the students with the required skills to overcome difficulties that prompt anxiety (e.g., syntax difficulties or interpreting CAS outputs).

The inclusion of emotions related to both CAS and P&P in a single instrument was a new approach. Analysing the number of students who experience an emotion is a coarse measure compared to the approaches of Alkhateeb (2002), W. L. Ng et al. (2005), and Orellana (2016). However, the instrument provided useful insights into these students' emotions related to CAS and P&P. In this small sample, we found that more students experience (i) anxiety related to CAS than P&P (ii) confidence related to P&P than CAS, and (iii) enjoyment related to P&P than CAS. These differences could be expected as students were novice CAS users who had many years of experience of P&P mathematics. However, as emotions may impact choices about CAS or P&P, it is important for teachers to support students to experience confidence and enjoyment related to CAS, and to limit feelings of anxiety.

## References


Ajzen, I. (2005). *Attitudes, personality and behaviour* (2nd ed.). Open University Press.

Alkhateeb, H. M. (2002). A preliminary study of achievement, attitudes toward success in mathematics, and mathematics anxiety with technology based instruction in brief calculus. *Psychological Reports*, *90*(1), 47–57. https://doi.org/10.2466/pr0.2002.90.1.47

Ashcroft, M. H. (2002). Math anxiety: Personal, educational and cognitive consequences. *Current directions in psychological science*, *11*(5), 181–185. https://doi.10.1111/1467-8721.00196

Ball, L. (2015). *Use of computer algebra systems (CAS) and written solutions in a CAS allowed Year 12 mathematics subject: Teachers' beliefs and students' practices* [Doctoral dissertation, the University of Melbourne]. Minerva. http://hdl.handle.net/11343/4223

Ball, L., & Stacey, K. (2005). Students' views on using CAS in senior mathematics. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice. Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia* (pp. 121–128). MERGA. https://www.merga.net.au/documents/RP82005.pdf

Cohen, L., Manion, L., & Morrison, K. (2018). *Research methods in education*. Taylor & Francis Group. https://doi.org/10.4324/9781315456539

Di Martino, P., & Zan, R. (2010). 'Me and maths': Towards a definition of attitude grounded on students' narratives. *Journal of Mathematics Teacher Education*, *13*(1), 27–48. https://doi.org/10.1007/s10857-009-9134-z

Graham, S., & Taylor, A. Z. (2014). An attributional approach to emotional life in the classroom. In R. Pekrun, & L. Linnenbrink-Garcia (Eds.), *International handbook of emotions in education* (pp. 256–288). Routledge. https://doi.org/10.4324/9780203148211

Hannula, M. S. (2006). Affect in mathematical thinking and learning. In J. Maasz, & W. Schlöglmann (Eds.), *New mathematics education research and practice* (pp. 209–232). Brill. https://doi.org/10.1163/9789087903510_019

Hannula, M. S. (2020). Affect in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 32–36). Springer. https://doi:10.1007/978-3-030-15789-0_146

Hine, G. (2023). Incentivising student enrolments in secondary mathematics courses: Is a 10% bonus enough? *International Journal of Mathematical Education in Science and Technology.* https://doi.org/10.1080/0020739X.2177895

Kissane, B., McConney, A., & Ho, K. F. (2015). *Review of the use of technology in mathematics education and the related use of CAS calculators in external examinations and in post school tertiary education settings*. Perth: School Curriculum and Standards Authority.

Meagher, M. (2012). Students' relationship to technology and conceptions of mathematics while learning in a Computer Algebra System environment. *International Journal for Technology in Mathematics Education*, *19*(1), 3–16.

Ng, L. K. (2012). Mathematics anxiety in secondary school students. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics education: Expanding horizons. Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia* (pp. 570–577). MERGA.

Ng, W. L. (2003). Effects of computer algebra systems on secondary students' achievement in mathematics: A pilot study in Singapore. *International Journal of Computer Algebra in Mathematics Education*, *10*(4), 235–251.

Ng, W. L., Kwee, T. C., Lau, H. S., Koh, Y.-H., & Yap, Y. S. (2005). Effects of using a computer algebra system (CAS) on junior college students' attitudes towards CAS and achievement in mathematics. *International Journal for Technology in Mathematics Education*, *12*(2), 59–63.

Orellana, C. (2016). *Investigating the use of CAS calculators by senior secondary mathematics students* [Doctoral dissertation, Monash University]. Figshare. https://figshare.com/articles/Investigating_the_use_of_CAS_calculators_by_senior_secondary_mathematics_students/4696885

Pierce, R., & Stacey, K. (2004). A framework for monitoring progress and planning teaching towards the effective use of computer algebra systems. *International Journal of Computers for Mathematical Learning*, *9*, 59–93. https://doi.org/10.1023/B:IJCO.0000038246.98119.14

Pierce, R., Stacey, K., & Barkatsas, A. (2007). A scale for monitoring students' attitudes to learning mathematics with technology. *Computers & Education*, *48*(2), 285–300. https://doi.org/10.1016/j.compedu.2005.01.006

Shuman, V., & Scherer, K. R. (2014). Concepts and structures of emotions. In R. Pekrun, & L. Linnenbrink-Garcia (Eds.), *International handbook of emotions in education* (pp. 13–35). Routledge. https://doi.org/10.4324/9780203148211

Victorian Curriculum and Assessment Authority. (2015). *Victorian certificate of education mathematics study design*. Victorian Curriculum and Assessment Authority. https://www.vcaa.vic.edu.au/Documents/vce/mathematics/2016MathematicsSD.pdf

# Teacher Reflections on Trialled Embodied Learning Principles

Katherin Cartwright
*University of Sydney*
katherin.cartwright@sydney.edu.au

Jennifer Way
*University of Sydney*
jennifer.way@sydney.edu.au

Teacher reflection as a practice in education provides a window into teachers' interpretations of what they are teaching. This paper analyses 8 teachers' reflections within the ELEMS project. Teachers recorded reflections on the trialling of embodied learning principles in Pre-school to Year 2 classrooms. Reflections were analysed through the lens of Teacher noticing. Results revealed that teachers identified and described the use of multiple embodied learning principles at once (what they noticed), often including examples of students' actions as supporting evidence (how they interpreted what they noticed). Several reflections translated what was noticed and interpreted (past tense) into future-directed teaching principles (future tense).

One of the key objectives of the Embodied Learning in Early Mathematics and Science (ELEMS) Project (2021-2024) is to produce a professional learning package for teachers to support their development of pedagogy inclusive of embodied learning principles. Way and Ginns (2022) presented a need for translational research on embodied learning where researchers "collaborate with teachers in naturalistic classroom settings to translate research findings into curriculum-connected pedagogy" (p. 538). They proposed that the project will potentially empower teachers to expand their 'repertoire of teaching practices'–focusing on embodied learning principles, enhancing young children's 'repertoire of representational modes' to communicate mathematical and scientific concepts. In 2022, the 'explore' phase of the ELEMS project was undertaken in one school. The purpose of Phase 1 was for teachers to trial embodied learning and teaching approaches and provide feedback as to what they tried. Teacher feedback and reflections were then used to develop a set of suggested embodied learning activities as a Teaching Guide in preparation for implementation in Phase 2 in 2023. The aim of this paper is to present initial findings drawn from teacher-reported reflection data collected via the online platform SeeSaw (https://web.seesaw.me/) during Phase 1 of the ELMS project. Reflections provided insights into the teachers' classroom practices during the trialling of a range of embodied learning principles. The teacher reflections included self-reflections on their own teaching in relation to the embodied principles, and student-reflections in relation to student engagement and learning.

Collectively, the embodied learning principles utilised in the ELEMS project draw on research in relation to gesture, touch-pointing or tracing (Alibali & DiRusso, 1999; Alibali & Nathan, 2012; Ginns et al., 2016; Martinez-Lincoln et al., 2019), conceptual body movement (Mavilidi et al., 2018; Shoval, 2011), and drawing (de Freitas & Sinclair, 2012; Machón, 2013; Preston, 2016; Way, 2018) as external representational modes that can enhance learning. Research suggests that these external representations can be both thinking and communication tools (Goldin & Shteingold, 2001) young children can engage with, assisting them to "notice and attend to the essential properties, structures and relationships of the mathematical and scientific ideas" (Way & Ginns, 2022, p. 539). The current paper shifts this action of noticing and attending to the teachers, employing *teaching noticing* as a critical lens through which to view the teachers' reflections on trialling the embodied learning principles to answer the following research question:

- What do teachers notice and attend to when reflecting on trialling embodied learning principles?

## Theoretical Underpinning

The theoretical construct of teacher noticing was employed to analyse reflections data from teachers trialling embodied learning principles in their classrooms. Although the teachers

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 139–146). Newcastle: MERGA.

themselves were not participating in a study on developing the intentional act of teacher noticing (Mason, 2011), their reflections may naturally illustrate aspects of teacher noticing. Sherin and van Es' (2005) earlier research on learning to notice focused on *what* the teachers noticed and *how* they interpreted what they noticed. More recently van Es and Sherin (2021) continue to focus on *what* teachers notice, "identifying key events that take place in a classroom" (p. 19) and *how* they interpret what they notice, drawing "inferences about particular features of instruction based on broader principles of teaching and learning" (p. 19). They also expanded teacher noticing to include shaping. Their research, along with research by Jacobs et al. (2010), highlight 'next steps' teachers notice regarding; potential decision-making (Jacobs et al.) that occurs *after* the fact; the shaping of classroom interactions that occur *during* teaching (van Es & Sherin, 2021) based on what they noticed. Adapting the act of teacher noticing to analysing teacher reflections within the current study provides space for 'noticing teacher noticing' from the researchers' perspective. As the teachers implement new knowledge and practices from the project's professional development sessions, noticing what teachers naturally notice and reflecting on is important. Eden (2020) makes this connection stating that "teacher noticing is an important element of reflective practice, and teacher reflection is key to strengthening the impacts of teaching by making sense of teaching/learning experiences and then using these to inform future practice" (p. 300). The critical point Jaworski (2003) makes in connecting noticing and reflection, mentioned by Eden (2020), is that reflecting "on aspects of past practice can support the development of enhanced noticing within future practice thus influencing classroom actions and potentially changing practice" (Eden, 2020, p. 301). Reflecting on the implementation of embodied learning principles by teachers in the current study has the potential to influence their classroom actions and change practice. The application of teacher-noticing connected with teacher reflections may shed light on how the teachers are translating research into practice aligned to the researchers' goal "to transform the findings into pedagogical knowledge and practice" (Way & Ginns, 2022, p. 542).

## Research Design

The aim of the three-year (3 phase) ELEMS project is to develop an evidence-based, classroom-ready professional learning resource centred on the use of embodied learning principles. The overall research approach is mixed methods including the collection of both quantitative and qualitative data components from teachers and students within each phase. The three phases follow an iterative *development-testing-upscaling* process. The collaborative nature of the project where researchers directly interact and cooperate with practising teachers to implement the embodied learning principles aligns to a design-based research methodology (Reimann, 2010). The focus of this paper is on Phase 1: translating the prior research, applying the collective findings from the field of embodied learning research, into early years classrooms (Preschool through to Year 2) to develop a professional learning resource.

### *Participants*

In Phase 1 the researchers partnered with a NSW Department of Education school situated in a low socio-economic area of South-Western Sydney, NSW, Australia. The school has 340 students, with an additional 38 students in an attached preschool. The students come from a diverse range of cultures and 78% of students are from Non-English Speaking Backgrounds (NESB). Eight teachers and their students participated in Phase 1. This paper focuses on the teacher reflections ($N$ = 40) collected during the project via the online platform SeeSaw. Teacher demographics and the number of reflections collected are presented in Table 1. Pseudonyms are used for the teachers.

**Table 1**

*Teacher Demographics and Number of Reflection Posts*

| Teacher | Class | Years of teaching experience | Reflection posts (N = 40) |
|---|---|---|---|
| Lauren | Pre-school | 6–10 years | 1 |
| Rhonda* | Kindergarten | >15 years | 1 |
| Isla | Kindergarten | 6–10 years | 14 |
| Melissa | Kindergarten | >15 years | 16 |
| Rosa | Year 1 | 6–10 years | 0 |
| Crystal | Year 1 | 4–5 years | 6 |
| Kathleen | Year 2 | >15 years | 0 |
| Elani | Year 2 | 6–10 years | 2 |

*Rhonda withdrew from the research study during Term 1.

## *Phase 1 of the Project: Teacher-Focus*

Phase 1 involved three professional learning days for the teachers (April, June and September) that presented the research background to embodied learning and the modes of gesture, tracing, body movement and drawing, and suggested activities to trial. Throughout the year, one of the project researchers acted as a mentor, visiting the school on a weekly basis to provide in-class support and to collect observational data. The classroom observations were a means to collect teachers' strategies and activities they were using in conjunction with the embodied learning principles. Note that these activities were later refined and developed into the Teaching Guide resource that will be implemented in the Phase 2–*testing* process of the research project.

## *Data Collection and Analysis: Teacher Reflections*

Teachers were encouraged to regularly reflect on the embodied learning principles they were trialling in their classrooms. The following questions were provided as a scaffold to guide teachers' reflection posts: What embodied learning did you try in your classroom? What was the lesson's mathematical or scientific focus? How did the embodied learning link to the concept you were teaching? Comment on how you felt the use of the embodied learning went (for you and for the students)

Teachers used the SeeSaw app as a journaling space to share what and how they were implementing the embodied learning principles, often accompanied by photographs and videos. Written reflections were transcribed from the app and were deductively analysed against Sherin and van Es' (2005) teacher noticing actions: *what* did they notice? and *how* they talked about what they noticed–descriptively, evaluatively, and interpretively. Additional iterative rounds of analysis occurred where the reflections were read and re-read considering any emerging ideas or themes not discovered through the deductive analysis.

This is the initial analysis of teacher data. Pre- and post-questionnaire data on embodied learning and post-project teacher interview data have also been collected and will be used in triangulating any emerging findings from the ELEMS project. In particular, noticing growth in the teachers' thoughts about embodied learning and any changes in their teaching practices.

# Findings

## *What Teachers Noticed*

Most reflections (37/40) included noticing of student actions. Of these 37 comments, 15 also included noticing of teacher actions or pedagogy. For example, Kindergarten teacher Isla wrote:

> Today I tried something that we trialled during the first PL—the mystery bag. I placed a 2D shape in the bag and students had to draw what they felt. After a few shapes I placed a cube in the bag. All students drew a square on their page. This started our discussion on 3D shapes. I felt like the drawing aspect made the students differentiate between 2D and 3D shapes easier. They could see that their drawings did not represent the whole cube. I have also started to notice that students are using gestures. I do have to refrain myself from saying "use your words" when asking students to describe things.

Of the 40 reflections, 33 mentioned embodied learning principles specifically, using terms such as tracing/pointing (n = 7), gesture (n = 13), body movement (n = 19), or drawing (n = 11). The number of references to embodied learning totals more than 40 as 17 of the reflections mentioned using combinations of embodied learning simultaneously in lessons, see Table 2.

**Table 2**

*Evidence of Combined Embodied Learning Principles*

| Combination of embodied learning principles | Excerpt from teacher reflection |
|---|---|
| Gesture, touch and tracing | Great fun with finger tracing and developing important use of touch and teacher gesture to recall lines and patterns [Melissa]. |
| Body movement and gesture | We continued our body movement and gesture theme this week, applying it to our science concepts. We have been looking at materials and their properties and used our bodies to represent these properties [Melissa]. |
| Body movement, gesture and drawing | Students were given the opportunity to make the patterns using their bodies … Here some students demonstrated the use of gestures to describe the patterns … Students were then given the opportunity to draw their own patterns [Crystal]. |

Year 1 teacher Crystal commented on how her lesson combined multiple embodied learning principles of body movement, drawing, and gesture by the students:

> I feel the body movement enabled students to gain a deeper understanding of patterns and that they can be more than just colours repeated. It was interesting to see the use of gestures, as this lesson was more about body movement and drawing.

## *How Teachers Noticed*

Reflection data aligned to Sherin and van Es (2005) teacher noticing actions in relation to *how* they noticed: *restating*–describing the classroom events, *investigating*–evaluating student actions as evidence of what they noticed, and *generalising*–making connections between student learning and teaching pedagogy. Presented in Table 3 are reflection excerpts, italics has been used to highlight evidence that aligns to aspects of the teacher noticing actions.

**Table 3**

*Teachers' Reflections Aligned to Teacher Noticing Actions (Sherin & van Es, 2005)*

| Teacher noticing action | Excerpt from teacher reflection |
|---|---|
| *restating*–describing the classroom events | Today we *used body movement* to learn the time on a clock. *We moved our arms* in the direction of the hour hand and identified each o'clock time [Melissa]. |
| *investigating*–evaluating student thinking/doing as evidence of what they noticed | I modelled at the beginning of the lesson what hard and soft might look like through gesture, *which some students were able to demonstrate again in our discussions* of our observations and summary concluding the lesson [Isla]. |
| *generalising*–making connections between student learning and teaching pedagogy | Last week we used movement by looking at numbers before and after … students took turns to be directed to a number and move backwards and forwards to state the numbers (before and after). *I believe having students up and moving really made them more aware of this concept* [Rhonda]. |

The reflections that included *investigating*–evaluating what the students were doing, and *generalising*– making connections between student learning and teaching pedagogy, were a rich source of data. These reflections illustrated the enactment of embodied learning principles in the classroom and their effect on student learning and teacher knowledge:

Student engagement while using embodied learning:

> I think the students were engaged and came up with a lot of movements themselves, making them more meaningful [Melissa].
>
> Students were engaged in the experience, and it was interesting to see those with less control (fine motor skills) complete the activity [Melissa].

Students' mathematical understanding revealed through embodied learning:

> … revealed a lot about their understanding of time and ability to recreate a clock! [Melissa]
>
> … we used this line to do subtraction as well. Students who don't normally respond to questions were able to confidently answer the questions [Crystal].

Students' misconceptions exposed through embodied learning:

> We did the numbers 1-4 using our bodies and looked at formations. Interestingly we had to correct some reversals (a huge focus on class) within body positioning [Melissa].

Teachers' developing understanding of embodied learning as a pedagogy:

> Rosa and I went outside to trial a body movement lesson. Students were given a simple equation and had to find the total then move that many steps forward. Students enjoyed moving but we realised this movement was not embodied learning [Crystal].

Challenges to teachers' conceptions related to embodied learning:

> … some students were frustrated by the limited time to draw and the fact they could only use pencil. Would it have been better to allow a longer time period and let them colour it in? Are students more invested when using colour? What is the best representation of true knowledge? [Melissa]

### *Noticing the Future, Not the Past*

After deductively analysing the reflections and aligning them to the teacher noticing *what* and *how* actions, evidence emerged that did not fit within Sherin and van Es (2005) actions. During a re-

reading of the reflections, several narratives made reflective commentary related to 'enduring understandings' in the form of future teaching and learning principles:

> This week I focused on simply 'being aware' of different ways I use gesturing and tracing principles within my lessons. Noticing how natural and embedded it is within my teaching practice … highlighting it gives me confidence that it is present within my classroom and will add value to my lessons. It is something I feel I can continue [Melissa].
>
> I would normally use drawing as a method to gather and organise data and more verbal conversations to determine students' knowledge on interpret results. I am beginning to understand how drawing can be an important tool in Mathematics [Crystal].

These initial findings provide illustrations of what researched embodied learning principles look like in classroom settings. The findings revealed *what* teachers noticed (Sherin & van Es, 2005) in terms of (a) teacher use of embodied learning principles "*I modelled at the beginning of the lesson what hard and soft might look like through gesture*" [Isla]; (b) student use of embodied modes *"I have also started to notice that students are using gestures"* [Isla]; (c) curriculum concepts "*We have been looking at materials and their properties and used our bodies to represent these properties*" [Melissa]; and (d) teachers' own pedagogy *"This week I focused on simply 'being aware' of different ways I use gesturing and tracing principles within my lessons"* [Melissa].

## Discussion

In 'noticing teacher noticing', data showed that teachers took a blended approach to using embodied learning principles in their lessons. This connection was seen for example when teachers utilised gesture and body movement together to explore concepts. The connection was likewise observed when embodied modes were used sequentially within a lesson, or to confirm what happened physically. For example, drawing was often the follow-up embodied learning principle used to 'check' understanding after gesture or body movement were used. These results of combining embodied learning modes will add new findings to research on embodied learning– particularly the enactment of this research into practice. In relation to *how* teachers noticed (Sherin & van Es, 2005), teachers wrote about the positive impact embodied learning principles had on student engagement and learning, as well as on their own teaching practices and pedagogy. In addition, the online platform appeared to be as a 'safe space' for teachers to acknowledge limitations or areas for improvement. This was visible in Crystal's reflection on co-teaching with Rosa reflecting that *"students enjoyed moving but we realised this movement was not embodied learning"*. Teachers also noted that students could now access content through embodied principles that perhaps was not accessible prior, such as Crystal's comment, "*students who don't normally respond to questions were able to confidently answer the questions*". Similarly, common misconceptions became visible through embodied principles, for example, "*interestingly we had to correct some reversals (a huge focus on class) within body positioning"* [Melissa]. Change in teacher pedagogy aligned with embodied teaching practices were also observed. This critical reflection on their own pedagogy is evident in Melissa's musing about the use of pencils and colour when drawing for mathematical purposes. Melissa posed important questions *"Would it have been better to allow a longer time period and let them colour it in? Are students more invested when using colour? What is the best representation of true knowledge?"* Melissa's self-reflection is a practical illustration of what Mason (2011) calls taking a stance of inquiry during noticing, in that noticing entails 'holding on to an observation and seeking multiple, preferably, conflicting possibilities' (p. 40).

An unexpected result, due to the limited length of time teachers had been implementing embodied learning, was their broader future-pedagogical reflections. For example Melissa stated *"it gives me confidence that it is present within my classroom and will add value to my lessons. It is something I feel I can continue"* and Crystal's comment, *"I am beginning to understand how drawing can be an important tool in mathematics"*. When compared to previous research on teacher noticing, specifically van Es (2011) framework for learning to notice, these reflections somewhat aligned to van Es' level 4 "extended noticing" (p. 139) category. Teachers made "connections between events and principles of teaching and learning" (p. 139), in the present study's case, the embodied learning principles. These translational comments were also compared with Jacobs et al.'s (2010) action of decision-making. However, teachers were not so much proposing "alternative teaching approaches" (p. 146) for the lessons being discussed, their comments were concerning the translation of embodied learning principles in general. The comments indicated a beginning growth in the teachers' understanding of the embodied learning principles and the benefits of embedding them within future practice. Their comments were not related to decision-making for teaching, as Mason (2002) would use the root of reflection, "'flection' to refer to noticing *in the moment* [emphasis added]" (p. 84). But appeared to be deeper *after-the-moment* comments, "preflection[s] meaning to look ahead" (Mason, 2002, p. 84), that may potentially impact their own teaching philosophy and future pedagogies. This new additional action of 'translating' could be attached to the teacher noticing construct. This suggestion aligns to van Es and Sherin's (2021) acknowledgement of the expanding and new aspects of teacher noticing emerging; "we have also come to the conclusion that restricting noticing exclusively to attending and interpreting does not fully embrace what teacher noticing involves" (p. 23).

A limitation of analysing this data set from Phase 1 of the ELEMS project is the small number of reflections ($N = 40$) the findings are drawn from. It is acknowledged that as the school year progressed, teachers found it difficult to find time to commit to posting reflections. A discovery not surprising due to the ongoing impact of COVID19 and illness within the school staff creating a time-poor teaching environment. These findings are not purporting to be generalisable. The findings are exploratory in nature and revealed what these specific teachers noticed when trialling embodied learning principles in their classrooms. These results however do provide illustrations of what embodied learning principles look like in the context of the classroom. Results shed light on teachers' noticings of the impact embodied learning principles had on student engagement and learning and their own teaching practices and pedagogies.

## Concluding Remarks

The theoretical construct of teacher noticing was applied as a lens through which to analyse teacher reflections of trialled embodied learning principles. The data provides illustrations of how asserted embodied learning principles from research can be translated into curriculum-connected pedagogy. These initial findings create a promising picture for how research on embodied learning can be transformed into classroom practice within a short time frame. Much of the embodied learning research reports on haptic modes (such as gesture, tracing or body movement) in isolation. However, the main finding from the teachers' reflections suggests that when enacted in a naturalistic classroom setting, embodied learning principles meld together. A second important finding relates to the potential addition of a new teacher noticing action–*translation*. Analysis of reflections through the act of teacher noticing showed *what* teachers noticed and *how* teachers noticed (reflection–past tense), and how teachers *translated* what they noticed (preflection–future tense). These initial findings contribute new knowledge to embodied learning research related to the interplay of haptic modes, and provide a potentially new observable action within the teacher noticing space that requires further investigation.

## Acknowledgments

The ELEMS project is funded by the NSW Department of Education, Strategic Research Fund (2021–2024). The University of Sydney research team: Dr Jennifer Way (Mathematics), Dr Paul Ginns (Educational Psychology), Dr Christine Preston (Science), Dr Amanda Niland (Early Childhood), Dr Jonnell Upton (TESOL), Dr Katherin Cartwright (Mathematics).

# References

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive Development*, *14*(1), 37–56. http://doi.org/10.1016/S0885-2014(99)80017-3

Alibali, M., & Nathan, M. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of Learning Sciences, 21*(2), 247–286. http://doi.org/10.1080/10508406.2011.611446

de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics, 80*(1–2), 133–152. http://doi.org/10.1007/s10649-011-9364–8

Eden, R. (2020). Learning together through co-teaching mathematics: The role of noticing in teachers' collaborative inquiry. *Proceedings of the 25th International Commission on Mathematical Instruction (ICMI) study: Teachers of mathematics working and learning in cooperative groups* (pp. 300–308).

Ginns, P., Hu, F. T., Byrne, E., & Bobis, J. (2016). Learning by tracing worked examples. *Applied Cognitive Psychology*, *30*, 160–169. http://doi.org/10.1002/acp.3171

Goldin, G., & Shteingold, N. (2001). Systems of representation and the development of mathematical concepts. In A. Cuoco, & F. Curcio (Eds.), *The roles of representations in school mathematics: NCTM 2001 yearbook* (pp. 1–23). NCTM.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Machón, A. (2013). *Children's drawings: The genesis and nature of graphic representation: A developmental study*. Fibulas.

Mason, J. (2002). *Researching your own practice: From noticing to reflection*. Routledge.

Mason, J. (2011). Noticing: Roots and branches. In M. G. Sherin, V. A. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing* (pp. 35–50). Routledge.

Mavilidi, M. F., Okely, A. D., Chandler, P., Domazet, S. L., & Paas, F. (2018). Immediate and delayed effects of integrating physical activity into preschool children's learning of numeracy skills. *Journal of Experimental Child Psychology, 166*, 502–519. http://doi.org/10.1016/j.jecp.2017.09.009

Martinez-Lincoln, A., Tran, L. M., & Powell, S. R. (2019). What the hands tell us about mathematical learning: A synthesis of gesture use in mathematics instruction. *Gesture, 17*(3), 375–416. http://doi.org/10.1075/gest.17014.mar

Preston, C. (2016). Draw like a scientist. *Teaching Science, 62*(4), 4–8.

Reimann, P. (2010). Design-based research. In L. Markauskaite, P. Freebody, & J. Irwin (Eds.), *Methodological choices and research designs for educational and social change: Linking scholarship, policy and practice* (pp. 37–50). Springer.

Sherin, M. G., & van Es, E. A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education, 13*(3), 475–491.

Shoval, E. (2011). Using mindful movement in cooperative learning while learning about angles. *Instructional Science, 39*(4), 453–466. http://doi.org/10.1007/s11251-010-9137-2

van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. A. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing* (pp. 164–181). Routledge.

van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. *ZDM–Mathematics Education*, *53*, 17–27.

Way, J. (2018). Two birds flew away: The 'jumble' of drawing skills for representing subtraction pre-school to Year 1. In J. Hunter, P. Perger, & L. Darragh (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 98–101). Auckland: MERGA.

Way, J., & Ginns. P. (2022). A call for translational research in embodied learning in early mathematics and science education: The ELEMS project. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical Confluences and Journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 538–545). Launceston: MERGA.

# Emergent Division Thinking on Entry to School

Jill Cheeseman
*Monash University*
jill.cheeseman@monash.edu

Ann Downton
*Monash University*
ann.downton@monash.edu

Kerryn Driscoll
*Monash University*
kerryn.driscoll@monash.edu

This paper contributes to the research literature describing young children's thinking about partitive and quotitive contexts of division. A task-based interview was conducted with 98 5-6-year-old children who had not received any formal instruction about division at school. Three main findings are reported: knowledge of division was exhibited by many children who could solve both partitive and quotitive problems; a range of emergent thinking was found in children's responses to tasks; and a substantial proportion of the children could reason about division with remainders.

## Introduction

Young children's number knowledge is a predictor of future mathematical success (Peter-Koop & Scherer, 2012). Research has established children's knowledge of addition and subtraction prior to school; yet there is little research about early multiplication and division knowledge. This paper reports the results of task-based interviews with 98 children who were 5-6 years old, in their first year of school and had not been formally taught division.

Australian and international research has established that young children engage with a range of mathematical concepts and processes prior to starting school (e.g., Aubrey, 1993; Carpenter et al., 1993; Gervasoni & Perry, 2015). Several research studies have reported young children's knowledge of multiplication and division prior to school (e.g., Bicknell et al., 2016; Carpenter et al. 1993; Desforges & Desforges, 1980; Frydman & Bryant, 1988). Recently Cheeseman and Downton (2021) found that young children think multiplicatively much earlier than is reflected in the research literature. More specifically, some children can imagine and draw equal group structures and, in doing so, recognize composite units; others could also enumerate the composite units. In addition, children's emergent ideas of division have been underestimated (Cheeseman & Downton, 2021).

As part of the Multiplication and Division Investigations (MULDI) research project we developed a task-based MULDI Division clinical interview. We used identical division questions to those presented in the pencil and paper test reported by Cheeseman and Downton (2021). Preliminary results indicate that even children as young as 4 years of age can use everyday equipment to solve division problems in ways that reveal their emergent concepts of division (Cheeseman et al., 2022). The aim of this paper is to present the findings of the larger data set and to question the assumption made by many educational authorities and teachers that young children have no early multiplicative concepts—by this term we mean rudimentary ideas of multiplication and division. The research question we posed was:

- What thinking about division concepts do young children have before they are formally taught division at school?

## Theoretical Background

Three themes emerged from a review of the literature relevant to our study: partitive and quotitive division; young children's intuitive strategies; and unequal grouping situations. We acknowledge that much of the literature reviewed was dated, highlighting a gap in research concerning young children's intuitive ideas of division thinking prior to instruction.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 147–154). Newcastle: MERGA.

## *Partitive and Quotitive Division*

The sematic structure of division word problems can be interpreted and represented in two different ways, as explained by Greer (1992, p. 176):

> Dividing the total by the number of groups to find the number in each group is called partitive division, which corresponds to the familiar practice of equal sharing […]. Dividing the total by the number in each group to find the number of groups is called quotitive division.

A key difference between the partitive and quotitive problems is that the action is different. In partitive division the action is one of sharing, whereas in quotitive division the action is forming equal groups. Previous studies advocate that young children are more successful on partitive than quotitive division due to their limited experience with quotitive division prior to formal school (e.g., Correa et al., 1998; Kouba, 1989; Squire & Bryant, 2002). However, from their study Squire and Bryant (2002) concluded that both partitive and quotitive experiences are important in young children's conceptual development of division concepts. In a more recent study, Ching and Wu (2021) reported that 5-6-year-olds could recognise and reason about multiplicative relationships in partitive and quotitive problems, and that explicit instruction is not a prerequisite for understanding division.

Three studies that investigated young children's approaches to partitive and quotitive division (Carpenter et al., 1993; Kouba, 1989; Mulligan & Mitchelmore, 1997) used different contexts and numbers (dividend, divisor) for each task. The contexts were familiar to the children, for example, sitting at tables, sharing toys or cakes. Some of the numbers included (e.g., 6, 3; 8, 2; 12, 3; 15, 3; 16, 2; 18, 3; 20, 4) were within the children's number experiences. Unlike the other studies, Kouba (1989) used the same numbers across the partitive and quotitive word problems as a way to control one of the variables. Of note, children in these studies had some experience of division, and two studies extended to children aged 6 and 7 (Kouba, 1989; Mulligan & Mitchelmore, 1997). In all of these studies it was reported that the children used different strategies for each division type.

## *Young Children's Strategy Use*

In studies examining how four and five year old children solved division tasks, it was found that most children used direct modelling of the situation using one-to-one correspondence or dealing out (e.g., Blevins-Knabe, 1988; Carpenter et al., 1993; Davis & Pitkethly, 1990; Desforges & Desforges, 1980; Frydman & Bryant, 1988). Although children as young as three and four could share a quantity of 12 equally between 2, 3, or 4 recipients, Frydman and Bryant (1998) found that only 41% of four year olds were able to articulate the number in each shared set and recognise the equivalence of the quantities without having to count each set. Other researchers found that for the partitive division tasks children estimated the number of items to put in each group and tested it out, or used trial and error sharing into groups of items and adjusting (Carpenter et al., 1993; Kouba, 1989; Mulligan & Mitchelmore, 1997). In contrast, strategies used to solve the quotitive division tasks included direct modelling of the situation, double count—keeping a running count of the items in the groups, at the same time as they counting out the items to form groups.

## *Unequal Grouping Situations*

Three of the aforementioned studies (Blevins-Knabe, 1988; Carpenter et al., 1993; Desforges & Desforges, 1980) included unequal grouping situations. Four strategies were evident in the Desforges and Desforges (1980) study: asking for one or more to make the shares equal; removing the excess to make the shares equal; breaking the remainder into two or three to equalise the shares; looking puzzled about what to do with the remainder; one to one sharing; and sharing of the dividend and ignoring the remainder (3.6 to 4.6 year old students). This last strategy was consistent with those used by 4 year olds in the Blevins-Knabe (1988) study. Common strategies used by the middle and older cohorts included asking for an extra item to make up fair shares, indicating their concern to make the sharing fair, and removing the excess (Desforges & Desforges, 1980). In contrast, Carpenter et al. (1993) included one division with remainder problem (19 children how many cars needed if 5 children could fit in each car) in their study. They found that more than half of the 5 year-olds could successfully solve the problem using materials. Findings from these studies suggest that young children can engage with division situations involving remainders with varying degrees of success. However, no recent studies have examined young children's performance on both partitive and quotitive division problems involving remainders.

In summary, some of these studies provided the same context and materials (e.g., cookies and dolls) for each interview task and the same dividend (e.g., 12) but varied the divisor (e.g., 2, 3, 4) (Blevins-Knabe, 1988; Davis & Pitkethly, 1990; Desforges & Desforges, 1980; Frydman & Bryant, 1988). Doing so provided the children with structure and consistency of materials and dividend. These studies focused on partitive division only. In contrast, Carpenter et al. (1993) took more of a problem-solving approach, used different contexts, dividends, and divisors (15, 3; 20, 4; 19, 5), and provided counters and paper and pencil for the children to use. Unlike these studies, we used different contexts in our study, included the same dividend and divisor for pairs of partitive and quotitive division tasks, and two tasks involved remainders. We were interested in the thinking the children exhibited as they solved the different tasks.

## Research Method

To investigate young children's intuitive ideas of division we interviewed 98 young children (5 years 4 months to 6 years 8 months) using a task-based interview. This form of interview involves the interviewee interacting with the interviewer as well as with carefully designed mathematical tasks (Goldin, 2000). In the present study, tasks were constructed to elicit children's developing division knowledge and problem-solving behaviours. According to Maher and Sigley (2020), task-based interviews are useful for investigating "existing knowledge, growth in knowledge, and [students'] representations of particular mathematical ideas, structures, and ways of reasoning" (p. 821). Our focus in the present research was on children's informal knowledge of division, their representation of ideas, structures, and ways of reasoning. Our interview protocols used a structured script to ensure reliability, replicability, and generalisability of data collected by seven interviewers.

### *Data Collection—Interviewing Children*

A sample of children, who were in their first year of school in Australia and had not formally been introduced to division, was obtained. There was a representation of schools across the Government, Catholic, and Independent sectors, providing data from 15 classes across 13 schools. Each child was interviewed for approximately 20 minutes by an experienced interviewer who was formerly a classroom teacher. The interviewers were trained to use the protocol. They were asked to read the script and to repeat, or to clarify, but not to explain the problems. Each interview task included equipment for children to manipulate. A video record documented the events, and the interviewer noted each child's responses on an interview record sheet for later analysis.

### *Data Analysis*

To summarise the data, each interviewer entered their interview results on a formatted spreadsheet and compiled a summary of their personal reflections. All video recordings were uploaded to a central digital file accessible only to the interviewers and researchers. Researchers met with the interview team to discuss their shared insights. The third author managed the data for analysis by collating a master file and deidentifying the data in the approved university ethics process (ID 18827). Each video was viewed by Author 3 to check the accuracy of the database. The thinking strategies children demonstrated for each problem were identified and categorised. When children's actions or words required interpretation, the three authors discussed the responses and consensus was reached about the appropriate category to apply. In an interactive way the categories were refined and applied to the data. To check for reliability the researchers then double coded 20% of the data with an 83% inter-rater reliability.

## Findings and Discussion

Our findings will detail the knowledge of division that the children's responses revealed, and the range of thinking young children displayed. Some findings related to children's partial understandings will also be outlined. Finally, children's knowledge of remainders will be raised, and children's application of partitive and quotitive thinking will be described.

In the analysis of the results of the interviews, item facility was calculated. The analyses are summarised in Table 1. As can be seen, the proportion of children correctly answering each question varied. The most difficult problem was 22 ÷ 4, presented in a quotitive context of 22 children, to be seated 4 at a table; 40% of the children reached the correct solution. The easiest problem for the children involved 12 ÷ 3, where the partitive context was of 12 candies shared equally between the 3 jars; a 90% success rate was achieved. In response to each task-based interview question between 40% and 90% of students correctly solved problems involving ideas of division. Therefore, the most important finding from the analysis of the interview results was that many young children have emergent understandings of division prior to formal school instruction. The finding that explicit instruction is not a prerequisite for understanding division echoes the argument of Ching and Wu (2021).

**Table 1**

*Proportion of Students Correctly Answering the Interview Question*

| Worded problem | Correct (n = 98) |
|---|---|
| These are twelve apples. Three apples fit in a bag. How many bags do you need to carry all apples home? | 58% |
| There are seven socks in the drawer. How many pairs can you put together? | 46% |
| Here are 22 children. Four sit at a table. All the children want to sit down. How many tables do you need? | 41% |
| These are twelve candies. Share all of them out equally between the three jars. How many candies go in each jar? | 90% |
| These are seven donuts. Share all of them out equally between the two children. How many will each child get? | 71% |
| Four children want to play cards. 22 playing cards are on the table. Share them out equally between the four children. How many cards will each child get? | 43% |

### *Range of Knowledge*

It would be misleading to claim that all young children on entry to school have informal or emergent ideas about division. We observed a range of children's thinking about division. In Table 2 we present the distribution of correct solutions achieved by individual children. As can be seen, 20 children answered no questions or only one question correctly. In contrast, 14 children answered all six problems correctly. The box around the figures indicates the 64 children who showed some partially correct thinking about division contexts.

**Table 2**

*Distribution of Students Achieving Correct Solutions*

| Number of correct solutions achieved by each child | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| Total number of children in each category (n = 98) | 5 | 15 | 10 | 14 | 19 | 21 | 14 |

Looking at the extremes of achievement brought to mind two children we interviewed who exemplified children who knew little about division and those who knew a lot. Johnny (pseudonyms are used) was able to share 12 candies (glass beads) between 3 jars by placing the candies into each jar one by one. This was the only problem he answered correctly. We came to know a little about Johnny, and that his prior-to-school opportunities to learn were limited. He had not attended kindergarten. His number knowledge was limited to counting to 10 and he could not say how old he was. However, he knew he was "the number after 5" and counted aloud to work out that he was 6 years old.

In contrast, Georgie, who was in the same school and the same class, could answer every task correctly. She was a sophisticated thinker who could share 12 ÷ 3 candies mentally using a known fact. Georgie could also recognise that 7 socks could not be put into pairs and visualised that one sock would be left over. These two children illustrate the extremes of division thinking we observed in our study.

### *Emergent Thinking*

While the two children described above exemplify the range of thinking about division in their responses to the interview questions, there were 64 children who exhibited partial understanding of division concepts. We consider these children are emergent division thinkers. In Table 2 the numbers of children in each category of correct solutions (2 to 5) provide evidence that the largest proportion of children in the sample group had partial understandings of division concepts. We have chosen to describe such partial understanding as *emergent thinking*. For example, in Table 3 the categories of response (thinking strategies) for the task for which children were asked to share 7 donuts between two children are listed, and the percentage of children responding to each is shown. We argue that each of these categories represents some correct thinking. The category that may not be self-explanatory is the second category in Table 3 for which 11 children made two plates of 3½ donuts (playdoh) and answered "four" when asked, how many will each child get? Our reasoning is that these children could divide partitively, they could recognise equal shares, but they did not know how to count the pieces (fractional parts) when they saw four pieces on the plate.

**Table 3**

*Categories of Strategic Thinking for Seven Donuts Shared Between Two Children*

| Thinking Strategy | Percent of correct responses |
|---|---|
| Answered "three for each and one left over" | 48% |
| Made 3 ½ and said "four" | 12% |
| Said "three and a half" | 5% |
| Said "split one donut in half" | 6% |

Their division thinking was partially correct. In a future publication we will report a detailed examination of the characteristics of emergent thinking. However, we make some general observations from an examination of our results. Young children's division concepts seemed to be reliant on children forming and recognising equal groups. Whether it was putting the same number in each group, such as was required for the quotitive thinking to put 3 apples into each bag, or the partitive thinking required to place candies equally into three jars, children had to recognise and construct equal groups to divide successfully.

Our finding concurs with earlier research that the acquisition of equal group structure is important to understanding multiplication and division (Clarke et al., 2002; Killion & Steffe, 2002). We found that knowledge of fair shares also influenced the knowledge of division concepts children were able to display in problem solving. In addition, the contexts that we anticipated would be familiar and meaningful to young children were not universally common, and the interview results were dependent on children's general life experiences. For example, as Amy told us, the socks problem was easy for her because her Mum gave her that job—to sort their socks and put them into pairs. Whereas some children did not know what a "pair of socks" was. These responses drew our attention to the importance of everyday life experiences where young children *mathematise* their world (van Oers, 2013) and develop informal mathematical concepts—in this case, division.

### *Remainders*

We found that young children recognised that division into equal sized groups was not always possible, that is, in situations involving remainders. For example, 71% of children correctly answered 7 ÷ 2 (see Table 1) when we accepted as correct any answer that showed an awareness that one donut of the seven donuts could not be shared equally between two children. The actions of the children indicated that they thought about the situation carefully. For example, it was clear from Georgie's solution—"three each and one for a spare" that she understood the remainder of one. Answering the same problem, Adriana worried about the extra donut. She placed three on each plate and tried several different ways of adding the extra donut to either of the two plates unsuccessfully. It seemed to us that she was testing whether the specific object (donut) mattered. She replaced one donut with another three times before she volunteered, "You have to do nothing with it." It seemed that this was the first time Adriana had been confronted with a sharing situation involving a remainder.

Four of the six interview problems involved remainders; the easiest one to solve was the donut problem (71%). This context for 7 ÷ 2 was easier than the context of pairing socks, which 46% of the children solved correctly. The two contexts for 22 ÷ 4 were correctly achieved by 41% and 43% (see Table 1) of the children. We noted that a substantial proportion of 5-6-year-olds were aware of the need to make equal shares and that some objects would be left over after the equal shares were made.

Our findings expand upon the earlier research (Blevins-Knabe, 1988; Carpenter et al., 1993; Desforges & Desforges, 1980) in that this study involved both partitive or quotitive division situations where each way of conceptualising division was contextualised using the same division operation. In addition, the problems posed in this study involved remainders.

### *Partitive and Quotitive Contexts*

An overview of the correct responses in Table 1 shows that many young children in this study could think about division problems solved in both partitive and quotitive contexts. While this finding concurs with research conducted decades ago (e.g., Carpenter et al., 1993; Kouba, 1989), it is unique in that the same children were asked to think about the same division calculation on the same occasion. In the present study, 68 children (see Table 1) could successfully interpret both partitive and quotitive division situations. We query the claim found in the literature that dealing is a natural way for young children to make equal groups (Davis & Pitkethly, 1990). It seems, based on our experience, that children can deal out the cards one by one. They continue until the cards are exhausted and consider the job done. When playing cards the children paid no attention to whether the dealt groups were equal. Johnny was more interested in playing with the cards. He asked, "How many should each person get?" Ann responded with, "I am sure that they would love whatever you give them." When the dealing was finished he announced that, "This lady has 7. She has more." Johnny dealt the cards and was unconcerned about forming unequal groups.

Georgie, when asked, "How many each?" counted the piles of cards and said, "6, 5, 6, 5". Asked, "Is that equal?" Georgie thought for a few seconds before saying, "Five each and two spare". Thereby she demonstrated that she knew about equal groups and what to do with amounts that were not divisible. Initially she had thought the process of dealing cards was completed when the cards were dealt out. As a result of the 22 cards divided between 4 people task, we learned that cards are a natural dealing context for these children but they can be a distraction.

In summary, we identified three major findings from our study.

1. Knowledge of division was exhibited by 95% of children in this study. The children had not been formally taught division at school at the time of interview. This result contributes to the literature by revealing 5-6-year-old children could solve both partitive and quotitive problems using informal mathematical knowledge.
2. A range of emergent thinking was found in the young children who were studied. This is an original finding as no research has been found that has considered the emergence of division concepts in young children's thinking.
3. Division involving reasoning about remainders was demonstrated by more than 40% of the children who realised that some objects could not be used to make equal groups. No earlier studies have investigated both partitive and quotitive division involving remainder thinking with young children.

## Conclusion

The limitation of this investigation is that it used a relatively small number of worded problems to assess division concepts. However, in the study young children's thinking about division was investigated, with some thought-provoking results. Our findings indicate that some young Australian children have emerging ideas of division in their first year of school, and before they are formally taught division. Unlike earlier studies, problems in our study were numerically matched (same dividend, same divisor) and presented in both a partitive and quotitive context to elicit children's thinking. Another distinguishing feature of this study was the inclusion of situations involving remainders. While earlier researchers reported the strategy use of children, they did not report children's emergent thinking. Thus, the findings of our study extend the research in this field. We think there is much to be added to the body of scholarly knowledge about key ideas in early division reasoning, as it is still an under-researched field of study. How children's informal knowledge of division productively develops into formal concepts is yet to be fully understood even though it has major implications for teaching early childhood mathematics.

## References

Aubrey, C. (1993). An investigation of the mathematical knowledge and competencies, which young children bring into school. *British Educational Research Journal, 19*(1), 27–41.

Bicknell, B., Young-Loveridge, J., & Nguyen, N. (2016). A design study to develop young children's understanding of multiplication and division. *Mathematics Education Research Journal, 28*(4), 567–583.

Blevins-Knabe, B. (1988). Preschool children's informal division concepts. *Paper presented at the meeting of the Southwestern Society for Research in Human Development*. https://files.eric.ed.gov/fulltext/ED308951.pdf

Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education, 24*(5), 428–441.

Cheeseman, J., & Downton, A. (2021). Investigating division concepts at entry to school. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Mathematics with distinction without discrimination. Proceedings of the 44th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 132). PME.

Cheeseman, J., Downton, A., & Roche, A. (2022). Ideas of early division prior to formal instruction. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 131–138). PME.

Ching, B, H.-H., Wu, H. X. (2021). Young children's knowledge of fair sharing as an informal basis for understanding division: A latent profile analysis. *Learning and Instruction, 73*(1). https://doi.org/10.1016/j.learninstruc.2021.101460

Clarke, D. M., Cheeseman, J., Gervasoni, A., Gronn, D., … Rowley, G. (2002). *Early numeracy research project final report*, February 2002. Fitzroy, Victoria: Australian Catholic University.

Correa, J., Nunes, T., & Bryant, P. (1998). Young children's understanding of division: The relationship between division terms in a noncomputational task. *Journal of Educational Psychology, 90*(2), 321–329 https://psycnet.apa.org/record/1998-02710-012

Davis, G. E., & Pitkethly, A. (1990). Cognitive aspects of sharing. *Journal for Research in Mathematics Education, 21*(2), 145–153 https://pubs.nctm.org/view/journals/jrme/21/2/article-p145.xml

Desforges, A., & Desforges, C. (1980). Number-based strategies of sharing in young children. *Educational Studies in Mathematics, 6*(2), 97–109.

Frydman, O., & Bryant, P. (1988). Sharing and the understanding of number equivalence by young children. *Cognitive Development, 3*, 323–339.

Gervasoni, A., & Perry, B. (2015). Children's mathematical knowledge prior to starting school and implications for transition. In B. Perry, A. MacDonald, & A. Gervasoni (Eds.), *Mathematics and transition to school: International perspectives* (pp. 47–64). Springer.

Goldin, G. (2000). A scientific perspective on structures, task-based interviews in mathematics education research. In Lesh, R., Kelly, A. (Eds.), *Research design in mathematics and science education*. Erlbaum, Hillsdale, pp. 517–545.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). Macmillan.

Killion, K., & Steffe. L. P. (2002). Children's multiplication. In D. L. Chambers (Ed.), *Putting research into practice in the elementary grades: Readings from journals of the NCTM* (pp. 90–92). Wisconsin Centre for Education Research.

Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division problems. *Journal for Research in Mathematics Education, 20*, 147–158.

Maher, C. A., & Sigley, R. (2020). Task-based interviews in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 821–824). Springer.

Mulligan, J., & Mitchelmore, M. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28*(3), 309–330. https://doi.org/10.2307/749783

Peter-Koop, A., & Scherer, P. (2012). Early childhood mathematics teaching and learning. *Journal für Mathematik-Didaktik*, 33, 175–179. doi:10.1007/s13138-012-0043-9

Squire, S., & Bryant, P. (2002). From sharing to dividing: Young children's understanding of division. *Developmental Science, 5(*4), 452–466.

van Oers, B. (2013). The roots of mathematising in young children's play. In U. Kortenkamp, B. Brandt, C. Benz, G. Krummheuer, S. Ladel, & R. Vogel (Eds.), *Early mathematics learning* (pp. 111–123). Springer https://pubs.nctm.org/view/journals/jrme/21/2/article-p145.xml.

# Use of Card Sorting Methodology to Characterise a Primary Teacher's Mathematical Knowledge for Teaching

Su Ngin Chia
*Nanyang Technological University*
nie20.csn@e.ntu.edu.sg

Ban Heng Choy
*Nanyang Technological University*
banheng.choy@nie.edu.sg

Many studies attempt to study teachers' mathematical knowledge for teaching (MKT) through instruments such as multiple-choice questions. Many of these instruments require considerable context information and often provide a static view of a teacher's knowledge. In this paper, we describe the design and use of a card-sorting instrument that can elicit a teacher's MKT. By illustrating how the card-sorting instrument was used to capture a primary teacher's knowledge about teaching division, we argue that card sorting, as part of a suite of other methods, can be a powerful approach to elicit teachers' knowledge in teaching. We conclude by highlighting possible directions in refining and developing card sorting instruments for future studies.

What mathematics teachers know about teaching and learning is an important attribute of teaching expertise. Even before Shulman's (1986) introduction of pedagogical content knowledge (PCK), and subsequently, the conceptualisation of mathematical knowledge for teaching (MKT) by Ball and Bass (2003), there had been efforts to investigate and measure teachers' knowledge (Hill et al., 2007). These early measurements of teachers' knowledge focused solely on mathematical content. However, as highlighted by Hill et al. (2007), teaching requires "more than the ability to do the mathematics in the school curriculum" (p. 125). Instead, Ball et al. (2008) argued that teaching needs knowledge about student errors, alternative algorithms, "rationales for procedures, meanings for terms, and explanations for concepts" (p. 398). Moreover, effective teaching requires coordination of appropriate representations (Shulman, 1987), selecting and using appropriate examples (Chick, 2009), as well as the sequencing and tailoring of content to specific profile of students (Shulman, 1987). Accessing and assessing these domains of knowledge often involve the use of observations, tasks and interviews, and test items, amongst others (Delaney et al., 2008; Hill et al., 2004; Hill et al., 2007; Li, 2007). Despite developments in assessing teachers' knowledge, many of these methods require substantial contextual information (Phelps & Howell, 2016), which may result in a rather limited view of a teacher's knowledge (Hill et al., 2008). We wonder about other alternative tasks that might be useful in uncovering a teacher's mathematical knowledge for teaching. One possible way is to consider the use of card sorting and sequencing tasks, commonly used to analyse participants' thinking and understanding of mathematical concepts as part of a task-based interview (Goldin, 2000). In this paper, we describe the design and use of a card-sorting instrument to elicit a teacher's knowledge in teaching and illustrate how the instrument was used to capture a primary teacher's knowledge about teaching division.

## Conceptualizations of MKT

In this section, we begin by recapping the notion of mathematical knowledge for teaching before we examine how teachers' MKT can be assessed. Ball et al. (2008) built on on Shulman's PCK to delineate three subsets of knowledge: knowledge of content and student (KCS); knowledge of content and teaching (KCT); and knowledge of content and curriculum. In addition, Ball and her colleagues expanded the notion of subject-matter knowledge (SMK) by considering the mathematical content needed for teaching. To that end, they characterised the idea of specialised content knowledge (SCK) as the mathematical knowledge and skills that are uniquely required in the work of teaching in contrast to common content knowledge (CCK). this may include knowledge of less common algorithms, different representations of concepts, and explanations of rules and procedures (Ball et al., 2008). In addition, they also developed the provisional idea of *horizon*

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 155–162). Newcastle: MERGA.

*content knowledge*, which "is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum", and this may include "vision useful in seeing connections to much later mathematical ideas" (p. 403).

## Common Methods for Assessing Teachers' MKT

Since Ball et al.'s (2008) description of MKT, researchers had explored different methods to measure teachers' MKT. However, the process of measurement is complex and challenging (Fauskanger, 2015; Hill et al., 2007). It is common for MKT to be assessed through pen and paper test such as multiple-choice items during teachers' examination (Hill et al., 2007) but such tests are usually focused on CCK and SCK (Hill et al., 2007). Other domains of MKT, such as knowledge of their students' common misconceptions and difficulties when learning a particular topic (KCS) or the knowledge as demonstrated by teachers' ability to select the appropriate materials and sequence their teaching examples (KCT) are generally not tested in these items.

With the aim of capturing a more comprehensive view of MKT, researchers have begun to explore the use of lesson observations, mathematical tasks which includes open ended questions, interviews, and teachers' responses to video clips of teaching (Fauskanger, 2015; Hill et al., 2007). Some of these more recent methods provide participants opportunities to explain a misconception or to produce alternative explanations of a mathematical concept (Fauskanger, 2015). For example, Roche and Clarke (2013) used division stories as a questionnaire item to assess teachers' MKT—participants had to name the form of division (partitive or quotative), drawing a picture and write a story problem to represent the form of division. Story problems are also used in the study conducted by Simon (1993) as one of their open-response problems to assess prospective elementary teachers' knowledge of division. In Simon's (1993) study, the participants were required to solve word problems, demonstrate the division algorithm, calculate remainders, and explain how they carried out long division as part of their responses. These methods have attempted to capture a more comprehensive view of teachers' MKT by capturing their pedagogical reasoning. However, the questions are often contextualised which limit teachers' responses and the nature of the questions also result in closed-ended and static responses which are unable to capture the depth of teachers' knowledge (Fauskanger, 2015). In this paper, we contribute to this ongoing effort by introducing another method—sorting and sequencing task—which has the potential to capture participants' MKT as demonstrated by their pedagogical reasoning and illustrate its use to elicit a teacher's MKT in the topic of division.

### *Sorting and Sequencing Tasks*

In sorting and sequencing tasks, participants are presented with a set of cards, carefully designed according to the structure or properties of the concept, for them to sort the cards into different categories or to sequence the cards according to some criteria or preferences (Galant, 2013; Hillen & Malik, 2013). These cards can be printed with different representations such as words, pictures, graphs, and mathematical equations related to the concept. The representations can be related or can be classified according to similar properties. For example, in a set of sorting cards on the topic of division, the representations can be designed based on key ideas of division such as quotient, divisor, and remainder. As the participants sort or sequence their cards, they have to make connections and identify relationships among the structure and properties of the mathematical concept, which are indicators of the level of their conceptual knowledge (Eli et al., 2011). For instance, if a participant is able to sort the cards according to the structures or mathematical features of the concept, we can infer that the participant may have a well-connected set of SCK (Galant, 2013). To gain more insights into the participant's knowledge, card sorting and sequencing tasks can be carried out together with a semi-structured interview (Goldin, 2000) to elicit their explanations and justifications. Using the sorting and sequencing task as a basis for discussion, researchers may ask their participants about the 'why' and 'how' of their sorting and sequencing. For example, in our study, participants may elaborate by describing how they will teach the concept based on students' profiles and highlight possible students' difficulties (KCS). Participants may also share the contexts that they will use in their teaching (KCT) and explain how they may make reference to the curriculum in their teaching. Hence, when compared to the current methods used to assess MKT, the use of sorting and sequencing tasks with semi-structured interviews can allow for more dynamic and open responses from participants, which can potentially elicit a more comprehensive perspective of a teacher's knowledge in the different domains of MKT. In the next section, we will describe how we designed a card sorting and sequencing instrument to capture teachers' MKT for the topic of division.

## Design of our Card Sorting Instrument on Division

Division at the primary level is a difficult topic for teachers to teach and for students to learn (Graeber, 1993; Holland, 1942; Pope, 2012; Sellers, 2010). For teachers, key ideas of division such as partitive and quotative concepts of division (Graeber, 1993; Martin, 2009; Pope, 2012), and its relation to other arithmetic operations (Holland, 1942) may be challenging for some to grasp. For students, they may lack the prerequisite knowledge of division such as multiplication facts, and other operations and face difficulties when learning division because of their emerging multiplicative thinking which is developed on the basis of partitioning (Holland, 1942; Pope, 2012). Furthermore, the foundational ideas of division form the basis for students in later grade levels to make sense of operations involving fractions, which often pose challenges for many teachers and students (Lamon, 2012). Hence, it is imperative for mathematics educators to gain insights into teachers' knowledge of the key ideas of division and the difficulties that students' face during learning (Holland, 1942; Hopkins et al., 2009) so that teachers can be better supported to design instruction for this foundational topic.

To capture teachers' knowledge for teaching division, we designed a set of sorting cards on division expressions as part of a semi-structured interview to elicit teachers' different domains of MKT in the topic of division. When designing the cards, we first considered the key division concepts such as quotient, divisor, remainder in relation to the long division algorithms. We referred to one of the coursebooks (Collars et al., 2015) used in Singapore to teach mathematics at the primary levels when designing the cards. According to the sequence of the content in the coursebook, the concept of quotient and remainder is first introduced, followed by division without renaming and division with renaming. 'Division without renaming' consists of dividends in which the digit in every place value can be easily divided by the divisor. For example, in '$42 \div 2$', the 4 tens and 2 ones can both be divided by 2 exactly. On the other hand, 'Division with renaming' consists of digits in the dividends that need to be renamed. For example, in '$42 \div 3$', 4 tens is not divisible by 3, and there is a need to rename 42 as 3 tens and 12 ones as we work through the long division algorithm. Based on the key ideas of renaming and remainder, four categories are created—with remainder and no renaming, with remainder and renaming, no remainder and renaming and no remainder and no renaming—to form the basis of our sorting cards.

Next, we decided on the choice of numbers in the expressions by applying the theory of discernment and variation (Kullberg, 2017). For example, we used the four expressions '$43 \div 2$', '$42 \div 2$', '$43 \div 3$' and '$42 \div 3$' to provide opportunities for the participants to discern the idea of remainder ('$43 \div 2$' and '$42 \div 2$') and the concept of remainder ('$42 \div 2$' and '$42 \div 3$'). In addition, smaller numbers are chosen so that participants can focus on the concepts of division, instead on the calculation of the answers. '205' and '204' were included as students often have difficulties working out division algorithm with zero in the place holders (Holland, 1942) and students have difficulties with the placement of digit when attempting long division (Holland, 1942; Martin, 2009). These considerations in the designing of the sorting cards expressions aim to provide a basis for participants to elaborate on their knowledge in the teaching of division during the interviews. For the semi-structured interview, we developed a protocol, as shown in Table 2, and provided guiding questions when participants needed more prompts to elicit their ideas. An example of guiding question asked is: What is the difference between ‘42 ÷ 3’ and ‘42 ÷ 2’? We used this question to prompt our participants to explain the difference based on their understanding of the concept of remainder, renaming or quotative/partitive concept.

**Table 1**

*Division Expressions Used in Sorting Cards*

| | With Remainder | No Remainder |
|---|---|---|
| Renaming | 7 ÷ 3<br>21 ÷ 2<br>43 ÷ 2<br>54 ÷ 4<br>205 ÷ 2<br>483 ÷ 2 | 20 ÷ 2<br>8 tens ÷ 4<br>9 hundreds ÷ 3<br>6 ÷ 3<br>42 ÷ 2<br>204 ÷ 2<br>482 ÷ 2 |
| No Renaming | 17 ÷ 4<br>21 ÷ 4<br>43 ÷ 3<br>54 ÷ 5<br>205 ÷ 4<br>482 ÷ 3 | 16 ÷ 4<br>20 ÷ 4<br>42 ÷ 3<br>54 ÷ 3<br>204 ÷ 4<br>483 ÷ 3 |

**Table 2**

*Key Questions Asked During Semi-structured Interviews*

| | |
|---|---|
| Sorting Task | How would you categorise the 25 division expressions?<br>Why do you categorise them this way?<br>Are you able to sort them further? |
| Sequencing Task | How would you sequence the categories in your teaching?<br>Why did you sequence it this way? |

When we worked with the teachers with this card sorting and sequencing task, we aimed to capture video and voice recordings of the teachers’ sorting. To do so, we set up the interview venue as shown in Figure 1. Referring to the set-up, iPad 2 (recording device) was positioned overhead to capture the movement and the grouping of sorting cards and iPad 1 mainly act as a backup to capture a different view of the participants. During the tasks, we asked questions to understand participants’ decisions behind their sorting and sequencing. During the interviews, we also took photos of the participants’ key steps when they completed their sorting.

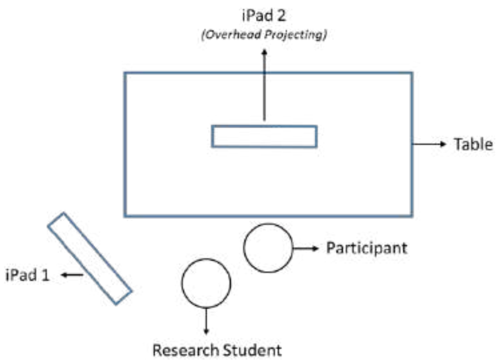



Figure 1. Set up of interview station.

Findings were developed from the transcripts of the video recordings of the card sorting and sequencing, voice recordings of the interviews, and the photographs of the different sorts and sequences as demonstrated by the participants. We analysed the card sort and sequence in chronological order and paid attention to the way the cards were sorted and sequenced. By making references to the interview for their sorting decisions, we matched the participants' understanding of the different domains of MKT using the descriptions developed by Ball et al. (2008). We also compared the ideas of division mentioned by the participants with the key ideas in extant literature related to the teaching of division (Graeber, 1993; Martin, 2009; Pope, 2012). To confirm or refute our assessment of the participant's understanding, we also referred to the coursebook used by the teacher and the syllabus document.

## What did the Cards Tell Us About Eleanor's MKT?

In this section, we will illustrate the use of our instrument by describing and analysing the responses of Eleanor, one of our participants, to the sorting and sequencing task. Eleanor is a Primary teacher with four years of teaching experience, a typical representative of an experienced teacher in Singapore. For Eleanor, only the first author was involved in the data collection. Through her responses, we characterised some domains of her MKT, in particularly, her KCT, KCS, and SCK.

### *Eleanor's Sorting and Sequencing of Cards*

We first present Eleanor's first sorting and further sorting as shown in Figure 2. When presented with the 25 cards, Eleanor first sorted the cards into nine groups according to divisors (2, 3, 4, and 5), followed by the number of digits in the dividends. Referring to Figure 2, we note that Groups 1 and 2 contain expressions involving 2 as a divisor; Group 3, 5 and 6 are expressions with 3 as a divisor; and Group 7 and 8 are expressions with 4 as a divisor. Furthermore, we see that Group 4 comprised expressions with dividends in words and Group 9 consists of an expression with divisor 5. When Eleanor was asked to further sort her expressions, she separated Group 7 into Groups 7a and 7b as shown in Figure 2, according to the number of digits in the quotients. Looking at the expressions in the two sub-groups, we note the 1-digit and 2-digit quotients in Group 7a and Group 7b respectively. For the sequencing task, Eleanor explained that she would start with Groups 3 and 7 as the expressions are linked to the standard multiplication facts covered in Primary Two. However, after the first author prompted her that some expressions in Groups 3 and 7 will result in remainders, she retracted and highlighted that she would begin with smaller number, and introduce expressions which answers have no remainder first, followed by expressions which answers consists of remainder without remainder. She also sequenced the cards according to the number of digits in the dividend in an increasing order.

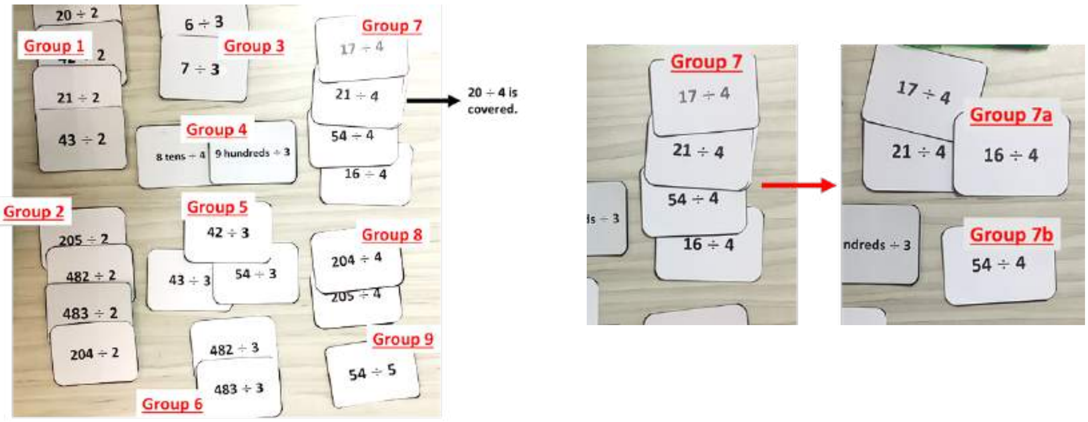



Figure 2. Eleanor's 1st sorting and further sorting.

## *Eleanor's KCT and KCS*

We observe that Eleanor would think about the sequencing of the expressions according to the different structures of the division expressions (KCT). In particular, she was cognisant of the number of digits in the dividends, i.e., a 3-digit number divided by 1-digit number, and 2-digit number divided by 1-digit number. When asked to explain her sorting for Groups 1 and 2, she elaborated:

> So, I [pointed at Group 1] look at it, the first, the first one is 2-digit. This is 3-digit [Group 2]. Because this one will lead to perform the algorithm. You know the long division thing. So, it's another time of skills also. For example, this one. This [Group 1] is quite straight forward.

Here, we see that Eleanor noticed the number of digits in the dividends and quotients and how students might respond to the different division problems (KCS). For instance, she noted that students would need to use long division ("algorithm") to find the answer for 3-digit dividends. Similarly, Eleanor emphasised on the importance of using smaller numbers because it would be easier for the students, and it is also easier for teachers to represent the information using other representation (KCT):

> I think I will start with this [group 3] because it's easier. Lesser number for them…Like what you said, I will bring in the simple numbers so that they are able to see. And, also easier for us to show visuals also because it is [a] smaller number.

Besides focusing on the number of digits in the dividends or quotients, Eleanor also demonstrated her KCS by identifying some of her students' difficulties when learning division:

> I realise that they are just memorizing the steps. That's why I always try to change it in such a way that they understand them, so I always use terms like how many 3 gets into this? Rather than what is divided by what. Because divided by what, sometimes they are also weak in the times table.

In the explanation above, Eleanor's pointed out that her students may be weak in their "times table" (multiplication facts) and some may have memorised the algorithm without understanding. She went on to explain that students were confused with the steps in the division algorithm (KCS) and provided explanation on how she tried to shift students away from rote memorisation (KCT). Instead of reading out the steps of the algorithm during teaching, she would attempt to explain the steps in a way that is meaningful to the students (KCT).

Eleanor also added that another common difficulty that students experienced when learning division was the concept of remainder (KCS). She shared that she used stories and the concept of equal grouping to teach the remainder concepts to help students to overcome this difficulty (KCT):

> Because normally right when you tell them like a story, like you share, then later on, you all want to be, let's say you all got a box of chocolate right, you all share, I give you one, then in the end, got left one. Do you put in either of the group or not? Then they will tell you 'no', must be equal. Then I say, this is remainder.

Hence, we can see that Eleanor had demonstrated several aspects of KCS and KCT because she not only understood students' learning difficulties in the long division algorithm but also highlighted how to help students overcome them.

### *Eleanor's SCK*

Eleanor's specialised content knowledge was also revealed during the card sort task. When asked about the difference between '42÷3' and '42÷2', she replied that "the number of groups are different". The first author continued to prompt and ask her how she would teach the division algorithm. For 42 ÷ 2, Eleanor explained that she would get them to think about equal grouping and asked the students ''like how many 2s can get into 4?', which is measurement model of division. Her response suggests that she was aware of the idea of division as equal grouping (measurement model). For '42 ÷ 3', she explained: "So, you have 3 groups, so they realise that only 1 group. That means 10 in each group". This was puzzling to us because "10 in each group" suggests an equal sharing concept of division. She then continued to explain that she would ask the pupils "how many 3s can get into 40?", which signify an equal grouping idea of division. Although she had wanted to use equal groupings to explain both questions, her use of two different models of division suggest that Eleanor might be confused about these two models of division. Again, the card sort task provided opportunities for us to understand a teacher's SCK and made it possible to identify the point of confusion that a teacher might have.

## Concluding Remarks

In this paper, we introduced the card sorting methodology as an alternative novel method to capture teachers' MKT and illustrated its design and use with a teacher, Eleanor, in the context of a division topic. As opposed to existing methods for measuring MKT, our card sort instrument together with the interview can potentially capture a wider and more dynamic view of a teacher's MKT. As seen in the case of Eleanor, we were able to capture her thought processes on how she sort and sequence her cards, and hence characterised some of the domains of her MKT. We believe that our card sort instrument on division can be refined by reviewing the division expressions selected, improving the interview procedures and analytical approaches, and possibly including other tasks to capture a more comprehensive view of a teachers' MKT. At this stage, how the method can provide a more usable snapshot of a teacher's MKT for a given topic remains unclear. How the instrument can be further refined and used will be an interesting area for research.

## References

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. 2002 annual meeting of the Canadian Mathematics Education Study Group, Edmonton, AB.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Chick, H. L. (2009). Choice and use of examples as a window on mathematical knowledge for teaching. *For the learning of mathematics*, *29*(3), 26–30.

Collars, C., Koay, P. L., Lee, N. H., & Tan, C. S. (2015). *Shaping maths coursebook 3A* (3rd ed.). Marshall Cavendish Education.

Delaney, S., Ball, D. L., Hill, H. C., Schilling, S. G., & Zopf, D. (2008). Mathematical knowledge for teaching:x Adapting U.S. measures for use in lreland. *Journal of Mathematics Teacher Education*, *11*(3), 171–197. https://doi.org/10.I007/s10857-008-907I2

Eli, J. A., Mohr-Schroeder, M. J., & Lee, C. W. (2011). Exploring mathematical connections of prospective middle-grades teachers through card-sorting tasks. *Mathematics Education Research Journal*, *23*(3), 297–319.

Fauskanger, J. (2015). Challenges in measuring teachers' knowledge. *Educational Studies in Mathematics,* (90), 57–73.

Galant, J. (2013). Selecting and sequencing mathematics tasks: Seeking mathematical knowledge for teaching. *Perspectives in Education*, *31*(3), 34–48.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, N.J.: L. Erlbaum.

Graeber, A. O. (1993). Misconceptions about multiplication and division. *The Arithmetic Teacher*, *40*(7), 408–411.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*(4), 430–511. https://doi.org/10.1080/07370000802177235

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal*, *105*(11–30).

Hill, H. C., Sleep, L., Lewis, J. M., & Ball, D. L. (2007). Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts? In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 111–155). Information Age Publishing, Incorporated.

Hillen, A. F., & Malik, L. (2013). Sorting out ideas about function. *The Mathematics Teacher*, *106*(7), 526–533.

Holland, H. (1942). Difficulties involved in long division and some suggestions for teaching the process. *The Elementary School Journal*, *42*(8), 585–596.

Hopkins, C., Pope, S., & Pepperell, S. (2009). *Understanding primary mathematics*. David Fulton Publishers.

Kullberg, A. R. K., Ulla ; Marton, Ference. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM Mathematics Education*, *49*(4), 559–569.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding* (3rd ed.). Routledge.

Li, X. (2007). *An investigation of secondary school algebra teachers' mathematical knowledge for teaching algebraic equation solving*. University of Texas at Austin.

Martin, J. F., Jr. (2009). The goal of long division. *Teaching Children Mathematics*, *15*(8), 482–487.

Phelps, G., & Howell, H. (2016). Assessing mathematical knowledge for teaching: The role of teaching context. *The Mathematics Enthusiast*, *13*(1–2), 52–70. https://doi.org/10.54870/1551-3440.1365

Pope, S. (2012). The problem with division. *Mathematics Teaching*, *231*, 42–45.

Roche, A., & Clarke, D. M. (2013). Primary teachers' representations of division: Assessing mathematical knowledge that has pedagogical potential. *Mathematics Education Research Journal*, *25*(2), 257–278.

Sellers, P. A. (2010). The trouble with long division. *Teaching Children Mathematics*, *16*(9), 516–520.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22.

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, *24*(3), 233–254.

# Design Principles for Raising Students' Awareness of Implicit Features of Ratio: Creating Opportunities to Make and Catch Mistakes

Sze Looi Chin
*Nanyang Technological University*
nie19.csl@e.ntu.edu.sg

Seeing a problem and immediately knowing how to solve it is typically desired in students. However, it can also lead to impulse thinking, where students are unconscious of critical features of concepts and consequently make negligent mistakes. This study investigated the design principles of a secondary mathematics teacher who designed instructional materials to raise her students' awareness of implicit features of ratio and proportionality. Analysis of her design and implementation revealed how she created opportunities for students to make mistakes in class and to "catch" them to address these implicit features. Implications of adopting design principles to manage impulse thinking are discussed, as well as an introduction of the notion of *catch tasks*.

Research on mathematics teachers' design work typically report on goals for developing students' conceptual understanding, procedural fluency, and metacognitive skills, amongst other things. To achieve these goals, design principles related to sequencing, choosing rich tasks, and forming connections are often used (Swan & Burkhardt, 2014). However, the goal of managing the pace of students' thinking (e.g., encouraging them to "slow down", to not act on impulse) is less reported, along with research on design principles that teachers use to achieve this goal. Although the ability to see a problem and immediately know how to solve it is desirable in students, acting on intuitions is not always conducive and can even be disadvantageous (Kahneman, 2012). This was the case for students in a class taught by the teacher in this study, Tanya (pseudonym). She noted that when her students are fixated on finding the solution and have prior knowledge of the content and procedures, they often act on impulse and are unaware of critical features of the task, consequently making negligent and costly mistakes that may impact future learning and assessment results. In the worst case, when *implicit features* that are tacit and rarely explicitly emphasised (e.g., proportionality is based on multiplicative reasoning, *not additive reasoning*) and errors (e.g., using addition to determine equivalent ratios) are not brought to students' attention and addressed, students can develop underlying misconceptions. Consequently, Tanya designed instructional materials with the goal of managing students' impulse thinking to raise their awareness of implicit features in the topic of ratio. The aim of this study is to examine the design principles she used in crafting her instructional materials to achieve this goal in the classroom.

## Background

There is a growing consensus on the need for more research on mathematics teachers' design work (Kaur et al., 2022; Watson & Ohtani, 2015). While Brown (2009) conceived of teachers' design work as occurring predominantly during instruction, emergent research has documented the significant design that can occur before the lesson, which includes not only lesson planning but also the design of *instructional materials* (IM) (Kaur et al., 2022). These teacher-designed IMs generally help to guide the flow of the lesson, and as such, they typically consist of a multitude of tasks that are selected, modified, or created by the teacher for the purpose of achieving their intended curriculum (Remillard & Heck, 2014). Hence, to design these IMs requires teachers to make multiple deliberate design decisions related to gathering and sequencing tasks so that they will be effective for teaching.

Schoenfeld (2010) proposed that to make sense of how teachers make these kinds of decisions, researchers should investigate the interactions between teachers' resources, orientations, and goals, and noted that decision-making is ultimately goal-oriented. One goal that is seldom reported in

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 163–170). Newcastle: MERGA.

mathematics education research is the goal of managing students' impulsive dispositions. This intuitive form of thinking, so-called *thinking fast* or *System 1* thinking (Kahneman, 2012), is automatic, well-rehearsed, and unconscious, but it becomes disadvantageous when individuals jump to conclusions without gathering accurate evidence and rational reasoning. In contrast, an analytic disposition, so-called *thinking slow* or *System 2* thinking, is deliberate, conscious, and logical; this is typically elicited when unexpected or unfamiliar situations arise that force individuals to reflect on and analyse the situation at hand. Lim and Wagler (2012) found that when students were familiar with the task situation and had some prior understanding, they tended to adopt an impulsive disposition, while not always producing correct solutions to mathematics problems. Despite its potential for supporting students' conceptual development, there is little research reported on what mathematics teachers do to engage and manage the pace of students' thinking through instructional design and in the classroom. If teachers do not consciously design opportunities for students to slow down to notice their errors, students are unlikely to notice important features of concepts and to realise their own mistakes (Watson & Ohtani, 2015).

Teachers' knowledge and their choice of examples and tasks is crucial for helping students to notice important features for solving problems (Ball et al., 2008; Goldenberg & Mason, 2008). While rich, collaborative, and higher-order tasks are usually touted for effective teaching (Swan & Burkhardt, 2014), typical problems like those commonly found in textbooks and examinations have also been shown to be promising for orchestrating productive classroom discussions. Choy and Dindyal (2021) demonstrated how this can be achieved when teachers perceive multiple affordances of a task and are therefore able to use seemingly typical tasks in rich and meaningful ways that extend beyond procedural solving. Furthermore, examples are an important resource for students' understanding of a concept, while non-examples serve a valuable role of demonstrating the boundaries of the concept (Mason & Watson, 2008), as "part of understanding a concept is knowing what it is *not* and when it does *not* apply", as stated by Lamon (2012, p. 5). Creating a so-called *cognitive conflict* can help students to identify and rectify their errors (de Bock et al., 2002). In particular, Warshauer (2014) suggested that providing opportunities for students to make mistakes and engage in struggle can be a productive way to strengthen their understanding of a concept. Thus, allowing students to see incorrect solutions and non-examples should also be a consideration for task selection.

It is evident that designing effective IMs can be complex, due to their central role in teaching and learning and the myriad of goals that are at times simultaneously desired, thus, teachers will necessarily employ several design principles (Kaur et al., 2022). By *design principles* (DP), I mean the guidelines for how teachers make decisions related to the selection, modification, creation, and sequencing of tasks that help to achieve their goals. Due to the recent emergence of research on teachers' design and scarcity of research on managing students' impulse thinking, there is little research that reports on teachers' DPs for managing impulse thinking and how they use these to support students' conceptual development. Hence, this study aimed to answer the following research question:

- What design principles does a secondary mathematics teacher use to design instructional materials to manage students' impulse thinking and raise their awareness of important features of concepts?

## Methods

To answer the research question, the methodological approach chosen must be able to develop in-depth descriptions of the teachers' processes to explain how their design decisions help to raise students' awareness of critical features. This requires collecting data from multiple sources as the teacher designs, redesigns, and implements their IMs. Hence, a *qualitative case study approach* (Yin, 2014) was chosen to accomplish these research design requirements.

The data presented in this paper is a part of a larger study on secondary mathematics teachers' design of IMs. Four teachers engaged in professional learning discussions (PLD) on teaching the topic of ratio with an emphasis on proportionality. The teacher chosen for this case study, Tanya (pseudonym), is an *exemplifying case* (Bryman, 2016) of a teacher who designed IMs with the goal of slowing students down to raise their awareness of important features of ratio and proportionality. Throughout the 14 weeks of the study, Tanya repeatedly mentioned in design interviews and PLDs that her students often hurried through tasks in her lessons and consequently made mistakes without realising. Furthermore, as Tanya was also familiar with designing IMs over her 15 years of experience teaching mathematics across all secondary year levels in Singapore, her design would likely reveal a multitude of deliberate DPs that could explain how she intended to manage students' impulse thinking and reveal the features she deemed important for students to notice. The topic of her IMs was ratio and proportionality for 12- to 13-year-old students in their first year of secondary school.

The data collected in this study included recordings of Tanya's participation in four PLD sessions (40-90 min each), five versions of her IMs, four one-on-one semi-structured interviews (20-40 min) after each design draft, recordings of her implementation of the IMs across three lessons (60 min) and three post-lesson reflection interviews (15-20 min). Ongoing data analysis was conducted throughout the recording of her design process that traced her design decisions throughout the revisions (e.g., her modifications of a task) and sought increasingly detailed explanations for her decisions in interviews. This included questions about overarching goals and goals of specific tasks, expectations of how the tasks will unfold, and anticipation of students' responses. On the other hand, lesson observations and post-lesson reflection interviews focused on how the implementation aligned with her goals and expectations. The ongoing data analysis revealed her recurrent goal of managing students' impulse thinking.

To determine the DPs for achieving this goal, content analysis was initially conducted for each of the tasks in her IMs to determine the key mathematical ideas present in the task. The definition of ratio that Tanya presented in an annexe of her IMs to her students was modified from the textbook and stated: "Ratio is a way of *comparing* 2 or more quantities of the *same* kind that either have *no* units or are in the *same* unit. The ratio $a$ : $b$, where $a$ and $b$ are positive numbers has *no units*. Note that $a$ and $b$ are *proportional*." After all recordings were manually transcribed, the transcripts were coded initially for when Tanya referred to her students' impulsive habits and her desires to slow them down, and the tasks within her IMs that would help her to address this. Subsequently, elaborations of how she wanted and anticipated her students would respond were coded for each task. This was used to identify the different features of ratio and proportionality that she deemed important and wanted students to notice, as well as features she expected they may not notice. These analysis steps combined were used to explain how her goals were manifested by her design decisions and the learning opportunities she wanted to afford her students through her design, hence, her design principles. The results of the analysis are presented in the following section.

## Results

While there may be several DPs that collectively help to achieve Tanya's goal of raising students' awareness of important features in the topic of ratio and proportionality, two recurrent DPs emerged throughout the analysis of four sets of tasks in her IMs.

### *Design Principle 1 (DP1): Create Opportunities for Making Mistakes*

In Example 1 (Figure 1), the aim was for students to determine equivalent ratios. As they had encountered the topic of ratios two years prior in primary school, the first two tasks acted as a review of how division and multiplication can be used to determine equivalence of ratios. However, in Question 3 the constant of proportionality to transform Ratio 1 into Ratio 2 using multiplication or division is not as immediately clear. Tanya had deliberately chosen the quantities of Ratio 2 to be +2 more than Ratio 1, to check if her students understood that transformations in ratio are based on *multiplicative reasoning*, *not additive reasoning* (Implicit Feature 1). As it turned out, during the lesson several students made the mistake of using additive reasoning to assert that the two ratios were equivalent. Although she was "stunned" (Reflection 1, #8) by her students' error, "I didn't expect that. I thought it was very clear cut that they would know that ratio is multiply or divide because it's not new to them" (Reflection 1, #13), it afforded her the chance to address this implicit feature during the lesson. She reminded students of the meaning of proportionality and asked them to write "Ratios cannot be added by the same number" in their IMs, an explicit statement of the implicit feature.

In a similar manner, Question 4 (Figure 1) was created by Tanya and introduced ratios with rational numbers, where the differing denominators would likely cause students to mistakenly multiply by different constants. Tanya's design allowed her to check if her students were aware of how to transform a ratio with differing denominators while maintaining the proportionality. While most of the students used the hint successfully, during the lesson there were still some students who made the vital mistake of assuming that multiplying each quantity in Ratio 1 by different constants was a valid way to transform the ratio. This consequently created the opportunity for Tanya to ask the class, "What's wrong here? … When you're simplifying ratio, can you multiply by different numbers? Cannot! Cannot by different numbers!" (Lesson 1, #91), drawing students' attention to the implicit rule that transformation of ratios requires multiplying or dividing by the *same constant, not different constants* (Implicit Feature 2).

In Example 3 (Figure 3), Questions 1 and 2 are typical problems that students would have encountered before in primary school and were not expected to pose any difficulty for students. Having led her students along for two tasks that affirm their use of multiplicative reasoning to determine an unknown value, the seemingly obvious solution to the subsequent question, Question 3, would be to assume the unknown value was the square root of the corresponding quantity, as squaring is also a multiplicative operation. Given that secondary students tend to have difficulty noticing the differences between linear and non-linear contexts (de Bock et al., 2002), Tanya's students were likely to make this mistake. Although most of them were quick to notice the critical difference between linear and quadratic operations, Question 3 would have caught students' attention and provided Tanya an opportunity to address that transformations in ratio are *linear, not quadratic* (Implicit Feature 3).

Likewise, Question 5 (Figure 3) was sequenced after a seemingly similar Question 4 and addressed another common mistake students make in ratio. In Question 4, the equivalent ratios were presented in fraction form and transforming the equation resulted in a solution that was finally presented as $y = 8$. However, in the subsequent Question 5, students' transformations of the ratios would lead them to the equation, $x = 6y$, which ultimately caused the class to stop and consider if the ratio $x\colon y$ should be $1\colon 6$ or $6\colon 1$. Almost the entire class (except two students) made the mistake of asserting the correct solution was $1\colon 6$ by taking the coefficients in the equation to be the corresponding quantities of the ratio. Following this lesson, Tanya reflected on this almost class-wide mistake, "I didn't expect that to happen … some of them catch it but there's still a group of them that still don't catch it" (Reflection 1, #39), meaning that some students eventually caught their own mistake by the end of the lesson. To explain their mistake, Tanya used the same feature students would have used in Question 4—a ratio $a\colon b$ in fraction form is represented as $\frac{a}{b}$, *not* $\frac{b}{a}$ (Implicit Feature 4). As they had confidently solved Question 4, if Tanya had not selected Question 5 from the textbook then it would have been likely that this significant misconception would not have otherwise been addressed.

**Example 1:** Without the use of calculator, determine if the following set of ratios are equivalent. Justify your conclusion using the table provided to help you.

| # | Ratio 1 | Ratio 2 | Justification (Show your workings) |
|---|---|---|---|
| 1 | 21 : 63 | 1:3 | Yes / No<br>21<br>63 |
| 2 | 3 : 7 | 24 : 56 | Yes / No<br>3<br>7 |
| 3 | 10 : 3 | 12 : 5 | Yes / No<br>10<br>3 |
| 4 | $2\frac{2}{5} : 1\frac{1}{4}$<br>Hint: Multiply by a common constant to convert both fractions into integers | 9 : 5 | Yes / No<br>$2\frac{2}{5}$<br>$1\frac{1}{4}$ |

Figure 1. Example 1—determine equivalent ratios.

**Example 2**: Without the use of calculator, express each ratio in the simplest form.

| # | Ratio | Justification (Show your workings) |
|---|---|---|
| 1 | 144 : 132 | 144<br>132 |
| 2 | $1\frac{1}{2} : 4\frac{1}{2}$ | |
| 3 | $0.48 : 1\frac{1}{5}$<br>Hint: Convert $1\frac{1}{5}$ to a decimal then simplify from there | |
| 4 | $850g$ *is to* $3.4kg$ | |
| 5 | 1.4 : 7 : 6.3 | |

If two quantities that are **proportional,** they can be expressed in ratio from such as, **a : b = ka :** _____

Figure 2. Example 2—simplify ratios.

Across Examples 1 (Figure 1) and 3 (Figure 3), Tanya created several opportunities for students to make mistakes and to encounter cognitive conflicts, thereby raising their awareness of implicit features of ratio and proportionality that would not have otherwise been addressed. While it is more conventional to teach what a concept *is*, Tanya's design suggests that she believed "part of understanding a concept is knowing what it is *not* and when it does *not* apply" (Lamon, 2012, p. 5) and "[p]art of what it means to understand proportionality is to recognise valid and invalid transformations" (Lamon, 2012, p. 7). In these two sets of tasks, and in subsequent sets, being aware of the implicit features was critical for answering the questions correctly. If, and when, students fell into the trap of making a mistake, Tanya's design ensured that these mistakes could be used productively to bring implicit features to students' attention.

### *Design Principle 2 (DP2): Sequence Tasks to Lead To and Catch Mistakes*

Beyond creating opportunities for mistakes to happen, analysis of Tanya's IMs also revealed how she strategically sequenced her tasks to *lead* and *catch* students in making mistakes. In Example 2 (Figure 2), Tanya included a variety of contexts for simplifying ratios, as well as the opportunity to make different types of mistakes. While the first three tasks were likely to be solvable by students, Questions 4 and 5 were intended to surprise students, causing them to slow down. Tanya wanted to remind her students that problems in ratio are not limited to numbers only, they can also contain units that need to be dealt with: "I want them to 'Eh?' Suddenly there's a need?' I wanted them to have a change of momentum" (Design 1, #31). Tanya later explained that her students "are interested in the answer … They'll probably [rush] with getting the right answer" (PLD 1, #191), hence the differing units would likely be overlooked, leading to students making the mistake of simplifying the ratio 850: 3.4 without recognising the need to convert the units. This was the only task in the worksheet with differing units, which suggests that it would be the sole opportunity in the lesson for students to make this mistake and for Tanya to remind students that ratio involves comparing two or more quantities of the same kind that have the *same units, not different units* (Implicit Feature 5).

In the final question in Example 2 (Figure 2), Tanya brought in a ratio with three quantities with decimals. This would indeed prompt students to reconsider how their existing procedures could be applied as they now had to maintain the proportionality between three quantities, two of which are decimals that required conversion. She noted that she wanted to use this task to "break the momentum" (Design 1, #34). If students were to be negligent of the decimals or unconscious of the need to maintain the proportionality between all quantities, Tanya's design of this task would remind students that transformation of ratios require multiplying or dividing by the same constant, *not different constants, even when there are more than two quantities* (Implicit Feature 6).

**Example 3:** Without the use of a calculator, answer the following questions

1. $3a : 7 = 18 : 21$, find a
2. $5 : 2b = 25 : 40$, find b
3. $3 : b = 3^2 : 5^2$, find b
4. $\frac{3y}{14} = \frac{12}{7}$, find y
5. $\frac{3x}{8} = \frac{9y}{4}$, find x : y

Figure 3. Example 3—find unknown value using ratio.

**Example 4:** Answer the following questions.

1. Find the simplest ratio of \$76 to \$84 to \$20
2. If $x : y = 5 : 6$ and $y : z = 4 : 9$, find $x : y : z$
3. If $x : y = 3 : 4$ and $y : z = 5 : 8$, find $x : y : z$
4. If $p : q = \frac{2}{3} : 2$ and $p : r = \frac{1}{3} : \frac{1}{2}$, find $p : q : r$.

(*Convert the fractions to integer by multiply by a same constant*)

Figure 4. Example 4—ratios with three quantities.

In the last set of tasks (Figure 4), Tanya's way of leading students to make mistakes for the purpose of catching them is evident when examining the structure of the last three questions. In general, two ratios are presented as $x\colon y$ and $y\colon z$ to be used to determine $x\colon y\colon z$, with the ratio quantity $y$ being common in both ratios, as can be seen in Questions 2 and 3. However, in the subsequent Question 4, Tanya deliberately made the common constant different, "I wanted to caught [sic] their moment, whether are they careful enough. Because when students do-do-do then they'll like 'ah!'. Because they'll just assume it's the same as the previous one, the sequence of the question. But I want to *catch* them and whether [they notice] this is another way of presenting the question" (Design 1, #77). Notably, because the sequence in Questions 2 and 3 are conventional, her design and sequencing of this question would likely lead and catch students making the mistake of focusing only on the numbers and not being conscious of the actual problem.

Design Principle 2 in Examples 2 and 4 can also be observed in Examples 1 and 3 and is necessarily two-fold, based on a deliberate sequencing of tasks. Firstly, Tanya created the opportunity for students to apply their existing procedures by beginning with tasks that could affirm their understanding of critical, yet explicit, features and solving procedures (e.g., using multiplication to determine unknown values), essentially allowing them to *think fast*. Once this procedure was anchored, subsequently within the middle or end of each set Tanya introduced tasks where these procedures may not be immediately applicable, where students may suddenly realise there was a need to *think slow*. Being aware of a specific critical, yet *implicit*, feature was necessary for solving these problems correctly. At these junctures, Tanya's sequencing was intended to cause students to either apply their previous understanding incorrectly, thereby making the mistake (e.g., squaring is multiplicative, so squaring and square-rooting are valid transformations) or pause to reflect on their understanding when confronted with cognitive conflicts ("Is there a difference between linear and quadratic?"). In both instances, DP2 will cause students to slow down, to "break the momentum" of their automatic and impulse thinking, and to increase her students' sensitivity to common mistakes and misconceptions they may have, while also demonstrating to them why they need to be more conscious of their thinking instead of fixating on solving the problem quickly.

## Discussion

In this paper, I presented a case of a teacher who designed IMs with the goal of raising her students' awareness of important implicit features. As they were often impulsive in their solving and prone to neglecting certain implicit features of concepts that Tanya deemed critical, she employed two DPs that collectively help to slow students down to achieve her goal. The two design principles she employed were: (1) *to create opportunities for students to make mistakes*, and (2) *to sequence tasks to lead to and catch mistakes*. To elaborate on Tanya's two principles and to discuss their implications, I introduce the notion of *catch tasks*.

From the analysis of Tanya's design of IMs, it is evident that certain tasks have the power to cause students to make mistakes. I term these *catch tasks*, which describe tasks that deliberately aim to engender incorrect solutions with the purpose of *catching* students in making mistakes, thereby raising their awareness to implicit features of concepts that may otherwise not be addressed. While this may appear to be a counter-intuitive goal, creating a form of cognitive conflict by embedding implicit features into tasks so that students would answer questions incorrectly has its advantages. Like non-examples (Goldenberg & Mason, 2008), catch tasks emphasise the boundaries of a concept and highlight common errors. However, instead of demonstrating the incorrect reasoning, catch tasks *lead* students to making those errors in their *own working*. They embrace mistake-making and acknowledge it as a natural element of learning—a belief that may be understood by teachers but is seldom enacted and prioritised (Warshauer, 2014), at least not as prominently as Tanya has made it in her IMs.

As for selecting, modifying, or creating a catch task, as demonstrated by Tanya a catch task is most useful when it is grounded in an implicit feature of a concept, often a complement of a feature that is hidden and tacit, and sequenced strategically within a set. They need not be particularly complex or rich collaborative tasks like those suggested by Swan and Burkhardt (2014); indeed, Tanya's catch tasks are seemingly typical tasks, like those reported in Choy's and Dindyal's (2021) study. Tanya's use of these tasks demonstrate that typical tasks can certainly go beyond being used for procedural practice, they can also be used to raise students' awareness of implicit features and cause students to *think slow* to reflect on their understanding of a concept. The key to achieving this was Tanya's knowledge of the content and students (Ball et al., 2008) that allowed her to notice the affordances of these tasks as individual items within a set. Her sequencing of tasks allowed students to *think fast* (Kahneman, 2012), such that students would likely have an anchoring and familiar procedure, then the catch task would likely engender an impulsive (and likely incorrect) response. From the students' mistakes in the lesson, this study demonstrated the potential of catch tasks for raising students' awareness of mathematical structure (Watson & Ohtani, 2015), which was achieved by creating variation and using sequencing to cause mistakes, rather than demonstrating correct solutions.

## Conclusion

Raising students' awareness of potentially implicit features of concepts is a valuable goal for teachers to have when designing lessons and IMs. In this paper, I presented a case of a teacher who achieved this by deliberately creating opportunities for students to make mistakes (DP1) and sequencing tasks to increase the likelihood for mistakes to happen (DP2). These two design principles productively engage students in a more conscious state of thinking but is contingent on the teacher's knowledge of the content and their students (KCS), and their ability to notice and harness the affordances of a task to be a catch task. While the notion of creating cognitive conflict has been historically recognised as a powerful way to help students to reflect on their understanding, the DPs underlying the notion of catch tasks afford opportunities for cognitive conflict to occur so that students who have the tendency to think fast may be forced to think slow to confront their errors. The deliberate and widespread use of catch tasks is not common amongst mathematics teachers in

Singapore, nor for teachers internationally. Despite appearing to be simply typical problems that may be commonly found in textbooks, catch tasks can be designed to develop conceptual ideas and to address students' errors. This presents a new possibility for teachers and educators when designing IMs. Hence, further research is needed to explore how catch tasks can be designed and implemented.

## Acknowledgements

The data presented in this paper was collected as part of larger project funded by the Office of Educational Research, National Institute of Education, Nanyang Technological University, Singapore (Grant number: OER 31/19 BK). This work was also supported by the Nanyang Technological University Research Scholarship.

## References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). Routledge.

Bryman, A. (2016). *Social research methods* (5th ed.). Oxford University Press.

Choy, B. H., & Dindyal, J. (2021). Productive teacher noticing and affordances of typical problems. *ZDM—Mathematics Education*, *53*, 195–213. https://doi.org/10.1007/s11858-020-01203-4

de Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics*, *50*, 311–334.

Goldenberg, P., & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics*, *69*(2), 183–194. https://doi.org/10.1007/s10649-008-9 143-3

Kahneman, D. (2012). *Thinking, fast and slow*. Farrar, Straus and Giroux.

Kaur, B., Leong, Y. H., & Attard, C. (2022). Teachers as designers of instructional tasks. *Mathematics Education Research Journal*, *34*(3), 483–489. https://doi.org/10.1007/s13394-022-00437-7

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (3rd ed.). Routledge Taylor & Francis Group.

Lim, K. H., & Wagler, A. (2012). Impulsive-analytic disposition in mathematical problem solving: A survey and a mathematics test. *Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia*, Singapore: MERGA.

Mason, J., & Watson, A. (2008). Mathematics as a constructive activity: Exploiting dimensions of possible variation. In M. P. Carlson, & C. Rasmussen (Eds.), *Making the connection: Research to practice in undergraduate mathematics education*. Mathematical Association of America.

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM—Mathematics Education*, *46*(5), 705–718. https://doi.org/10.1007/s11858-014-0600-4

Schoenfeld, A. H. (2010). *How we think: A theory of goal-orientated decision making and its educational applications*. Routledge.

Swan, M., & Burkhardt, H. (2014). Lesson design for formative assessment. *Educational Designer*, *2*(7), 1–41.

Warshauer, H. K. (2014). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education*, *18*(4), 375–400. https://doi.org/10.1007/s10857-014-9286-3

Watson, A., & Ohtani, M. (Eds.) (2015). *Task design in mathematics education: An ICMI study 22*. Springer. https://doi.org/10.1007/978-3-319-09629-2.

Yin, R. K. (2014). *Case study research: Design and methods* (5th ed.). SAGE Publications Inc.

# UDL: An Alternative to Ability Grouping in Mathematics?


Lisa Darragh
*University of Auckland*
l.darragh@auckland.ac.nz

Fiona Ell
*University of Auckland*
f.ell@auckland.ac.nz

Jude Macarthur
*University of Auckland*
jude.macarthur@auckland.ac.nz

Missy Morton
*University of Auckland*
missy.morton@auckland.ac.nz



Despite years of research into the limitations and negative consequences of ability grouping, the practice remains common in primary schools of Aotearoa New Zealand. In this paper we consider a potential alternative to ability grouping following our exploratory study into using Universal Design for Learning (UDL) as an approach for inclusive pedagogy in mathematics. Our case study of a Year 3-4 teacher over the course of one school year illuminated how planning with UDL inspired her to eliminate ability grouping from her mathematics pedagogy (with some relief) and yet still provided access to rich learning opportunities for all the children. However we also noted a tension between the pedagogies of productive struggle/challenge and scaffolding to support anxiety, and we invite discussion on this tension.


Fixed to the whiteboard was the "Maths tumble" and there I was, still in Yellow Two. Yellow was such a stink group to be in—all the smart kids were in Blue, and Green group always got the easy stuff to do. I hated Yellow. And today we were just doing worksheets. Soooo boring! I wanted to do the iPads, or see the teacher, or something from the maths centre, anything but worksheets….

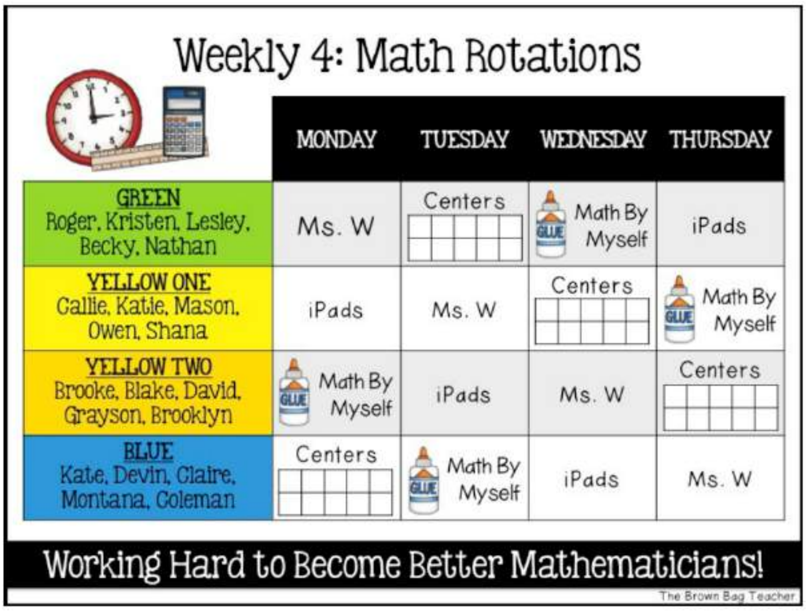



Figure 1. "Maths Tumble" image. Retrieved from https://brownbagteacher.com/reading-and-math-rotation-boards/

The opening vignette is fictional, but it captures the mathematics learning reality for many primary school children in Aotearoa New Zealand, where so-called "ability" grouping (and "maths tumbles" such as depicted in Figure 1) remains a common way to organise learning in mathematics (Anthony & Hunter, 2017; Darragh & Franke, 2021). We contend that the practice of ability grouping forms one of the biggest challenges to inclusive teaching in mathematics. By its design, ability grouping excludes children from mathematics learning opportunities, through limited access to rich content for some groups of learners, and through limited access to the teacher for all groups, who might be with other children 75% of the time.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 171–178). Newcastle: MERGA.

Despite decades of research into the negative impact of ability grouping (Wiliam et al., 2004; Zevenbergen, 2005), the practice appears hard to shake. It seems logical to assume that teachers will not break the ability grouping habit unless there is something pedagogically attractive and tangible to replace it. In this paper, we argue that Universal Design for Learning (UDL: Rose et al., 2014) may provide an alternative to ability grouping as it allows a way to plan and teach with everyone in mind, catering to the diversity of learners in our classrooms with the explicit aim of creating access to rich learning for all. In this paper we share the results of a small exploratory study that aimed to understand the challenges of teaching mathematics for inclusion and that trialled UDL as an approach to meet those challenges.

## Background: Ability Grouping—A Practice of Exclusion

Research into ability grouping, including streaming, setting, and within-class grouping by so-called 'ability', has a long history globally in mathematics education. Findings are generally consistent that the negatives outweigh any benefits—and for students placed in the lower groups the practice is even more detrimental as it tends to lower their achievement (Wiliam et al., 2004), widens the gap between students variously (and contestably) described as 'high and low achieving' (Hornby & Witte, 2014), narrows teaching (Barclay, 2021; Gervasoni et al., 2021; Wiliam et al., 2004), and promotes negative learner identities (Barclay, 2021; Gervasoni et al., 2021; Marks, 2014; Solomon, 2007; Zevenbergen, 2005).

In Aotearoa New Zealand primary schools, ability grouping is common-place (Darragh & Franke, 2021), particularly since the introduction of the Numeracy Development Project (Anthony & Hunter, 2017; Fitzgerald et al., 2021). In the secondary school context ability grouping practices include setting, streaming, and banding (Hornby & Witte, 2014). Research in Aotearoa New Zealand describes ability grouping as generating equity concerns, with Māori and Pacific students disproportionately placed in lower groups (Anthony & Hunter, 2017; Fitzgerald et al., 2021; Turner et al., 2015) as grouping by so-called ability is subject to teacher bias and low expectations for some students.

Given these negative aspects of ability grouping, why might the practice persist? Possibly it is due to a desire to cater for diverse learning needs, and an assumption on the part of teachers that ability grouping is the way to achieve this differentiation (Barclay, 2021; Fitzgerald et al., 2021). Anthony et al.'s (2019) position paper discusses the "slippery" notion of differentiation arguing that it might either be in the interests of marginalised children, or it might end up reinforcing that marginalisation (see also Webel et al., 2021). A recent special issue of *Mathematics Teacher Education and Development* has explored the topic of differentiation in mathematics (Russo et al., 2021), providing a collection of articles that trouble ability grouping and provide alternatives to this practice. Many of these suggest the use of open tasks with low floor and high ceiling and that promote productive struggle (Barclay, 2021; Ingram et al., 2020; Mellroth et al., 2021; Russo et al., 2021). Other practices include number talks and conversations about strategies (Webel et al., 2021), inquiry communities in heterogenous groups (Fitzgerald et al., 2021) and multi-level approaches involving both changing whole-class instruction and withdrawal interventions (Gervasoni et al., 2021).

Despite the body of research, it remains challenging for teachers to make these kinds of instructional changes (Fitzgerald et al., 2021; Mellroth et al., 2021). Webel et al., (2021) call for more primary school level research into "inclusively responsive instruction … where all students can be supported without grouping by ability" (p. 114). With this in mind, the research question we pose for this paper is:

- How might UDL provide an alternative to ability grouping in mathematics?

## Conceptual Frame: Inclusive Pedagogy and Universal Design for Learning

We draw from the concept of 'inclusive pedagogy', which refers to both a *discourse* as well as the *act of teaching* and entails a move away from "deterministic beliefs associated with bell-curve beliefs about ability" (Florian & Black-Hawkins, 2011, p.813). It involves a shift in thinking from an approach to teaching and learning that works for *most students* with additions for *some,* to planning for *all* from the outset (Florian & Black-Hawkins, 2011). It encourages teachers to critically reflect on their teaching decisions to consider whether they might create barriers to participation, or, on the other hand, generate a sense of belonging for all children and celebrate their diversity (MacArthur & Rutherford, 2016). Inclusive pedagogy means that teachers aim to support every child's relationships with others, sense of belonging, and positive identities as a learner (Florian & Black-Hawkins, 2011; MacArthur & Rutherford, 2016).

UDL (Rose, et al., 2014) provides a planning approach for teachers that is consistent with the idea of inclusive pedagogy and it is an approach endorsed in Aotearoa New Zealand (see for example *Te Kete Ipurangi*: Ministry of Education, n.d.). The UDL model sits within the broader field of inclusive education and includes three principles: multiple means of engagement, multiple means of representation, and multiple means of action and expression (Rose et al., 2014). However, research into UDL in mathematics education is still limited. Rachel Lambert has developed "UDL Math" and engaged in various studies to explore the approach (Lambert, 2020; Lambert et al., 2021). For example, during a summer course for mathematics educators the researchers promoted the use of "empathy interviews" together with UDL in order to understand the potential barriers and required supports for learning (Lambert et al., 2021). Paulo Tan (2017) has also promoted UDL as a planning framework in mathematics to ensure that the mathematics is both "for all" and "of all". One final example is Stephan and Dieker's (2022) inquiry into the use of UDL in co-teaching between the mathematics and special education teachers. Such studies highlight the potential for UDL in the context of mathematics.

## Methods

### *Context and Participants*

We employed case study methodology (Merriam, 2002) to look closely at one classroom teachers' practice. The teacher, who we name Anna Sunshine, had more than 20 years of teaching experience and had done postgraduate study in 'special' education. The school was situated in a mid-socioeconomic area of Auckland and had a reputation for being inclusive; the principal had completed a postgraduate diploma in specialist teaching with a complex educational needs endorsement. Anna's class in 2022 had 24 Year 3-4 children (aged 7-8 years) and was very diverse; there were 12 different ethnicities represented, 10 children were funded for ESOL support, and 16 children had learning or behaviour support needs identified. Further, the class had been considerably impacted by COVID-19, having only begun schooling shortly before the first in a long series of lockdowns and enforced distance learning. Whilst the class was typically diverse for the context of Auckland, the number of children receiving learning support were greater than usual due to the teachers' experience and expertise.

Our research team included academics from mathematics education, Lisa and Fiona, as well as academics from the field of disability studies and inclusion, Jude and Missy. Together with the school participants, we formed a triangle of expertise with particular areas of knowledge being mathematics, inclusion, and the students in the class.

### *Data Collection and Analysis*

Data collected included interviews, audio-recorded planning meetings, video-recorded lesson observations and reflections. Interviews were held at the start and end of the project, which ran over the full school year. We interviewed: Anna, her colleague and planning partner, the school principal, and the teacher aide who worked in Anna's classroom. Interviews were audio-recorded and ranged from 35 to 70 minutes.

The planning meetings followed a lesson study approach in that we identified a goal, planned the lesson, observed the lesson, and reflected on the goal (Murata, 2011). However, in contrast to typical lesson study, the goal for each lesson was always *inclusion* (defined by presence, participation, achievement, and belonging) for all students, rather than being a goal related to a particular mathematics learning outcome. The lessons were planned with alignment to UDL, that is, we identified possible barriers to participation and achievement and made sure that we planned adaptations to dismantle those barriers. These adaptations were made available to *all* the children in the class during the lesson. Following each lesson we held a reflection meeting (which was typically immediately prior to the subsequent iteration's planning meeting). Planning and reflection meetings ranged from 40 minutes to 1.5 hours and were always attended by Anna and two members of the research team, and variously attended by her colleague/planning partner, the teacher aide, the principal, the other members of the research team depending on their availability. We held four iterations of this cycle altogether.

The lesson observations were made via IRIS Connect technology. IRIS Connect is a platform that enables synchronised video-recording using two I-Pads and an audio device. The synchronised recordings may be shared to a group of users who are then able to make comments that are connected to a time-stamp in the recording. In this way, Anna had autonomy over which lessons to share with the research team, and the entire team were able to view her videos together with her comments in-time, and make their own comments in response. We recorded a total of 333 minutes of mathematics lessons over the course of the year which included seven lessons derived from the four lesson study plans.

Our data analysis was an iterative process. From the initial interviews we engaged in inductive coding to uncover challenges to inclusion, and we used the lesson observations to elaborate these. Prior to the study, Anna's approach to catering for the diversity in her class was to group by ability and use a 'maths rotation' similar to Figure 1, although her groups tended to be somewhat flexible with membership changing regularly. Anna's view that this kind of grouping generated a challenge for inclusive teaching emerged in our initial interviews and thus she was happy to use different strategies for the lesson study iterations. For the purpose of this paper we examine just one of the lesson study iterations and illustrate with some interview data. We share the final iteration of the project: a series of lessons on the topic of *finding fractions of a set*, held on the 17, 18, and 19 October. The lesson was inspired by the problem "Andy's Marbles" from https://nrich.maths.org/2421. In the findings section we briefly describe the lesson and discuss how UDL enabled a differentiated learning experience for the children in the class that was inclusive due to not using ability grouping.

## Findings

> I think also, this sounds crazy, but having permission to drop my ability grouping and to kind of go okay these people agree with me this isn't working I'm chucking it out and gosh they are from the university they must know. [...] I guess because I have got so much evidence now it is working, it makes it feel like okay for me to keep going in this way. (Anna, final interview).

By October, Anna had engaged in three iterations of lesson study cycles and was taking increasing ownership over the UDL planning process. Initially she chose to teach 'safer' lessons in the area of geometry, by this lesson she felt comfortable to tackle a riskier topic of fractions. As can be seen in the quote above, she was already happy to do so without any grouping by ability.

"Andy's Marbles" is a challenging problem for students even up to age 11 and thus posed a considerable challenge for the children in Anna's class. The research team certainly thought it was an ambitious task. Anna developed three lessons to build up students' skills and knowledge so that they would be able to access the problem. The first lesson introduced the children to the equipment of counters and fraction rectangles for solving exercises such as $^1/_5$ of 20, $^1/_3$ of 21 (see Figure 2). Anna explicitly modelled for the children "gathering resources" in which she asked herself which resources might be useful to solve the problem, such as selecting the fraction rectangle cut into fifths and counting out 20 of the counters, in the case of the first example. The children were sent away to work in small groups with the equipment and individual whiteboards to solve each problem, then returned to the mat to discuss their solutions after each. Finally, they created their own questions to solve.

Figure 2. Counters and fraction rectangles to solve questions about fractions of a set.

The second lesson introduced the "maths drain"—a piece of equipment Anna made in order to make the notion of 'drain' more accessible (this was an ice-cream container with drain grills cut into the lid—the lid was removable so children would be able to check the number of marbles that went down it). In this lesson Anna posed questions such as: "I had 30 marbles and $^2/_5$ went down the drain—how many marbles did I lose?" The reverse was also asked (e.g. $^2/_5$ went down the drain and I had 18 left, what did I start with?) The final lesson included multi-step questions similar to the one in "Andy's marbles" yet slightly simplified (e.g. "Pikachu had a bag of marbles. ⅕ of them rolled down the drain. Half of what was left dropped into the mud. Pikachu had only 10 marbles left. How many marbles did he have to begin with?"), and then Anna offered the full "Andy's marbles" problem as a final challenge.

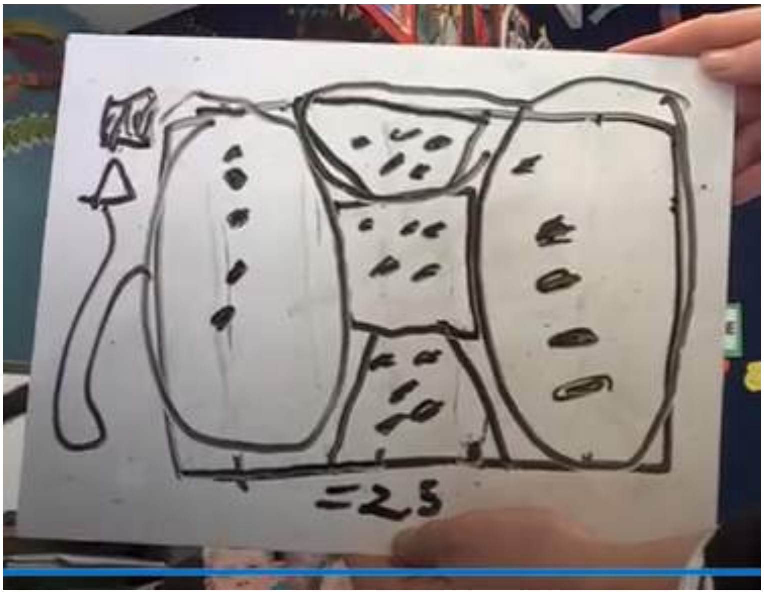



Figure 3. Solution to the Pikachu's marbles question.

UDL requires adaptations to be made to address any barriers to learning and these are then offered to all the students. During the lesson design process we discussed barriers that the children might face. These included the word "fractions" itself—a few of the children had been told by older siblings that 'fractions' was a topic they wouldn't like (Anna had previously solved this issue by teaching the topic of "pieces" instead). We talked about how drawing fractions can be difficult and decided to make some pre-drawn fractions rectangles available for the children to choose from so that they wouldn't have to draw their own. The context itself could have constituted a barrier—the children had difficulty visualising abstract contexts, before even having to pull mathematics from that context (this is the reason Anna built a "maths drain" from an ice-cream container—so that children could see something physical that marbles would drop into). The "maths drain" also enabled students to predict and then check inside as well. Oral language was a potential barrier, partly solved by physical representations of the mathematics and the context. Prior to the mathematics lesson Anna took one child outside to show him a drain and explain the word. Finally, for some of the children anxiety was a barrier that we identified early in the project. Anna knew that all examples would have to begin with easy fractions and small numbers otherwise some children would be "immediately overwhelmed". Yet a high ceiling was built into each lesson, first by allowing students to write their own questions (some wrote ½ of 104 or 1/3 of 99), and then with the increasing complexity of the challenge questions.

Despite the children approaching the series of lessons from very diverse levels of confidence and prior knowledge the lesson activities were successful in several ways. Children were visibly engaged, with high levels of interest and excitement in the tasks each day. Secondly, all children were able to access the problem-solving—we saw full inclusion in that all children were present, participating, achieving, and belonging regardless of their previous level of attainment in mathematics. Perhaps most crucially, by not ability grouping some children had greater access to more sophisticated problems than they might have had if they were taught in an ability group.

> I think I was putting a ceiling on them by ability grouping them because I had predetermined set outcomes for what I wanted in each lesson […] Whereas like yesterday for example looking at [the maths problem] and [one student] completely blew me away […] this sounds awful but yeah, I didn't think he would be capable of that [level], yet he has done it himself. (Anna, final interview)

In short, Anna was convinced that UDL was an alternative to ability grouping. This was further reinforced for her by the high levels of achievement in the end-of-year mathematics testing results Anna reported to us at the final interview.

## Discussion and Conclusions

It was clear to the entire research team, as well as the case study participants, that UDL enabled a lesson planning approach that catered to all the children in the class, giving every child access to rich mathematics learning opportunities. Our findings reflect the recent research looking at alternative ways to differentiate; for example the use of a rich task (Ingram et al., 2020; Mellroth et al., 2021; Russo et al., 2021), no ability grouping (Fitzgerald et al., 2021) and plenty of opportunity for discussion (Webel et al., 2021).

However, we have a few caveats. Firstly, UDL requires a teacher with in-depth knowledge of the children in her class. Anna's experience and connection to the learners in her class meant that she could easily anticipate barriers to a high level of specificity. She quickly identified the word "drain" as being likely to cause an interruption to learning—less experienced teachers may not have anticipated this. Using the IRIS technology enabled Anna to get to know the learners even further as it captured their talk when she was away from them. Here is where the anxiety felt by some of the learners emerged. We support Lambert et al.'s (2021) suggestion of using empathy interviews to help teachers get to know the learners in their classes and suggest that more research could explore this important aspect of planning with UDL. Equally, collaboration that supports the sharing of knowledge amongst teachers, children, families, and teacher aides provides further rich information to inform the question of "what works" for each child (Florian, 2017).

Because mathematics anxiety was found to be a key barrier to some children's engagement in the learning, Anna developed her lessons to reduce this anxiety with a carefully scaffolded approach that built up to the challenging task. This raised for us the question of how to balance the notion of struggle (Ingram et al., 2020) with the reality of students suffering anxiety during mathematics lessons. This concern echoes those of the teachers in Mellroth and colleagues (2021) study as they tried to resolve the tension between allowing for struggle without "funnelling" students answers (that is, showing them how to do it). Whilst we observed plenty of struggle in each lesson in the marbles series—every time the children were sent to solve a problem it required them to struggle with the concepts—we also saw funnelling in the very structure of the scaffolded lesson sequence. We suggest there is a tension here and it may well entail a considerable challenge for teachers using UDL to strike the 'right' balance between productive struggle without too much funnelling of the solution approach.

## References


Anthony, G., Hunter, J., & Hunter, R. (2019). Working towards equity in mathematics education: Is differentiation the answer? In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Educaiton Research Group of Australasia* (pp. 117–124). MERGA.

Anthony, G., & Hunter, R. (2017). Grouping practices in New Zealand mathematics classrooms: Where are we at and where should we be? *New Zealand Journal of Educational Studies*, *52*, 73–92.

Barclay, N. (2021). Valid and valuable: Lower attaining pupils' contributions to mixed attainment mathematics in primary schools. *Research in Mathematics Education*, *23*(2), 208–225.

Darragh, L., & Franke, N. (2021). Online mathematics programs and the figured world of primary school mathematics in the digital era. *Mathematics Education Research Journal*, (online). https://doi.org/https://doi.org/10.1007/s13394-021-00384-9

Fitzgerald, L., Hunter, J., & Hunter, R. (2021). Shifting teacher practices in relation to grouping: Gap gazing or strengths focused approaches. *Mathematics Teacher Education & Development*, *23*(3), 97–110.

Florian, L. (2017). The heart of inclusion is collaboration. *Pedagogika, 126,* (2), 248–253.

Florian, L., and K. Black-Hawkins. (2011). Exploring inclusive pedagogy. *British Educational Research Journal 37* (5), 813–828.

Gervasoni, A., Roche, A., & Downton, A. (2021). Differentiating instruction for students who fail to thrive in mathematics: The impact of a constructivist-based intervention approach. *Mathematics Teacher Education & Development*, *23*(3), 207–233.

Hornby, G., & Witte, C. (2014). Ability grouping in New Zealand high schools: Are practices evidence-based? *Preventing School Failure: Alternative Education for Children and Youth*, *58*(2), 90–95. https://doi.org/10.1080/1045988X.2013.782531

Ingram, N., Holmes, M., Linsell, C., Livy, S., McCormick, M., & Sullivan, P. (2020). Exploring an innovative approach to teaching mathematics through the use of challenging tasks: A New Zealand perspective. *Mathematics Education Research Journal*, *32*, 497–522. https://doi.org/10.1007/s13394-019-00266-1

Lambert, R. (2020). Increasing access to universally designed mathematics classrooms. *PACE Policy Analysis for California Education*, *February*, 1–11.

Lambert, R., Imm, K., Schuck, R., Choi, S., & McNiff, A. (2021). "UDL is the what, design thinking is the how:" Designing for differentiation in mathematics. *Mathematics Teacher Education and Development*, *23*(3), 54–77.

Marks, R. (2014). Educational triage and ability-grouping in primary mathematics: A case-study of the impacts on low-attaining pupils. *Research in Mathematics Education*, *16*(1), 38–53.

Mellroth, E., Bergwall, A., & Nilsson, P. (2021). Task design for differentiated instruction in mixed-ability mathematics classrooms: Manifestations of contradictions in a professional learning community. *Mathematics Teacher Education & Development*, *23*(3), 78–96.

Murata, A. (2011). Introduction: Conceptual overview of lesson study. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 1–12). Springer.

Rose, D. H., Gravel, J. W., & Gordon, D. T. (2014). Universal design for learning. In L. Florian (Ed.), *The Sage handbook of special education* (pp. 475–489). Sage Publications.

Russo, J., Bobis, J., & Sullivan, P. (2021). Differentiating instruction in mathematics. *Mathematics Teacher Education & Development*, *23*(3), 1–5.

Solomon, Y. (2007). Experiencing mathematics classes: Ability grouping, gender and the selective development of participative identities. *International Journal of Educational Research*, *46*, 8–19. https://doi.org/10.1016/j.ijer.2007.07.002

Stephan, M., & Dieker, L. (2022). Equitable co-teaching practices in mathematics. In Y. P. Xin, R. Tzur, & H. Thouless (Eds.), *Enabling mathematics learning of struggling students* (pp. 199–219).

Tan, P. (2017). Building inclusive mathematics classrooms for students with disabilities. *For the Learning of Mathematics*, *37*(3), 21–24.

Turner, H., Rubie-Davies, C. M., & Webber, M. (2015). Teacher expectations, ethnicity and the achievement gap. *New Zealand Journal of Educational Studies*. *50*, 55–69. https://doi.org/DOI 10.1007/s40841-015-0004-1

Webel, C., Dames, B., Woldruff, J., Lindman, L., Daugherty, K., & Brown, J. (2021). Field-based perspectives on enacting alternatives to ability grouping in elementary mathematics instruction. *Mathematics Teacher Education & Development*, *23*(3), 111–131.

Wiliam, D., Bartholomew, H., & Reay, D. (2004). Assessment, learning and identity. In P. Valero, & R. Zevenbergen (Eds.), *Researching the socio-political dimensions of mathematics education*. (pp. 43–61). Kluwer Academic Publishers.

Zevenbergen, R. (2005). The construction of a mathematical habitus: Implications of ability grouping in the middle years. *Journal of Curriculum Studies*, *37*(5), 607–619. https://doi.org/10.1080/00220270500038495

# Reflecting Together: Classroom Video as a Tool for Teacher Learning in Mathematics

Raewyn Eden
*Massey University*
r.eden@massey.ac.nz

With the aim of strengthening mathematics teaching | learning, classroom video is increasingly used as a tool for teachers to reflect on their practice. In the first phase of a two-phase design-based study, a group of primary teachers selected, viewed and discussed video excerpts from mathematics lessons. Engaging with and working to reconcile differences that emerged in the teachers' ideas about practice catalysed productive conversations for their learning. What was noticed and picked up for discussion mediated what conceptual resources were made available for teachers' reflections. Sharing classroom video provided teachers with both a window into the otherwise private practice of others, and a mirror in which to reflect on their own practice.

Groups of teachers using classroom video as a tool for professional inquiry to support reflection and strengthen mathematics teaching | learning is increasingly prevalent. In a two-phase, design-based study which explored how participation in collaborative inquiry generates teachers' knowledge for mathematics teaching | learning (see Eden, 2019), the author and a group of teachers used excerpts of classroom video in the co-design and enactment of an approach to collaborative teacher inquiry. The study is founded on a view that collaboration and inquiry are important for teachers' learning, and that teachers' collaborative inquiry activity is a productive site for developing adaptive expertise, needed to address inequities in mathematics classrooms. An underlying assumption is that teachers' noticing is important for reflection and that reflection is essential for understanding and enhancing classroom practice. The use of video as a tool for supporting teachers' noticing of classroom events was a particular characteristic of the teachers' shared activity in the first phase of the design process.

## Collaborative Teacher Inquiry

This study is premised on the idea that inquiry is a stance teachers take, and that collaborative activity can afford teachers opportunities to learn in the context of their work.

### *Collaborative Inquiry*

Collaborative inquiry situates teacher collaboration within the notion of an inquiry community (see Jaworski, 2003). Inquiry communities are characterised by systematic teacher inquiry whereby teachers "articulate questions, problems and dilemmas" and knowledge is developed through the structured analysis of data "and/or conversations which generate new, revised, or more explicit understandings" (Levine, 2010, p. 112). Central to the activity of inquiry communities are reciprocal processes of co-learning (Jaworski, 2008). Collaborative inquiry for teachers can be powerful in fostering the "kinds of thinking that lead to development" (p. 7). Collaborative teacher inquiry has been seen to promote expanded teacher knowledge, for example through lesson study (Hunter & Back, 2011), and improved student learning in mathematics (Ingvarson et al., 2004). It is widely accepted that teacher community represents productive conditions for improving teaching and raising student achievement (Eaker & Keating, 2012) wherein teachers engage interactively with colleagues in professional learning within the context of their work (Desimone, 2009). As argued by Fullan, Rincón-Gallardo and Hargreaves (2015), "collaboration focused on the improvement of teaching and learning is one of the highest-yielding strategies to boost student, school and system performance" (p. 8).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 179–186). Newcastle: MERGA.

### *Noticing*

The importance of teacher noticing as an aspect of teachers' expertise is commonly acknowledged (Kaiser & König, 2019; Scheiner, 2016), generating ongoing interest in the development of mathematics teachers' noticing (Santagata et al., 2021). Noticing has been described as a "hidden practice of in-the-moment decision-making that is needed to respond to children's … strategy explanations" (Jacobs et al., 2010, p. 197). Teaching mathematics is contingent; in any lesson there are many things happening simultaneously. Noticing is non-routine and complex, requiring a repertoire of interconnected skills including attending to the mathematical details in students' strategies; interpreting the underlying conceptual understandings; and determining how to respond (Jacobs et al.). Furthermore, as Deborah Ball (2011) suggests, a "paradox of expertise is that ideas that seem obvious are not so to the learner" (p xxi), pointing to both the importance and complexity of teachers' noticing. In other words, the more extensive a teachers' knowledge of a subject, the more challenging it can be to recognise and respond to students' understandings, and misunderstandings.

### *Reflection*

Reflection is fundamental to improving teaching as teachers interpret and derive meaning from classroom events to inform and shape future teaching. Teacher noticing is a key aspect of reflection. Accordingly, Jacobs, Philipp and Sherin (2018) argue that "noticing is a critical component of mathematics teaching expertise and thus a better understanding of noticing could become a tool for improving mathematics teaching and learning" (p. xxvi). Linking reflection with noticing, Jaworski (2003) argues that an increased awareness of classroom events supports critical intelligence, a kind of 'metaknowing' for teachers. Thus, she suggests, engaging in critical reflection beyond the classroom enhances teachers' explicit awareness of conditions and events within the classroom which can then expand possibilities for action. With a focus on challenging the tacit assumptions that underlie teachers' practice, Larrivee (2000) suggests that critical reflection entails:

> a deep exploration process that exposes unexamined beliefs, assumptions, and expectations … Reflective practitioners challenge assumptions and question existing practices, thereby continuously accessing new lenses to view their practice and alter their perspectives (p. 296).

Critical reflection in collaboration with others can support future noticing thereby affording new possibilities for teachers' decisions and actions (Mason, 2009). In line with a view of teaching as complex, this suggests that teachers may be better able to notice important aspects of both the practice and the impacts of their teaching (Jaworski, 2003). Furthermore, collaborative critical reflection on previous practice can promote enhanced noticing within future practice thus influencing classroom actions and potentially improving teaching (Jaworski, 2008).

## The Role of Video in Teacher Inquiry

Mathematics teaching | learning is complex and largely private, and there is increasing interest in video as a window through which to view classroom activity. Classroom video offers a tool that teachers, and students, can access and use with readily available equipment and little preparation. Unsurprisingly, perhaps, video is commonly regarded as a valuable source of evidence to support teachers' reflection on and responses to classroom events, and is increasingly used in teachers' professional development (Gaudin & Chaliès, 2015). Nevertheless, the use of video for the professional learning of teachers does not always align with research evidence about what and how teachers learn from video (Sherin & Dyer, 2017).

Classroom video is commonly regarded as a useful tool for teacher inquiry and, within inquiry communities, teachers reflecting together on classroom video has been seen to promote teacher learning (Borko et al., 2008). Video has been used to support the development of teachers' pedagogical content knowledge, provide a model of effective classroom practice, promote reflection and noticing, and examine equity-oriented approaches to teaching (Santagata, 2014). Accordingly, videos used for teacher reflection may range from commercially-produced videos depicting "exemplary" classroom practice (e.g., Santagata, 2014) to videos from teachers' own classrooms (e.g. van Es & Sherin, 2008). Research has tended to focus on excerpts of classroom video recorded and selected by researchers and/or by the teachers themselves. In many cases, teachers' reflections are guided by explicit tools and discussion protocols (Santagata et al., 2021).

A range of outcomes have been associated with using video for teacher learning, including the development of knowledge for teaching mathematics, shifts in teaching practice, and improved teacher motivation (Gaudin & Chaliès, 2015). The nature of the videos, how and why they are selected, and how teachers interact with them all influence their potential for supporting teacher learning (e.g., Blomberg et al., 2014). What aspects of classroom life can be perceived in video is influenced by decisions that are made as the video is captured. Ongoing, rapid technical advancement has expanded teachers' and researchers' possibilities for capturing classroom video whereby decisions about what to record and how to record it can be made "in the moment" as classroom events emerge (Santagata et al., 2021).

Learning from classroom video is generally assumed to occur as teachers are watching and discussing excerpts, however Sherin and Dyer (2017) found that there are important considerations and associated teacher actions for recording and selecting productive clips at three time periods: before, during and after the lesson. Accordingly, they highlight the importance of selecting excerpts that provide access to student thinking and, particularly, the depth of the thinking that is captured.

## The Study

This paper is concerned with the initial phase of a two-phase design-based study that explored the question: How does participation in collaborative inquiry generate teachers' knowledge for mathematics teaching | learning? The aim of the broader study was to co-design and enact an approach to teachers' collaborative inquiry, and explore opportunities for, and constraints to, the teachers' professional learning within their inquiry activity. A participatory and collaborative approach was taken in line with the research focus and Hintz and colleagues' (2013) observation that teachers can and should "build more detailed visions of ambitious teaching through [their] work together" (p. 10).

A group of four teachers and the author met three-weekly over a six-month period to co-design, enact and refine an approach to collaborative inquiry, with the teachers' inquiry activity focused on developing the practice of using talk moves to facilitate mathematically productive student discussions. At the study's end, the teachers' collaborative inquiry activity centred on co-teaching mathematics lessons involving co-planning, co-instruction, and co-reflection. This paper focuses on the initial phase wherein the inquiry activity concentrated on viewing and reflecting together on classroom video excerpts from the teachers' mathematics lessons.

Audio-recordings of the group's meetings and semi-structured interviews with each teacher at the start and the end of the project were the primary data sources. Research notes and informal classroom observations provided additional context to understand the events discussed in the meetings and interviews. The exploratory nature of the research meant that the design of the teachers' collaborative inquiry activity was continuously renegotiated. Cultural-historical activity theory (CHAT) provided a theoretical lens through which to analyse the complexity of the inquiry activity. Having identified both theoretically-derived and data-derived codes from the transcribed meetings and interviews, CHAT was used as a conceptual tool to analyse emerging themes in terms of the different elements of the teachers' collaborative inquiry activity, and to identify the nature of any contradictions and the impact of actions taken to resolve them (see Eden, 2019 for additional details of the process).

# Findings

During their first two inquiry meetings, the group had negotiated an overarching inquiry focus on using talk moves and discussed related research literature. The teachers then agreed to select excerpts of video from their classrooms to share at subsequent meetings:

> select something that you think the group could learn from, something that has that focus on student talk and in this case we're talking about trying out talk moves [Raewyn].

The following three examples are drawn from the group's discussions of the video excerpts, and focus on how the teachers made sense of the classroom events captured therein.

### *Example 1: Doing the Wrong Thing*

The first of the teachers to share a video excerpt with the group was Sam. In her lesson, students had been solving a problem that involved finding the number of dots on four dominoes missing from a set. Sam had deliberately chosen the excerpt as an example of a missed an opportunity to extend the thinking of her students:

> it's just me completely doing the wrong thing [laughs] which I thought was a really good one because as soon as I thought about it afterwards it was like "why did you do that?" and I thought it would be a good example of the opportunity that I could've used those [talk moves] but I didn't

The students had been working in small groups to solve the problem. Responding to a group that was finding the task challenging, Sam described how she had immediately provided a solution strategy rather than prompting students to share their thinking with one another:

> I just get up and say "oh you've got lots of different answers let's try and do it with multiplication" [laughter] when I should've just said "can someone show us? When I did it, I was like "ahhh that was so silly." The whole idea was for them to have different ideas and I went "once you've all got ideas I'm gonna show you" [laughs].

During the group's discussion of the video excerpt, Sam provided some commentary about the events and other group members asked clarifying questions and offered suggestions and feedback. Drawing the discussion to a close, the group affirmed Sam's decision to share classroom video that highlighted a problematic aspect of her practice:

| | |
|---|---|
| *Sam*: | I was just thinking I just completely shut [student A] down and he could've gone on and him and [student B] could've got there and they wouldn't've needed me. That was an 'aha moment' which I think will be good in the future. |
| *Pat*: | But that's part of the reflection process anyway, you see that as your next step and that's basically how we learn. |
| *Casey*: | And we all learn from that. |

Choosing an excerpt to share, and the subsequent discussion of the excerpt by the group together appeared to support Sam to reflect on her own practice. However, despite Casey's suggestion that "we all learn from that", there appeared to be less opportunity for the learning of other group members. Few explicit connections were made by the other teachers with their classroom practice and, despite an intended focus on students' thinking, the questions and comments that emerged in the discussion were for the most part directed at Sam's teaching. Interestingly, although Sam had noticed and commented on some of the students' responses to the task, these were not picked up for discussion by her colleagues. The lack of attention to the impacts of Sam's teaching appeared to be a missed opportunity for the group to reflect on their own teaching.

### *Example 2: A Picture Paints a Thousand Words*

During the same meeting, Pat shared an excerpt of video from her classroom. Posing the problem *"I had 24 lollies and I ate ¾ of them. How many lollies did I eat altogether?"* to her class of 7- and 8-year-olds, she invited them to draw a picture to help with solving the problem and explaining their thinking. Pat had chosen the excerpt to illustrate how students had used wait time to think about their contributions to the mathematical discussion in their group. She described different students' approaches to solving the problem with a focus on how students talked about the pictures they had drawn to represent the problem, and how this talk had elicited descriptions of the mathematics ideas they were engaging with. As the group discussed Pat's video excerpt, Kris asked her to justify her pedagogical actions:

> how do you know that them drawing the pictures is effective to help them understand?

Pat described how she had modelled the practice off a well-accepted strategy she used in literacy teaching of drawing pictures to understand ideas in texts. Casey endorsed the practice and affirmed Pat's justification:

> it's like a prompt isn't it for further explanation. It's the "hold it up and I can talk to it" and it lets other people see straight away

Describing her teaching supported Pat to elicit and make public her reasoning about aspects of her practice, the mathematics being explored, and the mathematical practices of her students. During the discussion of the video excerpt, Pat tended to focus her descriptions on her students' actions rather than her own. This appeared to support a robust discussion wherein Pat was pressed to justify aspects of her teaching which in turn deepened her descriptions. Attention was diverted away from Pat and her teaching which appeared to lessen the possibility of challenging questions being perceived as a personal attack. Interestingly, with the discussion more focused on Pat's descriptions of her teaching than on the video itself, the video appeared to serve as a spark and source of reference for a conversation in which space was created for members of the group to reflect on their own practice in light of Pat's.

### *Example 3: A Model of "Expert" Practice*

At the following meeting, three weeks later, Casey shared a video excerpt in which she was working with a group of 6-year-olds who were exploring "teen numbers" using an array of materials such as ice-block sticks, tens frames and Cuisenaire rods. Two aspects of Casey's lesson that became a focus for the group's discussion were the use of concrete materials and how Casey facilitated the students' talk.

Kris drew attention to Casey's use of a variety of materials, suggesting that the approach exemplified good practice. Others in the group agreed and identified this as something they would like to adopt in their own teaching:

> I'll use a lot more materials thinking about it, so that's what I've learnt. I'll use a lot more materials [and] use them differently [Sam].

For Casey, this was an aspect of her teaching that was connected to her expertise as a teacher of junior children:

> I think that in the early years you do need to because they are so hands on, to explore a concept you do need a variety.

Reflecting on Casey's use of concrete materials, the teachers made connections between the students' talk and their use of materials to represent ten, and how this might support conceptual understanding of place value.

Later, Pat described being struck by how effectively Casey used wait time when asking one student to repeat another's idea. Elaborating on what she had noticed, Pat suggested:

> It was almost like you were throwing in a bit of your EAL experience as well where you asked the child to add a bit more and clarify something and where one person said something and you telling someone else to add more information onto it.

Aware that Casey had initially been reluctant to share, or even view, video from her classroom, the group's comments were primarily aimed at affirming aspects of her practice that they saw as strengths. Nevertheless, the discussion highlighted differences in individual teacher's beliefs and practices related to student talk in mathematics, and the role of the teacher in that talk, and created space to share teachers' different perspectives. For instance, reflecting on approaches to supporting mathematics discussions in mixed-ability groups, Pat suggested providing additional instruction to students who were experiencing challenge whereas Casey and Kris suggested strategies that promoted opportunities for students to support one another such as strategically pairing students.

### *The Use of Video as a Window into the Classroom*

Sharing video excerpts gave the teachers access to view each other's classrooms and the usually private practice therein, and provided support to try new approaches. Kris noted the importance of discussing the excerpts together:

> There's such value in the discussion that comes from the videos and Casey deserves that. If we rush things then it loses the value of what we're doing.

Kris appeared to see the discussion both as a reward for Casey for sharing her video, and as an opportunity for the group to reflect and learn:

> Maybe we could think about what next for us, in our own teaching from watching Casey's footage. What can we take from her good practice into our practice or consider when we're planning or assessing?

In a subsequent meeting, Kris commented:

> With those videos, that was just the vehicle into the discussion, and the discussion actually then prompted you to think about your own practice. It's kind of getting that glimpse in and kind of taking, borrowing stuff.

Sharing video excerpts offered a resource for teachers to reflect on and think in new ways about their teaching. They saw potential for the video excerpts to be used as exemplars of effective practice, or as evidence of improvements in the impact of their teaching over time:

> My guess is that as time goes on and we video each other we would hopefully notice that there is a difference in who's doing the talking in our classrooms. So, the capturing video and keeping it is powerful and the talking to each other about what we notice and ways we could better include these is maybe the collaborative aspect that we're working on [Sam].

Viewing and talking about practices that were different from their own opened possibilities for teachers to extend their teaching repertoires by "borrowing" teaching ideas from one another. In some cases, the teachers noted practices that aligned with and expanded on their own current practices such as Chris who routinely used concrete materials in her teaching but saw new opportunities in adopting Casey's approach of offering a range of materials for students to choose from. In contrast, Casey's use of materials appeared very different to Pat's current teaching and the apparent effectiveness of the approach prompted Pat to suggest that she might adopt this easily observable aspect of Casey's practice.

## Discussion

Viewing and discussing classroom video excerpts has been found to support teachers' understanding of students' mathematical thinking by extending both what they notice about classroom events, and how they talk about what they notice (van Es & Sherin, 2008). The teachers in this study had opportunities to surface and discuss otherwise tacit aspects of their mathematics teaching practice. Eliciting differences in their teaching helped to spark conversations whereby teachers attempted to reconcile different perspectives and approaches, and new possibilities for their classroom practice arose. Highlighting differences in the teachers' practice prompted them to explain and justify aspects of their teaching thereby making their thinking about their teaching public and available as a resource for others to reflect on and develop ideas about their own teaching.

The teachers' roles and perceived levels of expertise appeared to mediate group members' access to one another's thinking about different aspects of their teaching. Casey's use of concrete materials to support her students' understanding of place value aligned both with her recognised expertise as a junior teacher and teacher of students with English as an additional language (EAL), and with accepted effective practice for mathematics teaching | learning in New Zealand primary schools (e.g., Ministry of Education, 2008a). In contrast, Pat's approach of having students draw pictures to help solve a problem was seen as a more innovative approach; something she had invented. In their discussions, the group tended to endorse Casey's approach whereas Pat's less well-established approach prompted lines of questioning that pressed Pat to justify the practice as the teachers wondered how she knew that drawing supported the students to develop their understanding of mathematics ideas. Positioning Pat as more of a novice appeared to open space for the teachers, including Pat, to critically examine her teaching in relation to potential and perceived impacts on students. Opening Pat's teaching to questioning, as Hunter (2007) observed, allowed different ideas about practice to emerge and the discussion created opportunities for teachers to notice and reflect on otherwise taken-for-granted aspects of their own teaching.

As contradictions surfaced among the teachers' different ideas and actions, conversations were sparked whereby they worked to reconcile these differences. As differences emerged, the teachers were pressed to explain and justify their ideas and classroom actions to a greater extent and in more depth than previously, thus expanding their access to diverse ideas about practice. The teachers' conversations, sparked by their sharing of classroom video, enabled these differences to be made available as conceptual resources and reflexive objects for thinking about their practice (Powietrzynska et al., 2015).

The use of classroom video was largely intended to provide a window into classrooms whereby teachers could view and engage with aspects of one another's classroom practices. However, sharing and discussing classroom video appeared to also act as a mirror, a tool for teachers to notice and reflect productively on the impacts of their own teaching in light of the differences that emerged in their conversations.

## References


Ball, D. L. (2011). Foreword. In *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. xx–xxiv). Routledge.

Blomberg, G., Sherin, M. G., Renkl, A., Glogger, I., & Seidel, T. (2014). Understanding video as a tool for teacher education: Investigating instructional strategies to promote reflection. *Instructional Science, 42*(3), 443–463. https://doi.org/10.1007/s11251-013-9281-6

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education, 24*(2), 417–436. https://doi.org/10.1016/j.tate.2006.11.012

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher, 38*(3), 181–199. https://doi.org/10.3102/0013189X08331140

Eaker, R., & Keating, J. (2012). Improving mathematics achievement: The power of professional learning communities. In J. M. Bay-Williams, & W. R. Speer (Eds.), *Professional collaborations in mathematics teaching and learning: Seeking success for all—Seventy-fourth yearbook.* National Council of Teachers of Mathematics.

Eden, R. (2019). *Restructuring mathematics teaching | learning: Co-teaching as a designed approach to teachers' collaborative inquiry* [Doctoral dissertation, Victoria University of Wellington]. Wellington.

Fullan, M., Rincón-Gallardo, S., & Hargreaves, A. (2015). Professional capital as accountability. *Education Policy Analysis Archives, 23*(15), 1–18. https://doi.org/10.14507/epaa.v23.1998

Gaudin, C., & Chaliès, S. (2015). Video viewing in teacher education and professional development: A literature review. *Educational Research Review, 16*, 41–67. https://doi.org/10.1016/j.edurev.2015.06.001

Hintz, A., Kazemi, E., Granger, J., Lind, T., Lewis, B., Simpson, L., & Crandall, D. (2013). Innovations in improving mathematics instruction: One school's story of implementing job-embedded school-wide professional development. *Washington State Kappan, 6*(2). https://journals.lib.washington.edu/index.php/wsk/article/view/14166

Hunter, J., & Back, J. (2011). Facilitating sustainable professional development through lesson study. *Mathematics Teacher Education and Development, 13*(1), 94–114.

Hunter, R. K. (2007). *Teachers developing communities of mathematical inquiry* [Doctoral thesis, Massey University].

Ingvarson, L., Beavis, A., Bishop, A., Peck, R., & Elsworth, G. (2004). *Investigation of effective mathematics teaching and learning in Australian secondary schools*. Australian Council for Educational Research.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169–202.

Jacobs, V. R., Philipp, R. A., & Sherin, M. G. (2018). Noticing of Mathematics Teachers. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 1–3). Springer International Publishing. https://doi.org/10.1007/978-3-319-77487-9_120-4

Jaworski, B. (2003). Inquiry as a pervasive pedagogic process in mathematics education development. *Proceedings of the Third Conference of the European Society for Research in Mathematics Education, 28*. http://fibonacci.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG11/TG11_Jaworski_cerme3.pdf

Jaworski, B. (2008). Building and sustaining inquiry communities in mathematics teaching development. In K. Krainer & T. Wood (Eds.), *Participants in mathematics teacher education: Individuals, teams, communities and networks* (Vol. 3, pp. 309–330). Sense Publishers.

Kaiser, G., & König, J. (2019). Competence measurement in (mathematics) teacher education and beyond: Implications for policy. *Higher Education Policy, 32*(4), 597–615. https://doi.org/10.1057/s41307-019-00139-z

Larrivee, B. (2000). Transforming teaching practice: Becoming the critically reflective teacher. *Reflective Practice, 1*(3), 293–307. https://doi.org/10.1080/713693162

Levine, T. H. (2010). Tools for the study and design of collaborative teacher learning: The affordances of different conceptions of teacher community and activity theory. *Teacher Education Quarterly, 37*(1), 109–130.

Mason, J. (2009). Teaching as disciplined enquiry. *Teachers and Teaching, 15*(2), 205–223. https://doi.org/10.1080/13540600902875308

Ministry of Education. (2008). *Book 1: The number framework*. Ministry of Education.

Powietrzynska, M., Tobin, K., & Alexakos, K. (2015). Facing the grand challenges through heuristics and mindfulness. *Cultural Studies of Science Education, 10*(1), 65–81. https://doi.org/10.1007/s11422-014-9588-x

Santagata, R., & Guarino, J. (2011). Using video to teach future teachers to learn from teaching. *ZDM, 43*(1), 133–145. https://doi.org/10.1007/s11858-010-0292-3

Santagata, R., König, J., Scheiner, T., Nguyen, H., Adleff, A.-K., Yang, X., & Kaiser, G. (2021). Mathematics teacher learning to notice: A systematic review of studies of video-based programs. *ZDM—Mathematics Education, 53*(1), 119–134. https://doi.org/10.1007/s11858-020-01216-z

Scheiner, T. (2016). Teacher noticing: Enlightening or blinding? *ZDM, 48*(1–2), 227–238. https://doi.org/10.1007/s11858-016-0771-2

Sherin, M. G., & Dyer, E. B. (2017). Mathematics teachers' self-captured video and opportunities for learning. *Journal of Mathematics Teacher Education, 20*(5), 477–495. https://doi.org/10.1007/s10857-017-9383-1

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education, 24*(2), 244–276. https://doi.org/10.1016/j.tate.2006.11.005

# Mathematics Teachers' Perceptions of a STEM Approach for Selected Student Outcomes

Emma Every
*MGSE, University of Melbourne*
everye@student.unimelb.edu.au

Lynda Ball
*MGSE, University of Melbourne*
lball@unimelb.edu.au

A STEM approach in mathematics lessons may offer affordances to mathematics teaching, with potential benefits for students' outcomes. A survey of 30 mathematics teachers identified the perceived importance of selected student outcomes (related to cognitive, affective, and STEM capabilities), for mathematics lessons with a STEM approach compared to those considered 'typical'. Some teachers perceived that a STEM approach in a mathematics lesson places importance on developing student outcomes such as critical thinking and problem solving and contributes to mathematical understanding. A STEM approach in mathematics lessons may be beneficial if teachers can envisage, and realise, potential gains for student outcomes.

Science, Technology, Engineering, and Mathematics (STEM) is seen as vital to meet social, cultural, and economic challenges in Australia (Office of Chief Scientist, 2013). Education is identified as key to increasing STEM skills and capabilities both nationally (Office of Chief Scientist, 2013) and in the state of Victoria (DET, 2016) which is the context for this study. A STEM approach is widely assumed to involve teaching and learning through the integration of two or more STEM disciplines (English, 2016); Wang et al. (2011) suggested that connections across disciplines can result in broader and deeper student understanding. However, the National Academy of Engineering and National Research Council (NAENRC) (2014)) cautions that integration may impact the development of disciplinary knowledge, highlighting a tension between the potential benefits of integrated learning through a STEM approach and the development of disciplinary knowledge (i.e., mathematics in this paper).

The relationship between mathematics and STEM education raises numerous issues with Maass et al. (2019) suggesting that STEM activities have focussed on science. Mathematics can be incidental to STEM activities (Fitzallen, 2015) or play a service role (Tytler et al., 2019), where mathematics learning and teaching are not a priority. The NAENRC (2014) highlighted the difficulty of enhancing mathematics achievement when integrating with another discipline. Despite these challenges, there are common student outcomes for STEM and mathematics education (e.g., critical thinking and problem-solving skills; Gravemeijer et al., 2017).

Wang et al. (2011) found that different teacher perceptions of a STEM approach can lead to different classroom practices in aspects like integration, communication, and interactions. Margot and Kettler (2019) similarly noted that differences in teacher perceptions influence their design and delivery of a STEM approach. However, there is limited research on mathematics teachers' perceptions of the affordances of a STEM approach in their lessons. This paper presents findings from a survey of thirty secondary mathematics teachers about the perceived importance of eight selected student outcomes when teaching mathematics with a STEM approach, A comparison is also made with what these teachers consider important in a typical mathematics lesson (refer to Methodology section).

The research questions were:

- What are Victorian mathematics teachers' perceptions of the importance of selected student outcomes when a STEM approach is used in a mathematics lesson?
- What are the perceptions of the importance of the same outcomes in their typical mathematics lessons?
- Are there similarities and differences in the outcomes perceived to be important?

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 187–194). Newcastle: MERGA.

## Literature Review

Understanding teacher perceptions in STEM education is important due to its potential influence on teaching. Thibaut et al. (2018), in a survey of 135 secondary teachers, noted a negative correlation between attitude towards a STEM approach and experience in teaching mathematics. Conversely, Sevimli and Ünal (2022) found positive views on the usefulness of STEM tasks in mathematics (study of 36 secondary mathematics teachers). Research on STEM education and teachers' perceptions has focussed on the teaching of science concepts or views of science teachers, with less focus on the views of mathematics teachers (Sevimli and Ünal, 2022). Further, there is little literature on Australian teachers' perceptions of a STEM approach.

The intent of this paper is to determine student outcomes that the participating teachers perceived to be important when adopting a STEM approach in a mathematics lesson and how these may differ from those in a typical mathematics lesson. Hence the focus is on student outcomes from a STEM approach. As a result, the literature considered papers related to STEM outcomes to identify those for inclusion in the survey. Specifying student outcomes is important in enabling the effectiveness of a STEM approach to be determined (NAENRC, 2014). Although a STEM approach aims to achieve broad and deep student understanding (Wang et al., 2011), research specifying student outcomes from a STEM approach is developing (English, 2016).

Many researchers (e.g., NAENRC, 2014; Thibaut, 2018; Martín-Páez et al., 2019; Gao et al., 2020) have identified and categorised a list of student outcomes for a STEM approach, with a degree of overlap. These can be grouped as cognitive (including disciplinary content knowledge and understanding); affective or attitudinal (e.g., interest and engagement); and STEM capabilities or skills (including 21st century skills). Attard et al., (2020) noted that increasing student interest and engagement is a key driver of a STEM approach. The importance of a STEM approach for contributing to critical thinking and creativity has also been highlighted (Yildirim and Türk, 2018). This aligns with a key goal of STEM education which is to develop 21st century skills despite these not having a common definition (Maass et al., 2019). Increasing students' generic skills through adapting traditional curriculum can support students in responding to a changing global world (Millar, 2020).

Student outcomes from a STEM approach might also be guided by policy and curriculum documents. In Australia, the National STEM School Education Strategy (National STEM Strategy) refers to STEM education as an umbrella term that includes the teaching of science, technology, engineering, and mathematics (Education Council, 2015). While not specifying STEM student outcomes, the National STEM Strategy has the goals of ensuring students achieve strong foundational knowledge in STEM subjects and related skills such as critical thinking and problem solving and choose to study more challenging STEM subjects.

Although the Victorian Curriculum and Assessment Authority (VCAA) has few references to 'STEM' in the Victorian mathematics curriculum (VCAA, 2019), it incorporates many student outcomes of STEM education identified in the literature including problem solving, communication, and connections to other disciplines. Other STEM skills such as critical and creative thinking, and inquiry-based learning are general capabilities and are expected across the curriculum. The Victorian Department of Education and Training (DET) released STEM in the Education State (DET, 2016) which recognised the need to equip Victorian students with STEM capabilities and skills which it advises are incorporated in the curriculum. Based on the emphasis on STEM in Australia and the inclusion of STEM outcomes in the curriculum, Victorian teachers should be aware of student outcomes related to a STEM approach.

# Methodology

This study used a mixed method, embedded design approach allowing for a quantitative data set to be the focus and a supplemental role played by qualitative data. Mixed methods design provides broader evidence than either a quantitative or qualitative approach (Creswell & Plano Clark, 2011). Quantitative analysis focused on teachers' perceptions of student outcomes, and qualitative analysis centred on teachers' perceptions of the potential benefits and drawbacks of a STEM approach in a mathematics lesson with inference drawn about student outcomes. Data collection was via a survey based on identified student outcomes.

### *Student Outcomes for Survey Inclusion*

A list of outcomes was identified through a literature review undertaken in 2020. Papers were identified through education databases including ERIC and Google Scholar using the search term 'STEM education'. Of these, twelve papers were selected which had a focus on student outcomes. Thirty-two student outcomes were named in these papers with those most frequently mentioned identified. There were inconsistencies in the terminology used (e.g. flexibility /adaptability), so the authors used one term to represent terms with common meanings. The first named author checked to ensure that student outcomes in each category (cognitive, affective, STEM capabilities) represented those identified through the literature review. 'Cognitive' outcomes included outcomes related to cognition in mathematics rather than specific content areas (e.g. Pythagoras' theorem). For this category, we focussed on fluency and understanding (i.e., procedural and conceptual knowledge) and problem solving, encompassing thinking and application of mathematics, which is a noted outcome in STEM literature (Maass et al., 2019).

A survey was developed using Qualtrics and trialled by academic peers, three of whom have experience teaching secondary mathematics. Feedback suggested the need for a shorter survey so items that were mentioned fewest times in the literature were removed. The student outcomes included in the survey were:

- Fluency, understanding, problem solving (i.e. cognitive outcomes)
- Interest, engagement (i.e., affective factors)
- Critical thinking, creative thinking, and transfer of understanding across disciplines (i.e., STEM capabilities).

### *Survey*

The research question compared teachers' perceptions of 'a mathematics lesson with a STEM approach' to what they considered a 'typical mathematics lesson'. Teachers opted into the survey and so contexts were not expected to be consistent. Thus these terms were not defined as they would be expected to vary from teacher to teacher. Similarly, each of the student outcomes were not defined. However, as all teachers were Victorian mathematics teachers some understanding might be expected to be in line with the Victorian mathematics curriculum where these terms are defined (for example, understanding and fluency).

For each of the eight student outcomes, teachers were asked two questions:

- In a typical mathematics lesson how important is it for students to develop [Student Outcome] (e.g. critical thinking)?
- In a mathematics lesson with a STEM approach how important is it for students to develop [Student Outcome] (e.g. critical thinking)?

Teachers indicated the importance on a 5-point Likert scale (1 = Not at all important; 5 = Extremely important). Teachers were also asked the open question, "What do you see are the potential benefits of adopting a STEM approach in a mathematics lesson?" which was then replicated for drawbacks.

The survey was advertised to Victorian secondary school mathematics teachers via social media, the Mathematical Association of Victoria (MAV) (a mathematics teacher organisation), and personal networks (mathematics teachers). Participants included 30 Victorian secondary school mathematics teachers (non-government, 17%; government, 83%), currently teaching at least one mathematics class, and a range of year levels from Years 7–12; 60% taught a subject other than mathematics. While all 30 teachers responded to the Likert style questions, only 27 teachers responded to the open questions.

Data was collected between March and August 2021, a period of disruption for Victorian schools due to Covid restrictions and potentially contributing to a small sample size. Teachers self-selected so may have strong positive or negative perceptions of a STEM approach. The survey participants are not considered to be a representative sample of Victorian secondary school mathematics teachers. Results were analysed using Microsoft Excel. Each graph presents the frequency of response for each level of importance for each outcome and compares a typical mathematics lesson and a mathematics lesson where a STEM approach is used.

## Results and Discussion

This section reports Victorian secondary school mathematics teachers' perceptions of the importance of eight student outcomes in either a mathematics lesson that is perceived to be typical by the teacher (referred to as a "typical maths lesson"); or one that adopts a STEM approach ("STEM-approach lesson"). Findings (Figure ) are discussed for the student outcomes within categories of cognitive (mathematics), affective, and STEM capabilities.

### *Cognitive (Mathematics)*

Figure 1a shows teachers believed that developing Understanding was important for both typical mathematics lessons and those with a STEM approach. In the Victorian curriculum (VCAA, 2019) understanding relates to knowledge of mathematical concepts (relevant to a typical mathematics lesson) and to connecting mathematical ideas and interpreting mathematical information), (relevant to a STEM approach). As noted, research has highlighted that mathematics may be incidental in a STEM activity (Fitzallen, 2015) or plays a service role (Tytler et al., 2019). Hence there may not be an expectation that the development of conceptual understanding of mathematics is a focus when adopting a STEM approach. However, 87% of teachers indicated mathematical understanding was very or extremely important for a mathematics lesson with a STEM approach, highlighting a focus on mathematical understanding when adopting a STEM approach, rather than mathematics being incidental.

Fluency is a student's ability to undertake procedures flexibly, accurately, and efficiently (VCAA, 2019). Figure 1b shows that the importance placed on developing fluency in a typical mathematics lesson was high with over 85% indicating it was very or extremely important, compared to just over 50% for a STEM approach lesson. Fluency includes 'flexibility', which might be achieved through the solving of complex problems (a feature of a STEM approach, Gao et al., 2020). It was noted that a STEM approach has the benefit of "... apply(ing) mathematical skills in a range of situations" (Teacher 7). Despite this, the results suggest that some teachers perceived that a STEM approach is not as important for mathematical fluency with Teacher 19 noting that the drawback of a STEM approach is "... a lack of fluency in basic skills such as times tables, fraction addition, algebra…". Thus, while some teachers might perceive mathematical fluency as highly important in a STEM approach, this was not universal.

Problem solving is referenced in both the Victorian mathematics curriculum and within its general capabilities (i.e., Critical and Creative thinking), thus a high degree of importance could be expected of this student outcome for a typical mathematics lesson. Figure 1c indicates that while over 80% of teachers consider it very or extremely important for a typical mathematics lesson there is also a high percentage (93%) for a STEM approach lesson. This highlights that a STEM approach might represent a pathway for teachers to focus on the development of problem-solving skills of students. A STEM approach in a mathematics lesson can be of benefit as "Students can focus on learning basic fundamentals and then use problem solving skills to expand on these fundamentals" (Teacher 7). Problem solving reflects student ability to interpret, choose, investigate, and communicate solutions to problems (VCAA, 2019), with a STEM approach seen by some teachers as a way to achieve this.

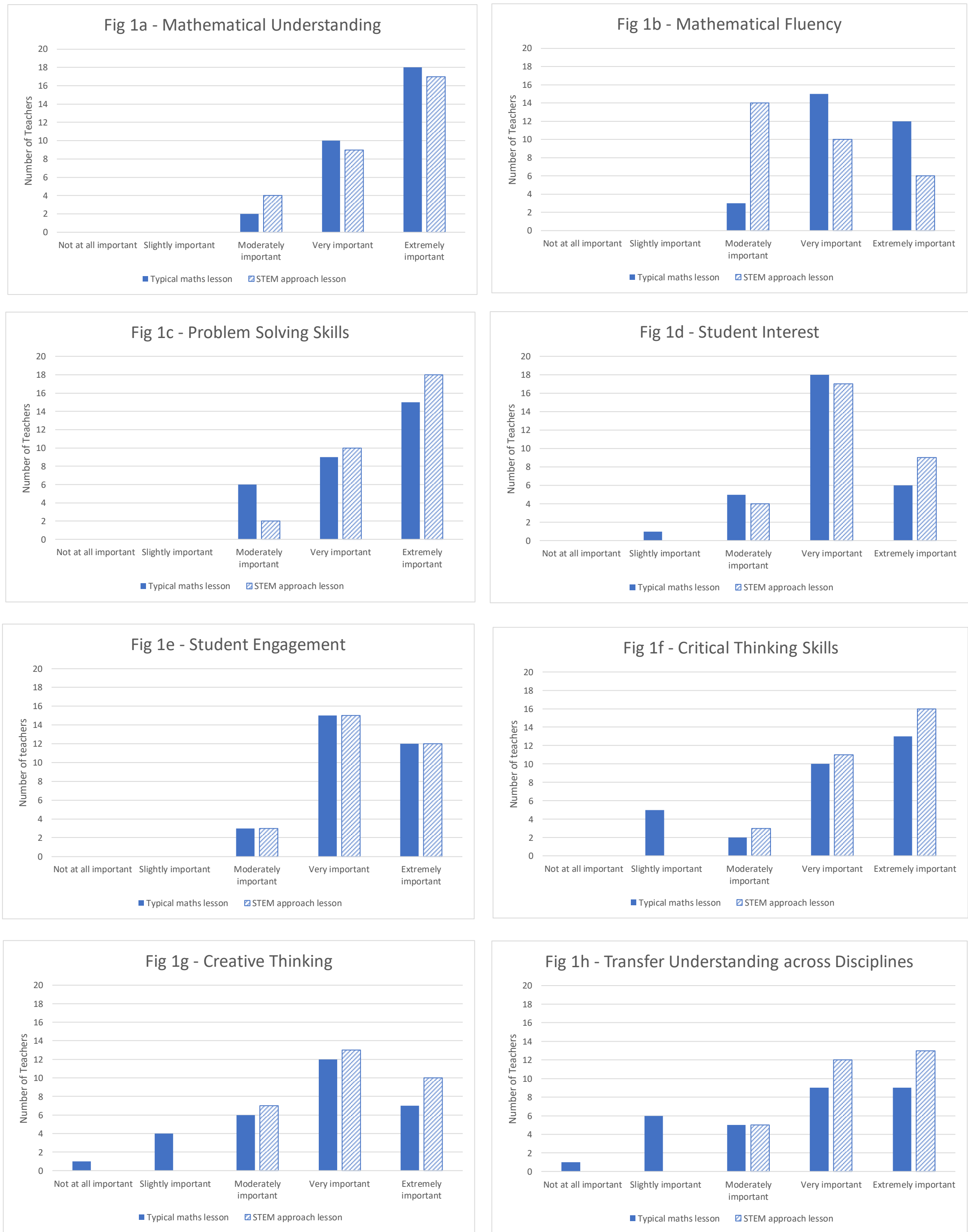



Figure 1. Importance of student outcomes for a STEM approach and typical mathematics lesson.

Mathematical understanding and problem solving were highly important (very or extremely) for either a STEM approach or a typical mathematics lesson by over 80% of teachers. A number of teachers regarded mathematical fluency to be of lower importance when using a STEM approach despite Fluency incorporating flexibility, which is connected to 21st century skills (Maass et al., 2019) and a feature of a STEM approach.

### *Affective Factors (Interest and Engagement)*

Increased student interest and engagement is a goal of a STEM approach (Attard et al., 2020) and an area for national action in school education (Education Council, Australia, 2015). Teachers identified student interest (Figure 1d) and engagement (Figure 1e) as important for both typical mathematics lessons and a STEM approach lesson. For both approaches, 90% of teachers considered engagement to be very or extremely important reflecting Millar's (2020) observation that teachers in the STEM disciplines have long recognised the need to engage students. While teachers indicated affective factors are of similar importance for both approaches, it is not clear whether teachers consider the two approaches to be equally capable of achieving this. Teacher perceptions of interest and engagement vary with Teacher 9 suggesting a STEM approach results in "increased engagement, developing a deeper understanding of mathematics …". Conversely Teacher 8 noted "if students are not engaged in the lesson, then the learning may be completely lost". Somewhat reflecting this, Tytler (2020) has suggested that teachers are motivated to adopt STEM initiatives to improve student engagement, however, Attard et al. (2020) noted there is little to support how a STEM approach might be successful in addressing interest or engagement.

### *STEM Capabilities*

Critical and creative thinking in the Victorian Curriculum (VCAA, 2019) is a general capability that involves students developing thinking processes and how to apply these to support logical, strategic, and flexible thinking across a range of contexts. Teachers' perceived importance of critical and creative thinking are presented in Figures 1f and 1g respectively. For a typical mathematics lesson, 77% of teachers consider critical thinking skills very or extremely important with 23% considering them only slightly important. For a STEM approach lesson, more teachers (90%) consider it very or extremely important. More teachers identified creative thinking as very or extremely important in a STEM approach lesson (77%) compared to a typical mathematics lesson (63%). For critical and creative thinking, two 21st century skills, the importance of these outcomes was slightly greater for a STEM approach than a typical maths lesson with Teacher 11 noting that a STEM approach provided "more opportunities for creativity, critical thinking, etc." (). This aligned with Yildirim and Türk (2018) findings on Turkish secondary school science and mathematics teacher views. Maass et al., (2019) noted that while capabilities such as critical and creative thinking are recognized as being of increasing importance in education, teachers have been provided little guidance on how to promote these 21st century skills. Given teachers' perceptions that critical and creative thinking might be more important in a STEM approach lesson, this might be motivation for teachers to consider implementing a STEM approach in mathematics classes.

Figure 1h shows variability in teachers' perceptions of the importance of transferring understanding across disciplines in a typical mathematics lesson; 23% considered it not at all or only slightly important and 60% considered it very or extremely important. Although Tytler (2020) suggested mathematics curricula do not prioritise interdisciplinary tasks, the Victorian curriculum (VCAA, 2019) expects mathematics to be used to solve problems in other contexts. This points to the importance of transferring mathematical understanding across disciplines, hence more teachers might have been expected to place greater importance on this outcome in a typical mathematics lesson. For a STEM approach, transfer of understanding was considered very or extremely important by 83% of teachers. Teacher 11 noted a STEM approach was advantageous for "Breaking down the subject-based silos/emphasising multidisciplinary nature of learning and work". This benefit for transferring understanding across disciplines reflects the interdisciplinary or multidisciplinary nature of a STEM approach; a commonly noted feature (e.g., English, 2016). These findings suggest that for some mathematics teachers, the use of a STEM approach may support students' transfer of understanding across disciplines.

## Conclusion

This study identified mathematics teachers' perceptions of the importance of eight selected student outcomes and compared what was perceived to be important in a STEM approach lesson and a typical mathematics lesson. Understanding teachers' views is important as it can influence their teaching (Thibaut et al., 2018).

The three 'Cognitive' (mathematics) student outcomes: understanding, fluency, problem solving, varied in the perceived importance for a STEM approach compared to a typical mathematics lesson. Fluency is perceived as markedly more important in a 'typical' mathematics lesson while problem solving skills are perceived as moderately more important for a STEM approach. Understanding is perceived as similarly important for both.

The 'Affective' student outcomes: interest and engagement were perceived by teachers to be of similar importance for both options. While Attard et al. (2020) suggested that findings related to a STEM approach increasing student interest and engagement were inconclusive, it was unclear if surveyed teachers thought that student interest and engagement could be achieved through a STEM approach. Further research would be needed to determine if a STEM approach represents a potential to increase student engagement and interest.

The three student outcomes for 'STEM Capabilities': transfer of understanding across disciplines, critical thinking, and creative thinking, were perceived as more important for a STEM approach compared to a typical mathematics lesson. These outcomes all feature in the Victorian Curriculum with critical thinking considered an important student outcome for mathematics (Gravemeijer et al., 2017) and a student outcome needed for a changing world. The perceived lower importance in a typical mathematics lesson could result from a focus on discipline specific content (e.g., mathematics concepts or skills) rather than outcomes (e.g., 21st century skills) that go across disciplines. This could reflect a potential difference between the intended curriculum (which includes discipline specific content and cross curriculum goals) and the enacted curriculum.

In this study of 30 teachers, several teachers recognised potential affordances of a STEM approach in mathematics lessons for contributing to student outcomes. This suggested that a STEM approach in mathematics lessons may be worth pursuing if teachers can envisage, and consequently realise, potential gains for student outcomes. Positive perceptions of a STEM approach in mathematics lessons may contribute to teachers' willingness to employ STEM approaches and consequently impact student outcomes.

## References

Attard, C., Grootenboer, P., Attard, E., & Laird, A. (2020). Affect and engagement in STEM education In A. MacDonald, L. Danaia, & S. Murphy (Eds.), *STEM education across the learning continuum: Early childhood to senior secondary*. Springer Singapore. . https://doi.org/10.1007/978-981-15-2821-7_11

Creswell, J. W., & Plano Clark, V. L. (2011). *Designing and conducting mixed methods research* (2nd ed. ed.). SAGE Publications.

Department of Education and Training, Victoria. (2016). *STEM in the education state*. Melbourne: DET. https://www.education.vic.gov.au/Documents/about/programs/learningdev/vicstem/STEM_EducationState_Plan.pdf

Education Council, Australia. (2015). *National STEM school education strategy 2016–2026*. Canberra: Department of Education, Australia. https://www.education.gov.au/ education-ministers-meeting/resources/national-stem-school-education-strategy

English, L. D. (2016). STEM education K-12: Perspectives on integration. *International Journal of STEM Education*, *3*(1), 3. https://doi.org/10.1186/s40594-016-0036-1

Fitzallen, N. (2015). STEM education: What does mathematics have to offer? In Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 237–244). Sunshine Coast: MERGA.

Gao, X., Li, P., Shen, J., & Sun, H. (2020). Reviewing assessment of student learning in interdisciplinary STEM education. *International Journal of STEM Education*, *7*(1), 24. https://doi.org/10.1186/s40594-020-00225-4

Gravemeijer, K., Stephan, M., Julie, C., Lin, F.-L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future? [Report]. *International Journal of Science & Math Education*(1), 105. https://doi.org/10.1007/s10763-017-9814-6

Maass, K., Geiger, V., Ariza, M. R., & Goos, M. (2019). The role of mathematics in interdisciplinary STEM education [journal article]. *ZDM*, 1-16. https://doi.org/10.1007/s11858-019-01100-5

Margot, K. C., & Kettler, T. (2019). Teachers' perception of STEM integration and education: A systematic literature review. *International Journal of STEM Education*, *6*(1), 1–16. https://doi.org/http://dx.doi.org/10.1186/s40594-018-0151-2

Martín-Páez, T., Aguilera, D., Perales-Palacios, F. J., & Vílchez-González, J. M. (2019). What are we talking about when we talk about STEM education? A review of literature [Article]. *Science Education*, *103*(4), 799–822. https://doi.org/10.1002/sce.21522

Millar, V. (2020). Trends, issues and possibilities for an interdisciplinary STEM curriculum [Article]. *Science & Education*, *29*(4), 929–948. https://doi.org/10.1007/s11191-020-00144-4

National Academy of Engineering, & National Research Council. (2014). *STEM integration in K–12 education: Status, prospects, and an agenda for research*. The National Academies Press. https://doi.org/doi:10.17226/18612

Office of the Chief Scientist. (2013). *Science, technology, engineering and mathematics in the national interest.* Canberra: Australian Government. https://www.chiefscientist.gov.au/wp-content/uploads/STEMstrategy290713FINALweb.pdf

Sevimli, E., & Ünal, E. (2022). Is the STEM approach useful in teaching mathematics? Evaluating the views of mathematics teachers. *European Journal of STEM Ed.*, *7*(1), 1. –11. https://doi.org/10.20897/ejsteme/11775

Thibaut, L. (2018). *Implementing integrated STEM: Teachers' attitudes, instructional practices and students' learning outcomes.* [PhD dissertation, KU Leuven]. Leuven.

Thibaut, L., Knipprath, H., Dehaene, W., & Depaepe, F. (2018). How school context and personal factors relate to teachers' attitudes toward teaching integrated STEM. *International Journal of Technology and Design Education*, *28*(3), 631–651. https://doi.org/10.1007/s10798-017-9416-1

Tytler, R. (2020). STEM education for the twenty-first century. In J. Anderson, & Y. Li (Eds.), *Integrated approaches to STEM education: An international perspective* (pp. 21–43). Springer International Publishing. https://doi.org/10.1007/978-3-030-52229-2_3

Tytler, R., Williams, G., Hobbs, L., & Anderson, J. (2019). Challenges and opportunities for a STEM interdisciplinary agenda. In B. Doig, J. Williams, D. Swanson, R. Borromeo Ferri, & P. Drake (Eds.), *Interdisciplinary mathematics education: The state of the art and beyond* (pp. 51–81). Springer International Publishing. https://doi.org/10.1007/978-3-030-11066-6_5

VCAA. (2019). *The Victorian curriculum F-10*. Melbourne: Victorian Curriculum and Assessment Authority. https://victoriancurriculum.vcaa.vic.edu.au

Wang, H.-H., Moore, T. J., Roehrig, G. H., & Park, M. S. (2011). STEM integration: Teacher perceptions and practice. *Journal of Pre-College Engineering Education Research*, *1*(2), 1–13. https://doi.org/10.5703/1288284314636

Yildirim, B., & Türk, C. (2018). Opinions of secondary school science and mathematics teachers on STEM education. *World Journal on Educational Technology: Current Issues*, *10*(1), 52–60. https://doi.org/10.18844/wjet.v10i1.3120

# Enjoyable Mathematics Lessons can be Contagious


Maggie Feng
*University of Sydney*
mfen5873@uni.sydney.edu.au

Janette Bobis
*University of Sydney*
janette.bobis@sydney.edu.au

Bronwyn Reid O'Connor
*University of Sydney*
bronwyn.reidoconnor@sydney.edu.au

Jennifer Way
*University of Sydney*
jennifer.way@sydney.edu.au


The current study explored the reasons for students' preferences for the teach-first and task-first lesson structures, and whether students' preferences were influenced by their perceptions of the teacher's preference. Students (n=18) from two composite Year 3 and 4 classes (aged 8-10 years) completed a post-lesson drawing task and participated in a semi-structured interview following a series of lessons. Findings indicated students had a variety of reasons for their preference of lesson structure. Most focus students reported noticing aspects of the teacher's enjoyment during instruction. The results have implications for the way teachers inadvertently influence their students' own enjoyment of and preferences for instructional approaches.

Numerous interactions occur between teachers and students each day. Such interchanges can range from individual discourses to those between a teacher and all students within a classroom. During these exchanges, teachers and students inevitably affect each other. For instance, if teachers are experiencing enjoyment while teaching, they may project their enthusiasm by speaking faster, and exaggerating their gestures and expressions (Frenzel et al., 2017). Students commonly notice their teacher's excitement and approach lessons with the same level of excitement and engagement (Keller et al., 2016). This type of influence on student engagement is of significance not only across various subjects or topics within a subject, but also regarding the implementation of a specific pedagogical approach including the way in which lessons are structured. The study reported in this paper is part of a larger project designed to investigate Year 3 and 4 students' engagement with teach-first and task-first lesson structures that incorporate challenging mathematical problem-solving tasks. The aims of the current study were to investigate the reasons for students' mathematics lesson structure preferences and explore whether their perceptions of the teacher's preference influenced their own preferences.

## Literature Review

### *Teach-first and Task-first Lesson Structures*

Mathematics lessons can be structured in various ways. For example, a lesson can be structured to begin with teacher directed explanation and discussion, followed by students independently solving tasks (teach-first). A lesson can also begin with independent student exploration followed by discovering key mathematical ideas as the lesson unfolds (task-first). There are those who advocate the effectiveness of the task-first lesson structure (Sullivan et al., 2020), but also those who argue that different benefits can be derived from both teach-first and task-first structures (Russo & Hopkins, 2019). See Table 1 for the components of the task-first and teach-first lesson structure. Indeed, the way lessons are structured to integrate the various instructional strategies is an important consideration for optimising mathematics learning. This is because student engagement is directly impacted by how tasks are located within the structure of a lesson as well as the teacher's implementation (Sullivan et al., 2016). However, studies that explicitly explore primary aged students' preferences for different lesson structures that include challenging mathematical tasks are rare. Because students are directly impacted by the extent to which they are engaged in a lesson, it is important to explore their perspectives of what they find enjoyable and engaging.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 195–202). Newcastle: MERGA.

**Table 1**

*Components of the Task-first and Teach-first Lesson Structure*

| Task-first lesson structure | Teach-first lesson structure |
|---|---|
| *Introduction*: Tuning-in activity. | *Introduction*: Tuning-in activity. |
| Launch challenging-task (without telling). | Model and explain possible strategies for solving challenging task. |
| *Body*: Students explore challenging task independently and are spotlighted as the lesson unfolds. | *Body*: Students explore similar challenging task independently while teacher monitors and helps. |
| *Conclusion*: Whole class summary. | *Conclusion*: Whole class summary. |

### *Student Perceptions of Teacher Enthusiasm and Transmission of Enjoyment*

It has been suggested that enthusiastic teaching behaviours are displayed verbally and nonverbally when teachers feel enjoyment (Frenzel et al., 2017). When higher levels of enjoyment are felt during teaching, it is more likely for such emotions to be observable by students via enthusiastic teaching behaviours. Furthermore, students who perceive their teacher display higher levels of enthusiasm tend to enjoy learning more (Frenzel et al., 2009). When students experience positive emotions such as enjoyment in the classroom, they become more intrinsically motivated and interested to engage in learning and academic content (Renninger & Hidi, 2016). The control-value theory of achievement emotions (Pekrun, 2000) implies that people's emotions are influenced by their perceptions of the behaviours noticed from their interaction partner. Frenzel et al. (2017) interpreted the mechanisms responsible for the reciprocal relationship between the teachers' and students' emotions by using Pekrun's (2000) control-value theory of achievement emotions. The implications of gaining a deeper understanding of how enjoyment is transmitted between teachers and students in the classroom includes providing valuable insight into how teachers can best manage and shape social interactions to maximise student engagement in the classroom.

Although the idea of enjoyment transmission between teachers and their students makes intuitive sense, the phenomenon of enjoyment transmission in the classroom remains underexamined. In a study involving 149 Grade 9 students across four different subject domains in Switzerland, Becker et al. (2014) found that student perceptions of their teachers' emotions and instructional behaviours significantly shaped their own emotions in class. Similar findings were established in a study by Bakker (2005) with 178 music teachers and 605 students from 16 different music schools in the Netherlands. His findings demonstrated an emotional contagion between students and teachers who reported experiencing flow (absorption, work enjoyment, and intrinsic work motivation) (Csikszentmihalyi, 1997). Furthermore, a positive correlation between teachers' and their students' enjoyment was found by Frenzel et al. (2009) using self-reported enjoyment data of 1542 Grades 7 and 8 students from 71 mathematics classrooms in Germany. Despite these studies providing insight into the positive relationship between a teacher's and their students' enjoyment, the questionnaires with Likert scale items designed for students to self-report (e.g., "This teacher teaches with enthusiasm") does not give qualitative information about the students' reasons for their perceptions. Moreover, studies have mostly been conducted in European countries, with secondary aged students, and across various subjects. Therefore, little is known about emotional transmission with primary aged students in the Australian context, and even less so during lessons in which the content and tasks are identical but presented using different instructional approaches.

The study reported in this paper was guided by Pekrun's (2000) control-value theory of achievement emotions and designed to answer the following research questions:

- Do students prefer one lesson structure over another? If so, why?
- Are students' preferences for a task-first or teach-first lesson structure influenced by their perception of their teacher's preferences? If so, what aspects of the teachers' behaviours do they perceive?

## Methodology

The intervention study adopted a qualitative, exploratory design with multiple data sources including a post-lesson drawing task and semi-structured interviews. Two classes of students were initially randomly allocated to one of two intervention conditions–the task-first lesson structure and the teach-first lesson structure. After the initial allocation of each of the two classes for the first half of the unit of work (3 lessons), the task-first and teach-first lesson structures were inverted such that each class participated in the other condition for the second half of the unit of work involving the same mathematics content (3 lessons). There was a total of 12 mathematics lessons (6 lessons x 2 topics) spread across 4 weeks of instruction. All lessons for both classes were taught by the same teacher who has considerable expertise with teaching with challenging mathematical tasks across both lesson structures due to prior involvement in another research project.

### *Participants and Data Collection*

A purposive sampling method (Polkinghorne, 2005) was used to select participants for this study. Two multi-aged classes of Year 3 and 4 students (aged 8-10 years) (class A, $n = 21$; class B, $n = 19$; $N = 40$) from a Catholic primary school in Victoria participated in this investigation. The school was invited based on the criteria: current or recent student experience with challenging tasks in the classroom; willingness of principal, teachers, students, and their parents to participate. A group of nine focus students per class ($N = 18$) was invited to participate in individual semi-structured interviews with the researcher to allow for the in-depth exploration of students' perceptions of each lesson structure. Potential focus students with various mathematics engagement and performance levels were identified from their responses to a teacher constructed content pre-test and a Motivation and Engagement Survey (Martin et al., 2015) completed prior to the intervention.

Administered at the end of the first lesson of the final week (Lesson 9 out of 12), the post-lesson drawing task consisted of a blank sheet of paper on which all students drew either a happy, neutral, or sad face to represent the extent to which they enjoyed the lesson (see Figure 1). Students were encouraged to provide a reason for their response. Shortly after the lesson, the focus students participated in an individual interview with the researcher in which they were asked questions about their preferences for each type of lesson structure, including the questions: "Which lesson structure do you think is better for your learning (and why)?" and "Which lesson structure do you think Ms J (the teacher) prefers teaching (and why)?".

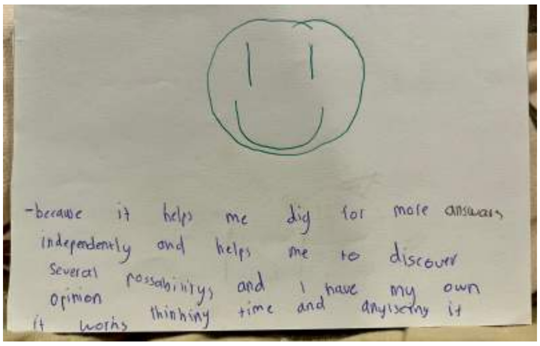



Figure 1. Post-lesson drawing task demonstrating enjoyment in the task-first lesson.

### *Data Analysis*

Individual interviews with the focus students were transcribed. Student responses were first coded as indicating a preference for: (1) the task-first lesson structure; (2) the teach-first lesson structure; (3) equally preferred; or (4) unsure of their preference. The total number of students that preferred each lesson structure and their perception of the teacher's preference for a particular lesson structure were calculated. Next, an inductive thematic analysis (Braun & Clarke, 2006) was used to identify patterns of meanings for students' reasoning towards their preference for the lesson structure. All data were collated on an Excel spreadsheet.

## Results and Discussion

Due to length limitations, only data from the focus students will be reported in this paper. The results are reported in three parts. The first part examines the number of focus students that perceived either the task-first or teach-first lesson structure to be better for their learning and their corresponding reasons for such preferences. The second part reports the focus students' perceptions of the teacher's lesson structure preference and corresponding reasons for such perceptions. Finally, focus students' lesson structure preferences are compared with their perceptions of the teacher's lesson structure preference.

### *Focus Students' Lesson Structure Preference*

From both classes, the focus students' interview responses corroborated their post-lesson drawing task responses. This consistency is likely due to both data sources being completed at roughly the same time as each other and for the same lesson. Of importance is the fact that students seemed to report preferring the lesson structure they had just experienced. Recency is likely a factor for their preference. However, of interest in the current study were their reasons for these preferences and whether their perceptions of the teacher's preference were influential on their preferences.

Six focus students from the class that just finished their first lesson of the final week of teach-first lessons (Class A) reported preferring the teach-first lesson structure more than the task-first lesson structure. Students' reasons for favouring the teach-first structure predominantly included being better equipped for independently solving the challenging tasks that occurred later in the lesson. For example:

> I'd say maybe the mini lesson (teach-first) a bit more because when you start something new at first, you're not so good at it so maybe the mini lesson can help me by giving me more ideas for what I'm going to do. (S9)

By contrast, a few focus students from Class A either preferred the task-first lesson structure (*n* = 1), preferred both lesson structures to the same or a similar extent (*n* = 1), or were unsure of their preferences (*n* = 1). One student preferred the task-first lesson structure because it enabled hard thinking and challenged them: "…But for my opinion, I like sweaty brain time more because I feel it challenges my brain a lot more because it makes me think more and I like thinking hard and challenging my brain" (S3). The student who reported preferring both lesson structures to the same or a similar extent provided a mix of both reasons described above. Figure 2 provides a graph of Class A focus students' lesson structure preferences.

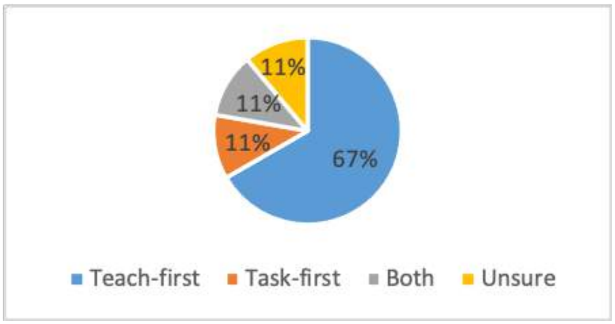



Figure 2. Class A focus students' lesson structure preferences.

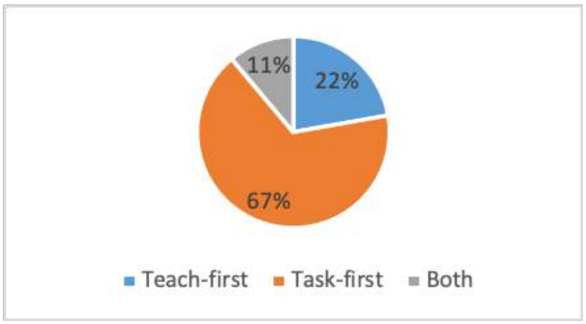



Figure 3. Class B focus students' lesson structure preferences.

On the other hand, six focus students from the class that had just finished their first lesson of the final week of task-first lessons (Class B) reported preferring the task-first lesson structure more than the teach-first lesson structure. Students' reasons for preferring the task-first lesson structure were similar to the reason provided by S3, but also include opportunities for independent thinking and learning. For example:

> I think the sweaty brain (task-first) lessons because it helps me dig for my own answers and work independently. Because Ms J doesn't give us answers, it makes it more open for us because there's a lot of options, instead of when your teacher shows you some ways to do something because that makes all of your other options shut down. You can put it on paper, discuss it with your teacher, and think about it. (S15)

Unlike the preferences of Class A focus students in which most students preferred the teach-first structure, only two focus students from Class B preferred the teach-first structure more than the task-first structure. The reasons for this preference include the low level of difficulty and the choice to think out loud. As one student indicated:

> I think it is the teach-first because it's easier for me but with sweaty brain time (task-first), it's a bit hard for me to stay quiet for a while. It's hard to think in my head when I can't think out loud. I think the mini lesson is helpful for my learning. (S12)

Furthermore, identical to Class A, one student reported preferring both lesson structures to the same extent with reasons surrounding differing benefits perceived in both lesson structures–teach-first being less stressful but task-first effective for improving your brain. Figure 3 shows Class B focus students' lesson structure preferences.

Overall, students' reasons for their lesson structure preferences resonate with findings by Russo and Hopkins (2017), which revealed students that preferred the teach-first lesson structure considered it to activate their cognition in preparation for solving challenging tasks, while students that endorsed task-first lessons perceived it to be more cognitively demanding.

### *Focus Students' Perceptions of the Teacher's Lesson Structure Preference*

Sixteen focus students developed a variety of opinions around the lesson structure preferred by Ms J. The perceptions of Class A students were almost identical to that of Class B, with the only difference being one more student from Class B perceiving Ms J preferred teaching the task-first lessons, while one less student from the same class perceived Ms J preferred teaching the teach-first lessons. Two students perceived Ms J preferred teaching both lesson structures to the same or a similar extent, while two students did not have an opinion on her preference. Figure 4 provides a graph of Class A focus students' perceptions of Ms J's lesson structure preference and Figure 5 provides a graph of Class B focus students' perceptions of Ms J's lesson structure preference.

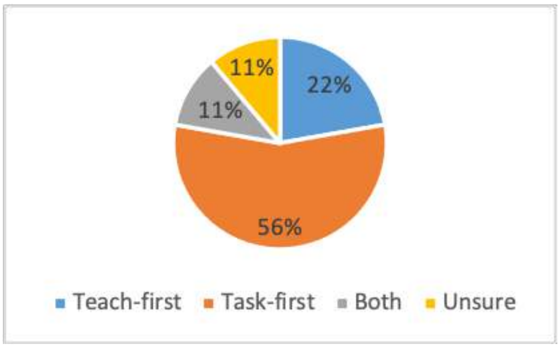



Figure 4. Class A focus students' perceptions of the teacher's lesson structure preference.

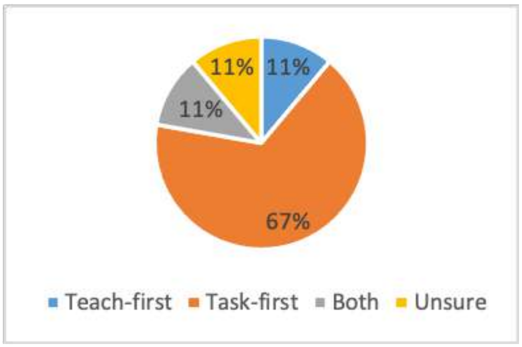



Figure 5. Class B focus students' perceptions of the teacher's lesson structure preference.

When prompted to explain the reasons for Ms J's perceived lesson structure preference, a variety of responses were given. While a few students that perceived Ms J to prefer the task-first lesson structure over the teach-first lesson structure offered reasons that reflect their perceptions of her work-related preferences (e.g., quietness during sweaty brain time, less likely to feel overwhelmed because the students are working more independently, option to talk to her friends while the students work), most students were aware of her reactions towards teaching the task-first lessons (e.g., looking more relaxed, smiling when spotlighting students and listening to their ideas, enthusiasm towards seeing students challenge their own thinking and deepening their learning in the absence of telling answers). One student indicated:

> I think she likes the sweaty brain (task-first) lessons too because every time she goes around, she puts a smile on her face and she always feels very interested about all the ways we're doing things so instead of doing teach-first, she gets to know where we're coming from with our opinion. I noticed she's very excited to learn about our thinking. (S15)

All students that perceived Ms J to prefer the teach-first lesson structure over the task-first lesson structure gave reasons that reflect their perceptions of her work-related preferences (e.g., more involvement with teaching, actively leading the whole class to show students possible solutions and strategies for the problem, keeping occupied during the lesson) but did not notice any display of positive behavioural reactions during those lessons. Such students were perhaps less tuned-in to Ms J's enjoyment. As described by one student: "I think she prefers teaching the teach-first because she can explain to us and she has something to do during class. But during task-first, she doesn't get to explain anything except fishbowls which the children explain themselves" (S14). The data shows that Year 3 and 4 students are insightful enough to notice what the teacher prefers and enjoys. Corresponding with findings from Frenzel et al. (2017), there seems to be a positive link between the reasons given for the students' lesson structure preferences, and the reasons given for Ms J's lesson structure preference. Despite a relatively small number of students not having an opinion, most students believed they figured out what Ms J felt was more enjoyable to teach.

### *Focus Students' Lesson Structure Preferences and their Perceptions of the Teacher's Lesson Structure Preference*

From Class A, a slightly larger number of focus students shared the same or a similar lesson structure preference as Ms J ($n$ = 4). Included in this data are students who presented reasons for preferring both lesson structures and/or perceiving Ms J to prefer both lesson structures. The remaining students either had a different preference from Ms J ($n$ = 3) or did not have an opinion on either their own lesson structure preference or Ms J's lesson structure preference ($n$ = 2). Figure 6 shows a graph comparing Class A focus students' lesson structure preferences with their perceptions of Ms J's lesson structure preference.

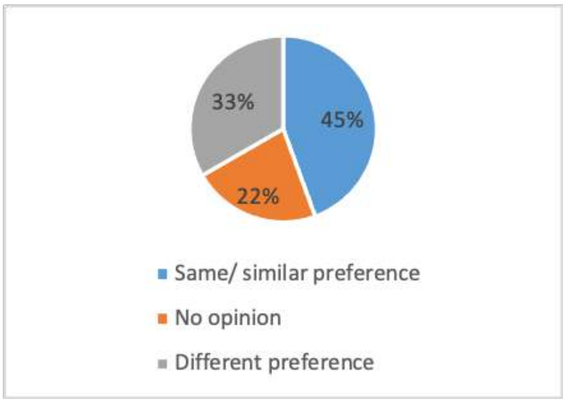



Figure 6. Comparison of Class A focus students' preferences and their perceptions of the teacher's preference.

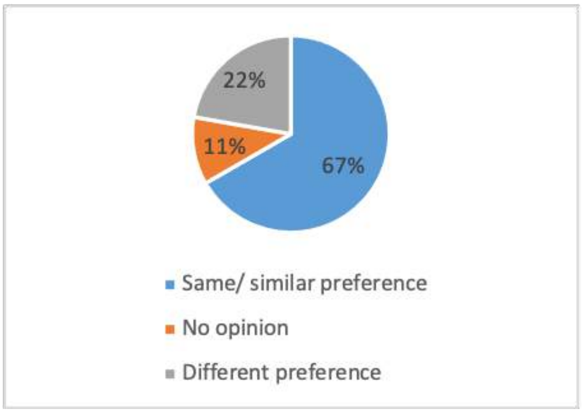



Figure 7. Comparison of Class B focus students' preferences and their perceptions of the teacher's preference.

From Class B, an even larger number of focus students shared the same or a similar lesson structure preference as Ms J ($n$ = 6). This data includes the student that presented reasons for perceiving Ms J to prefer both lesson structures. The remaining students either had a different preference from Ms J ($n$ = 2) or did not have an opinion on her preference ($n$ = 1). A graph representing the comparison between Class B focus students' lesson structure preference and their perceptions of Ms J's lesson structure preference is represented in Figure 7.

Interestingly, of the focus students that formed an opinion, 10 reported favouring the same lesson structure as they perceived Ms J did. Given the prevalence of students preferring the same lesson structure that they perceived their teacher preferred, it is possible that students' preferences were influenced by their perception of Ms J's lesson structure preference. This possibility resonates with Pekrun's (2000) control-value theory of achievement emotions, which claims that emotional experiences are affected by individuals' perceptions of their interaction partners' behaviours. Because students from both classes participated in both lesson structures across both topics throughout the intervention, students' preferences towards the lesson structure itself may have only partially contributed to their opinion. Perhaps some students preferred and enjoyed a particular lesson structure because they perceived their teacher to prefer and enjoy teaching it. As established in findings by Frenzel et al. (2017), there is a positive reciprocal relationship between the teacher's enjoyment and the students' enjoyment which are noticed through observations of each other's classroom behaviours.

## Conclusion

This study aimed to explore in-depth the reasons for the students' preference for a lesson structure based on how it supported their learning, and whether their perceptions of the teacher's preference for a lesson structure appeared to influence their own preference. Overall, the results suggest that students preferred either the teach-first or task-first lesson structure for various reasons relating to perceived benefits to their learning, that most students had formed an opinion on their teacher's lesson structure preference, and that there seems to be a positive reciprocal relationship between the lesson structure the students enjoy and what they perceive their teacher to enjoy. While limitations of this study include a small number of students who were either frequently absent, unable to form an opinion and/ or provide informative and sensible responses, the majority of students were aware of their perceptions. Therefore, implications of this study include the importance of teachers knowing that their own enjoyment of teaching a lesson can inadvertently influence their students' enjoyment of mathematics.

## Acknowledgements

EMC[3] was funded by the Australian Research Council, Catholic Education Diocese of Parramatta and Melbourne Archdiocese Catholic Schools (LP 180100611).

## References

Bakker, A. B. (2005). Flow among music teachers and their students: The crossover of peak experiences. *Journal of Vocational Behavior, 66*(1), 26–44. https://doi.org/10.1016/j.jvb.2003.11.001

Becker, E. S., Goetz, T., Morger, V., & Ranellucci, J. (2014). The importance of teachers' emotions and instructional behavior for their students' emotions—An experience sampling analysis. *Teaching and Teacher Education, 43*, 15–26. https://doi.org/10.1016/j.tate.2014.05.002

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Csikszentmihalyi, M. (1997). *Finding flow: The psychology of engagement with everyday life* (1st ed.). BasicBooks.

Frenzel, A. C., Becker-Kurz, B., Pekrun, R., Goetz, T., & Lüdtke, O. (2017). Emotion transmission in the classroom revisited: A reciprocal effects model of teacher and student enjoyment. *Journal of Educational Psychology, 110*(5), 628–639. https://doi.org/10.1037/edu0000228

Frenzel, A. C., Goetz, T., Lüdtke, O., Pekrun, R., & Sutton, R. E. (2009). Emotional transmission in the classroom: Exploring the relationship between teacher and student enjoyment. *Journal of Educational Psychology, 101*(3), 705–716. https://doi.org/10.1037/a0014695

Keller, M. M., Hoy, A. W., Goetz, T., & Frenzel, A. C. (2016). Teacher enthusiasm: Reviewing and redefining a complex construct. *Educational Psychology Review, 28*(4), 743–769. https://doi.org/10.1007/s10648-015-9354-y

Martin, A. J., Way, J., Bobis, J., & Anderson, J. (2015). Exploring the ups and downs of mathematics engagement in the middle years of school. *The Journal of Early Adolescence, 35*(2), 199–244. https://doi.org/10.1177/0272431614529365

Pekrun, R. (2000). A social-cognitive, control-value theory of achievement emotions. In J. Heckhausen (Ed.), *Motivational psychology of human development* (pp. 143–163). Oxford, UK: Elsevier. http://dx.doi.org/10.1016/S0166-4115(00)80010-2

Polkinghorne, D. E. (2005). Language and meaning: Data collection in qualitative research. *Journal of Counselling Psychology, 52*(2), 137–145.

Renninger, K. A., & Hidi, S. (2016). *The power of interest for motivation and engagement*. New York, NY: Routledge. https://doi.org/10.4324/9781315771045

Russo, J., & Hopkins, S. (2017). How does lesson structure shape teacher perceptions of teaching with challenging tasks? *Mathematics Teacher Education and Development, 19*(1), 30–46.

Russo, J., & Hopkins, S. (2019). Teaching primary mathematics with challenging tasks: How should lessons be structured? *The Journal of Educational Research, 112*(1), 98–109. https://doi.org/10.1080/00220671.2018.1440369

Sullivan, P., Bobis, J., Downton, A., Hughes, S., Livy, S., McCormick, M., & Russo, J. (2020). Ways that relentless consistency and task variation contribute to teacher and student mathematics learning. In A. Coles (Ed.) *For the learning of mathematics monograph 1: Proceedings of a symposium on learning in honour of Laurinda Brown* (pp. 32–37). FLM Publishing Association.

Sullivan, P., Borcek, C., Walker, N., & Rennie, M. (2016). Exploring a structure for mathematics lessons that initiate learning by activating cognition on challenging tasks. *The Journal of Mathematical Behavior, 41*, 159–170. https://doi.org/10.1016/j.jmathb.2015.12.002

# Hypothetical Data Analysis and Representation in Year 4

Noleine Fitzallen
*University of New South Wales*
n.fitzallen@unsw.edu.au

Jane Watson
*University of Tasmania*
jane.watson@utas.edu.au

This paper reports on students' experiences of describing and representing variation in hypothetical data. Fifty-six students (8–9 years-old) experienced collecting and working with quantitative data for two years as part of a STEM education project. The task described here was an end-of-year survey question, with three parts about a hypothetical context for surveying students in two different Australian cities: recording the data, describing the potential variation in the data, and creating a representation of what the data might look like when only a descriptive account of the context and variables were provided. The data analysis framework utilised provides a means of determining students' readiness for further development of statistical ideas.

Interest in primary school students' understanding of data, appreciation of the existence of variation in data, and representation of that variation visually has grown tremendously over the past 30 years, as national school curricula have included statistics. Often activities in the early years involved providing students with contexts and data sets considered appropriate for their year level. For example, students were presented with a set of "data cards," each including data for several common variables, which provided a hands-on starting point for representation, as students placed the physical cards on a table in a manner to represent at least one of the variables on the cards (e.g., Watson et al., 1995). This was extended as students were encouraged to create representations on paper for the variables that interested them, with quite diverse consequences (e.g., Chick & Watson, 1998). Because technology has become available more recently, much research has focused on student use of data analysis software to create representations for exploring data (e.g., Fitzallen, 2012). There continues, however, to be interest among early childhood and primary school educators and researchers in hand-drawn representations. Estrella (2018), for example, considered pre-school and Year 2 students' creations of representations as part of early transnumeration, and Leavy and Hourigan (2018) examined the initial representations and creative explanations of 5–6-year-olds when asked to "collect data" while listening to a story about selecting animals for a zoo. Watson (2018) also reported on 6-year-olds' creations of representations with concrete materials and drawings of representations on paper in chance and weather contexts. In each of these studies, data were provided in a context for students to represent in some way, in anticipation of providing a summary or answering a question.

Using data to answer statistical questions is one component of the *Practice of Statistics* (Watson et al., 2018): Pose a Question, Collect Data, Analyse the Data, and Interpret the Results. The above studies created specific opportunities for students to work through the "analyse data" component related to creating visual representations for data they were given or collected. Visual representations are an avenue for displaying the variation that occurs in data, variation that is needed for explanation of the results of the investigation. As part of the context for posing a question and collecting data, each of the studies noted above provided data for the beginning of the analysis to occur. The purpose of the practice of statistics is for it to be applied in contexts where investigators encounter novel ideas to study, which require imagining the question to ask and what the associated data will look like; in particular, identification of the variation that underlies the practice of statistics, as without variation in data, there is no need for the practice!

Earlier research conducted by Fitzallen (2012), asked Grade 4–6 students to draw a graph for any data and context of their choosing. Outcomes from that research suggested that students often address context and data separately and make few connections among relevant contexts and data that may be generated. Moritz (2004) also asked students to speculate about the data represented in

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 203–210). Newcastle: MERGA.

graphical representations. No other studies were found that had the approach of providing a context but no data. There are times when hypothesising about the context and its potential provides the starting point for posing one's questions and deciding on actual data to be collected. English et al. (2017), however, found students experience difficulties writing statistical questions that have the potential to collect meaningful and relevant data.

In the research reported in this paper, interest was in students' capacity to hypothesise in terms of foundational elements of the practice of statistics. Students were provided a context and given the opportunity to imagine the possibilities for recording data, recognising variation, and creating a representation, but given no data with which to do so. To explore this aspect of students' development half-way through a 4-year project when the students were in the final stages of Year 4, it was decided to choose a topic believed to be very familiar to the students but not dependant on or reflected in the topics covered previously in the project. Hence the next section describes the topics students had thus far encountered.

## Background on Student Experience

The students in this study were part of a 4-year project from Year 3 to Year 6 integrating data and the informal practice of statistics in STEM (Science, Technology, Engineering, Mathematics) contexts. See Fitzallen and Watson (2020) for a summary of all activities over the four years. Up until the time of the task explored here at the end of Year 4, five other experiences with data representation and variation had been encountered by the students, one in a pre-study survey, and four in classroom activities as part of the project.

In the Year 3 pre-study survey, students were asked a three-part question on data: (a) What do you think "data" means? (b) Give an example of data you have seen or collected. (c) Sketch a graph of the data (Watson & Fitzallen, 2021). The graph responses to part (c) reflected the suggestions of the *Australian Curriculum: Mathematics Version 8.4* for Year 4 (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2018) with 24% being tables/lists, 9% being pictographs, and 67% being column graphs. Nearly half of the students related their representations meaningfully to the examples they had provided in part (b), mostly related to food. This indicated their experiences with varying contexts may have been limited.

The first activity in the study involved making "licorice sticks" with Play-Doh™ two different ways, by hand and with a Play-Doh™ "machine," to introduce the concept of variation (Watson et al., 2020). Again, using the representations suggested in the curriculum, 56% of students were able to represent the three data values for each of the three people in their groups for the hand-made sticks; 81% of students could later do so for the machine-made sticks. At the end of the activity when asked to tell the story of the class investigation in a picture and in words, 21% presented two appropriately labelled graphs and 11% explained the difference in the two methods using a version of the word "variation." On the end of Year 3 survey when asked again about the Play-Doh™ activity, 43% could picture the difference of the two methods of production with 11% labelling the axes, whereas 23% described the difference in variation.

The other activity in Year 3 involved the topic of heat from the Science curriculum with students in groups of three testing the cooling rate of hot water in insulated and non-insulated plastic cups, placed in a cold-water bath (Fitzallen et al., 2017). Temperature (°C) was the measure of change. For this activity students did not create the graphing format but gained experience in plotting data values on stylised graphs created from images of thermometers, and using the variation seen over time to explain the difference in the way in which the water cooled for the three conditions. The results indicated young students have the capacity to describe differences in variation among and between treatments from graphical representations.

At the beginning of Year 4 the students were introduced to the complete practice of statistics (Watson et al., 2018) to compare their life experiences with those of students in another city: posing and refining questions; collecting data on-line; making and analysing representations of their data in terms of variation and trend; and drawing conclusions, acknowledging uncertainty (Watson et al., 2019). The questions posed were critiqued and refined by the students with help from the teachers and researchers (English et al., 2017), and then 22 questions of six types (e.g., numerical, multiple choice, yes/no) were chosen by the students and answered by 85 students across the two cities. In terms of the representations created, 87% realised the importance of displaying all the data, with 64% including a written summary of the data in the representation. The second activity in Year 4 took place over two terms, carrying out fair tests involving testing catapults. After initial tests the force on the throwing mechanisms was increased and trials were repeated (Watson et al., 2023). In the initial trials, before entering data into the software program *TinkerPlots* (Konold & Miller, 2015), students created graphical representations of their initial trials by hand. At this point, 11% of the representations were considered idiosyncratic, with 15% being tallies, lists, or pictographs. Other representations were either bar or column graphs. Of the remaining column/bar graphs, four sub-types were identified: sequential trial ordering with distance on the y-axis, sequential trial ordering with distance on the x-axis, ordered data with distance on the y-axis, and frequency bar charts in 10 cm intervals along the x-axis. Outcomes from these activities indicated that, given the opportunity to engage in statistical investigations, young students have the capacity to create graphical representation that are meaningful to them, albeit at varying levels of understanding (Watson et al., 2023).

Following the background of students with these highly structured data-based activities and in particular their experience with the complete practice of statistics at the beginning of Year 4, the question arises about student ability to hypothesise about data, variation, and representation in a new imagined context. This leads to the following research question:

- How do students hypothesise about a new context for investigation without data being provided?

## Research Approach and Participants

The research adopted a pragmatist paradigm (Mackenzie & Knipe, 2006) to explore young students' capacity to hypothesise in terms of foundational elements of the practice of statistics. The methodology was chosen to capture specifically the students' level of understanding gleaned from responses to a survey question. Following the suggestion of Ballou (2008) that open-ended questions are appropriate to gain insight into how terms are understood and how associated ideas are developed, the three-part task summarised in Figure 1 was developed. The question was included in a survey administered at the end of Year 4, reflecting to some extent the initial activity in Year 4, where students compared lifestyles in their city with another city. Note that the cities named in the task in Figure 1 are the capitals of a state and territory in Australia, and it was reasonable to assume students appreciated their locations on opposite sides of the continent with different climatic conditions and environments for native wildlife. Further, the teachers confirmed that this was a reasonable task to present to the students. Subsequently, data were collected from 56 students in two classes from a parochial school in an Australian capital city, whose parents gave permission for the data to be collected. The average age of the students was 10.5 years, and the gender split was 60% male and 40% female. The project had ethics approval from the Tasmanian Social Sciences Human Research Ethics Committee (H0015039). Each student's work was assigned a code to maintain anonymity.

| |
|---|
| A Grade 4 class in Melbourne and a Grade 4 class in Darwin were asked to fill in a survey. As part of the survey, they were asked to:<br>***List three of the native wildlife that are found in your state.***<br>(a) How would you record the data for this survey item?<br>(b) Describe the variation you might get in the data for this survey item.<br>(c) If you were asked to present the data, what would your representation look like? Use the space below to sketch what your representation might look like. |

Figure 1. Task completed at the end of Year 4.

## Data Analysis

The data coding scheme for responses to the questions in Figure 1 was informed by the SOLO model of Biggs and Collis (1989) as extended by Groth et al. (2021). The aspects of the model employed here relate to the Ikonic and Concrete Symbolic modes. The Ikonic (IK) mode (from about 18 months) precedes the Concrete Symbolic (CS) mode (from about 6 years). In the CS mode, the basis for beginning learning in school, there are potentially three levels of reasoning: Unistructural (U), where a single element relevant to the task is employed in a response; Multistructural (M), where two or more elements are employed in sequence; and Relational (R), where links are created among two or more elements relevant to the task (Biggs & Collis, 1989). In considering the IK mode, Groth et al. found it useful to distinguish two types of responses: those that are "normative incompatible," IK(ic), with the task, such as myths, superstitions, and subjective ideas out of context, or "normative compatible," IK(c), with the task, such as personal experience, imagery, or intuition in the context but not employing specific elements of the task. Distinction in the IK mode has the potential to identify IK(c) students, who are ready for transition to the CS mode (Groth et al., 2021; Watson & Fitzallen, 2021; Watson et al., 2022). Using the coding scheme described in Table 1, the second author and a research assistant coded the responses to the task separately. Agreement on coding was 84% and differences were agreed by negotiation. Tables 2, 3, and 4 provide examples of responses to each of the three questions in Figure 1, coded using the rubric in Table 1, along with the percentages of response at each SOLO level.

**Table 1**

*Coding Rubric for the Task in Figure 1*

| Level | Part (a) Record data | Part (b) Variation in data | Part (c) Represent data |
|---|---|---|---|
| IK(ic) | Data unrelated to question | Not addressing variation | No evidence of context |
| IK(c) | Noting data in context but not recording of data | Variation not related to wildlife context | Contextual drawing without reference to data |
| CS(U) | Reference to a single type of representation | Single reference to difference in the context | Single aspect of data or variables represented |
| CS(M) | Reference to more than one type of representation | One aspect of potential difference in the data | Multiple aspects of data represented |
| CS(R) | NA | NA | Multiple aspects of both variables and data represented |

**Table 2**

*Levels of Response for Part (a): Record the Data*

| Level | How would you record the data for this survey item? | % |
|---|---|---|
| IK(ic) | What different foods do they eat? [ID121]<br>I would find out if the places are for warmer or colder areas. [ID149]<br>A lot because a lot escape in zoos. That's why. [ID154] | 14.3 |
| IK(c) | Because the wildlife can help you record. [ID159]<br>Write down your animal's name and bring in a photo of it. [ID128]<br>I would collect data from that survey. [ID147] | 23.2 |
| CS(U) | I would draw a table. [ID101] You could use a tally. [ID131]<br>A graph. [ID146] I would use TinkerPlots. [ID130] | 48.2 |
| CS(M) | A graph or a tally sheet. [ID142] Picture chart with tallies. [ID139]<br>In a column chart or a number chart. [ID111] | 14.3 |

**Table 3**

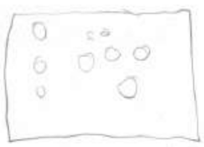
*Levels of Response for Part (b): Variation in the Data*

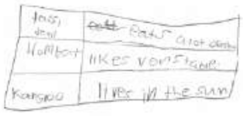
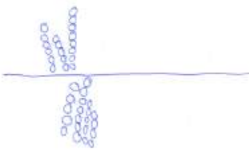
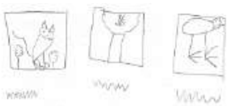
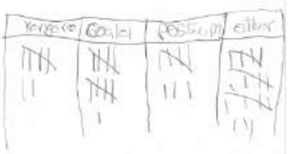
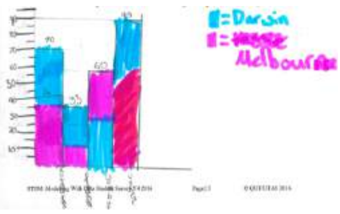
| Level | Describe the variation you might get in the data for this survey question* | % |
|---|---|---|
| IK(ic) | What one is hotter and what is colder and how many people live there. [ID139]<br>You would go on the computer. [ID147] I am going to make a bar graph [ID155] | 13.7 |
| IK(c) | How many different foods each one eats. [ID121] How fast they are. [ID141] Some people might not have native animals. [ID143] | 27.5 |
| CS(U) | Darwin might not have the same wildlife as Melbourne. [ID102] I think it would be quite close to each other. [ID111] No one will get the same answer. [ID131] They both might have kangaroos which would be a similarity. [ID158] | 33.3 |
| CS(M) | You might get different animals or different habitats. [ID144]<br>Darwin might have more kangaroos than Melbourne or in Darwin they might see cockatoos while in Melbourne they don't see them at all. [ID114]<br>You might only get 1 animal in the group or No animals in a group. [ID125] | 25.5 |

*Five students did not answer this question.

**Table 4**

*Levels of Response for Part (c): Represent the Data*

| Level | If you were asked to represent the data, what would your representation look like? Use the space below to sketch what your representation might look like.* | % |
|---|---|---|
| IK(ic) |  [ID147]  [ID140]  [ID160] | 25.0 |

| Level | If you were asked to represent the data, what would your representation look like? Use the space below to sketch what your representation might look like.* | % |
|---|---|---|
| IK(c) |  [ID106]  [ID157]  [ID164] | 9.6 |
| CS(U) |  [ID154]  [ID128]  [ID148] | 15.4 |
| CS(M) |  [ID109]  [ID105]  [ID101] | 36.5 |
| CS(R) |   [ID104]   [ID116] | 13.5 |

*Four students did not answer this question.

## Discussion

The levels of response in Tables 2, 3, and 4 illustrate the differences in the students' responses when asked to imagine data for a context that was generally familiar to them. For each part of the question, over a third of students responded in the IK mode. In these cases, having the distinction of normative compatible or incompatible was useful in distinguishing those who appreciated the context of the question, a survey on wildlife. Of these IK responses, 47% were IK(ic), and 53% were IK(c), indicating as suggested by Groth et al. (2021), the potential for over half of these students to move to the CS mode. Across the questions, 35% of responses were CS on all three, whereas 18% were all IK. Of interest is the comparison of these results with the results of the pre-study survey question from two years earlier, where students were asked specifically to name and create a graph of some data of their choice (Watson & Fitzallen, 2021), where IK responses were below 20% for all three questions. Perhaps this was related to the opportunity to choose any context to answer the question in conjunction with experience gained from hands-on activities in contexts where they collected, discussed, and represented data.

No other studies were found with results that could be compared directly with those of this study. The studies referred to in the Introduction were analysed with different methods. This study illustrates a way of identifying when student learning moves developmentally from the IK mode to the CS mode. The SOLO levels used here, and by Watson et al. (2022), has the potential to be used as a framework for structuring learning trajectories and developing assessment hierarchies. Young children need to be given opportunities to hypothesise interesting data and create representations to tell the stories related to the context of the data they imagine are in the data. Such opportunities may support them to develop statistical thinking that enhances and goes beyond the outcomes expected from regular classroom activities that usually require students to draw conclusions from data and graphical representations provided (Estrella, 2018; Leavy & Hourigan, 2018).

## Practical Implications

The framework displayed in Table 1, provides a means of determining students' readiness for further development of statistical ideas and the research results show that there are benefits to giving young students opportunities to explore, create, and represent data in unconventional ways, even though some students may experience difficulties enacting elements of the practice of statistics when not provided with specific data to analyse. Such opportunities may support students to be able to apply the creative skills needed to imagine the data needed to answer statistical questions, to make projections about how the data may be collected, and to suggest various ways in which the data can be represented, which are all required to address Statistics outcomes in the *Australian Curriculum: Mathematics Version 9* (ACARA, 2022). In Year 1, for example, students are expected to "review data collected and explain how they might change the way they collect data next time." In Year 4, there is the expectation students will engage in the practice of statistics by "constructing graphs of data collected through observation during science experiments, recording, interpreting and discussing the results in terms of the scientific study."

As well as supporting learning in mathematics, opportunities abound to apply the practice of statistics in cross-curriculum activities. As students begin to apply their developing understanding of the practice of statistics in other subjects across the curriculum, particularly in planning project work, the ability to hypothesise about different contexts and how data are to be recorded and represented to show variation is going to become invaluable, particularly in the initial stages of an investigation that requires the posing of a question to explore. At each year level from Foundation to Year 7 across the Humanities and Social Sciences (HASS) curriculum (ACARA, 2022), the content descriptors for Inquiry and Skills includes "Researching," which itself says, "Locate and collect information and data from different sources," including "sources provided," "observations," "primary sources," and/or "secondary sources." Across the Science curriculum content descriptions for the same years require students to be working with data and undertaking experiments that generate data. Also, the Technologies curriculum at Years 3–4 expects students to develop the skills to "recognise different types of data and explore how the same data can be represented differently depending on the purpose" (Digital Technologies), and at Years 5–6, gather relevant data to "evaluate design ideas, processes and solutions" (Design & Technologies). Developing foundational understanding of initiating and following through with a statistical investigation in mathematics in the early years of schooling has the potential to foster student learning across many learning areas of primary and secondary education.

## Acknowledgments

The research was funded by an Australian Research Council (ARC) Discovery Grant (DP150100120). The authors thank Ben Kelly for assisting with the data analysis and giving feedback on early drafts of the paper.

# References


Australian Curriculum, Assessment and Reporting Authority. (2018/2022). *The Australian curriculum. Version 8.4 and Version 9.0.* https://www.australiancurriculum.edu.au/f-10-curriculum/

Ballou, J. (2008). Open-ended question. In P. J. Lavrakas (Ed.), *Encyclopedia of survey research methods* (pp. 548–550). SAGE Publications.

Biggs, J., & Collis, K. (1989). Towards a model of school-based curriculum development and assessment using the SOLO taxonomy. *Australian Journal of Education*, *33*(2), 151–163.

Chick, H. L., & Watson, J. M. (1998). Showing and telling: Primary students' outcomes in data representation and interpretation. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times*. *Proceedings of the 21st annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 153–160). MERGA.

English, L., Watson, J., & Fitzallen, N. (2017). Fourth-graders' meta-questioning in statistical investigations. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning. Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 229–236). Melbourne: MERGA.

Estrella, S. (2018). Data representations in early statistics: Data sense, meta-representational competence and transnumeration. In A. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in early childhood and primary education: Supporting early statistical and probabilistic thinking* (pp. 239–256). Singapore: Springer.

Fitzallen, N. (2012). *Students' reasoning about covariation with TinkerPlots* [Doctoral dissertation, University of Tasmania]. Tasmania: Australia.

Fitzallen, N., & Watson, J. (2020). Using the practice of statistics to design students' experiences in STEM. In B. Shelley, K. te Riele, N. Brown, & T. Crellin (Eds.), *Harnessing the transformative power of education* (pp. 74–99). Brill. https://doi.org/ 10.1163/9789004417311_007

Fitzallen, N., Watson, J., & Wright, S. (2017). The heat is on! Using a stylised graph to engender understanding. *Australian Primary Mathematics Classroom*, *22*(2), 3–7.

Groth, R. E., Austin, J. W., Naumann, M., & Rickards, M. (2021). Toward a theoretical structure to characterize early probabilistic thinking. *Mathematics Education Research Journal*, *33*(2), 241–261.

Konold, C., & Miller, C. D. (2015). *TinkerPlots: Dynamic data exploration* [Computer software, Version 2.3.2].

Leavy, A., & Hourigan, M. (2018). Inscriptional capacities and representations of young children engaged in data collection during a statistical investigation. In A. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in early childhood and primary education: Supporting early statistical and probabilistic thinking* (pp. 89–107). Springer.

Mackenzie, N., & Knipe, S. (2006). Research dilemmas: Paradigms, methods and methodology. *Issues in Educational Research, 16*, 193–205. https://www.iier.org.au/iier16/mackenzie.html

Moritz, J. B. (2004). Reasoning about covariation. In D. Ben-Zvi, & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 227–255). Kluwer Academic Publishers.

Watson, J. (2018). Variation and expectation for six-year-olds. In A. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in early childhood and primary education: Supporting early statistical and probabilistic thinking* (pp. 55–73). Springer.

Watson, J. M., Collis, K. F., Callingham, R. A., & Moritz, J. B. (1995). A model for assessing higher order thinking in statistics. *Educational Research and Evaluation, 1*, 247–275.

Watson, J., & Fitzallen, N. (2021). What sense do children make of "data" by Year 3? In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in mathematics education: Foundations and pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 409–416). Singapore: MERGA.

Watson, J., Fitzallen, N., English, L., & Wright, S. (2020). Introducing statistical variation in Year 3 in a STEM context: Manufacturing licorice. *International Journal of Mathematical Education in Science and Technology*, *51*(3), 354–387.

Watson, J., Fitzallen, N., Fielding-Wells, J., & Madden, S. (2018). The practice of statistics. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 105–137). Springer International Publishing

Watson, J., Fitzallen, N., & Wright, J. (2019). Practicing statistics in year 4. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 739–746). MERGA.

Watson, J., Wright, S., & Fitzallen, N., Kelly, B. (2022). Consolidating understanding of variation as part of STEM: Experimenting with plant growth. *Mathematics Education Research Journal.* http://doi.org/10.1007/s13394-022-00421-1

Watson, J., Fitzallen, N., Wright, S., & Kelly, B. (2023). Characterizing the ways in which young students recognise describe, explain and employ variation when analysing data in a STEM context. *International Journal in Science and Mathematics Education, 21*, 943–968. https://doi.org/10.1007/s10763-022-10281-7

# Utilising the Expertise of Specialist Intervention Teachers in Primary Mathematics Classrooms

Ann Gervasoni
*Monash University*
ann.gervasoni@monash.edu

Kerry Giumelli
*CSPD*
kgiumelli@parra.catholic.edu.au

Linda Flanagan
*CSPD*
lflanagan@parra.catholic.edu.au

Ann Downton
*Monash University*
ann.downton@monash.edu

Anne Roche
*Monash University*
anne.roche@monash.edu

Owen Wallis
*CSPD*
owallis@parra.catholic.edu.au

Typically, more primary students qualify for mathematics intervention support than schools have the capacity to assist. This highlights the importance of every primary mathematics teacher having the expertise to design inclusive and responsive curricula and instruction for students who may experience difficulty. Our study addresses this issue through exploring how specialist intervention teachers can support the mathematics professional learning of teachers using co-teaching cycles. The findings provide insight about the challenges experienced by classroom teachers when teaching students who experience difficulty with mathematics, and the actions of the specialist intervention teachers that assisted the teachers' professional growth.

Most primary teachers face the dilemma of how best to support the students in their class who are not yet thriving with learning mathematics. Typically, in Australia, primary teachers are generalists who have not had the opportunity to develop the specialist knowledge associated with diagnosing the mathematics difficulties that some students encounter, nor planning responsive teaching. To assist teachers to support students who experience difficulty, some principals employ a specialist mathematics teacher to run intervention programs or engage mathematics leaders to collaboratively plan with teachers and provide professional learning. Our study proposes that the school's specialist mathematics intervention teacher has the expertise to support classroom teachers with whole class mathematics planning and instruction.

The pilot study presented in this paper aimed to gain insight into how the expertise of the school's specialist mathematics intervention teacher might be utilised to support classroom teachers to increase their capacity to assist students who experience difficulty with learning mathematics. Within the context of the broader study, the specific research questions addressed in this paper are:

- What are the challenging aspects of teaching students who experience difficulty with mathematics for classroom teachers; and
- What actions of the specialist intervention teachers do classroom teachers report help them to support students who experience difficulty with mathematics.

## Background Literature

According to the OECD's Programme for International Student Assessment (PISA), 22% of 15-year-old Australian students are low performers in mathematics and around 46% do not attain the National Proficient Standard (Thomson et al., 2019). In response, schools employ a range of intervention approaches that typically fall into three tiers (Bryant et al., 2008). Sonnemann and Hunter (2023) described Tier 1 as high-quality classroom instruction that meets the needs of all students, Tier 2 as targeted small group support for about 15% of students who fall behind, and Tier 3 as intensive one-on-one support for students who make minimal progress in Tier 2. While intensive

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 211–218). Newcastle: MERGA.

interventions have been shown to be effective (Nickow et al., 2020), not all students who may benefit are able to access these. Hence, building the capacity of teachers to provide high-quality mathematics instruction for all is paramount.

### *Professional Learning*

Many schools employ a mathematics leader or coach to provide in-situ professional learning for teachers focused on developing high-quality instruction. Previous studies have demonstrated the effectiveness of this approach for enhancing mathematics teaching (Anstey & Clarke, 2010; Sexton & Downton, 2014), and for increasing student achievement (Bruce et al., 2010). One model for leading classroom embedded professional learning is using the co-teaching cycle (Sharratt & Fullan, 2012) comprising: co-planning, co-teaching, co-debriefing, and co-reflecting. Cobb et al. (2019) also identified modelling instruction, co-teaching, co-planning, and debriefing as potentially productive activities for mathematics coaches, and teachers have indicated that "modelling, observation, and debriefing were the most valuable components" of a professional learning model (Butler et al., 2004, p. 447).

Typically, professional learning based on co-teaching cycles address aspects of practice that teachers wish to improve. Previous research has highlighted that high-quality student-centred mathematics instruction promotes collaboration, problem solving, dialogue, and using tasks with enabling and extending prompts through which all students can access the task (Russo et al., 2020). However, teachers and mathematics leaders have indicated that one of the most difficult aspects of teaching mathematics for students with diverse abilities is knowing how best to differentiate instruction to meet the needs of all students (Downton et al., 2022; Gervasoni et al., 2021). Further insight is needed about the particular challenges that teachers face when addressing the needs of students who are struggling to learn mathematics within inclusive, student-centred settings, and how teachers' professional learning can be supported.

## Context for the Pilot Study

The pilot study took place in three primary schools situated within a system of 58 schools in Sydney. Twelve years earlier this school system launched a mathematics initiative that included professional learning for principals and teachers (Roche & Gervasoni, 2017) based on constructivist aligned and inquiry-based mathematics teaching (Russo et al., 2020). The initiative included an annual task-based mathematics assessment (Clarke et al., 2002) enabling each student's growth to be monitored, and those who were mathematically vulnerable to be identified. The *Extending Mathematical Understanding* (EMU) intervention program (Gervasoni et al., 2021; Gervasoni, 2015) was introduced for students who were mathematically vulnerable. The EMU program is taught by certified specialist teachers (ST) who complete an initial course and annual professional learning. The theoretical underpinnings of the intervention, teaching approach, and lesson structure are described in detail in Gervasoni (2015). Each lesson focuses on whole number learning, mathematical problem-solving, engagement with open tasks, and reflection on the mathematical focus of the lesson. More than 400 teachers in the system qualified as EMU STs over the past 12 years.

An issue faced by school leaders in this system is that not all eligible students are able to access an EMU intervention program. For example, in a 2018 study involving 57 schools in the same system, Gervasoni et al. (2019) found that only 23% of 1471 Grade 1 students who were mathematically vulnerable were able to access the EMU intervention program in their school. Our study responds to this situation through harnessing the expertise of EMU STs to support classroom teachers' professional learning. Through experience with EMU intervention, the STs had developed expertise in differentiating instruction for groups of three students, guided by diagnostic assessment and the Early Numeracy Research Project (ENRP) growth point framework (Clarke, 2013). They had designed lessons based on problem-solving and engagement with open tasks, and were experienced with: selecting concrete models to assist students' construction of knowledge; prompting students to visualise and explain their thinking and strategies for each other; and developing students' confidence and positive dispositions for mathematics. These were all examples of high-quality instruction relevant for Tier 1 mathematics teaching.

## Method

Mixed methods were chosen as most relevant for addressing the research questions in this pilot study. The research design involved EMU STs leading co-teaching cycles for Grade 1 and Grade 2 classroom teachers for at least 10 weeks during Term 2 and Term 3. At the conclusion of the co-teaching cycles, participants were surveyed using an online platform (Qualtrics) and interviewed (via zoom). The research followed the approved ethical guidelines, and pseudonyms are used for the classroom teachers. Results and findings for classroom teachers are the focus for this paper.

### *Data Collection Instruments and Data Analysis*

The teacher survey included Likert style items in which they rated their confidence for teaching mathematics, any change in knowledge and pedagogical approaches following the co-teaching cycles, and the extent to which the actions of the EMU ST during the co-teaching cycles contributed to their professional learning. Open response items investigated what teachers considered most challenging about teaching students who were struggling with mathematics, and any other support from the EMU Specialist that they found valuable. The semi-structured interviews aimed to provide greater depth and clarity about the nature of the EMU ST support that teachers received, and their perceptions of the impact of the support. The Likert-style survey responses were summarised. Open response items and the transcribed interview data were analysed using constructivist grounded theory methods (Charmaz, 2014). A narrative is used to present the findings concerning the challenges teachers described and insights about the support provided by the EMU ST.

### *Co-Teaching Cycles*

The co-teaching cycles for the study took place during Term 2 and Term 3, and involved weekly planning meetings, co-teaching at least twice each week, and time for co-debriefing and co-reflection. Each school selected classes for the pilot based on analysis of their school mathematics assessment data (Gervasoni et al., 2021) and the proportion of students who were mathematically vulnerable. School A selected Grade 1 for Term 2 and Grade 2 for Term 3, and 21% and 39% of students in these classes, respectively, were vulnerable in at least one number domain. Both School B and School C selected Grade 1 classes for both terms. The proportion of students who were vulnerable in School B and School C was 19% and 25%, respectively.

Each EMU ST in the pilot received professional learning prior to implementing the co-teaching cycles, which was facilitated by the system EMU Professional Learning Leaders (PLLs) in 2021. The professional learning focused on the key components of the co-teaching cycle (Sharratt & Fullan, 2012), the features of high-quality instruction for student-centred learning (Russo et al., 2020) and processes for monitoring the progress of students. In 2022, ongoing support from the PLLs included two check-ins via zoom or email each term, collegial visits, providing professional readings, and discussing student assessment.

### *Participants*

The three schools for the pilot were purposefully selected based on: (1) support and commitment of the school principal; (2) having an EMU intervention program in place; and (3) having an experienced EMU ST on staff who gave consent to participate in the study and was released from classroom teaching. The participants were an EMU ST from each school, four Grade 1 teachers, and one Grade 2 teacher. One classroom teacher participated in the survey but chose not to participate in the interview. The classroom teachers had between one- and seven-years teaching experience. The EMU Specialist Teachers in School A, School B, and School C had implemented EMU intervention programs for 7, 4, and 5 years, respectively.

## Results and Findings

### *Challenging Aspects of Teaching Mathematics to Students who Experience Difficulty*

To gain insight about the challenges that classroom teachers faced when teaching students who experienced difficulty in mathematics, the five participating classroom teachers were invited to describe their challenges in an open response item in the survey, and in the semi-structured interview. Two themes emerged from the analysis of data: (1) Differentiating instruction to respond to student difficulties in the moment; and (2) Finding time within a lesson to work with students who are mathematically vulnerable. These themes are described below, using illustrative examples from the survey responses and interview transcripts. It is important to note that all teachers typically implemented a student-centred approach for mathematics using challenging tasks with enabling and extending prompts (Russo et al., 2020) to differentiate instruction. The teachers also used the ENRP growth point framework to identify students in their class who were mathematically vulnerable and to inform their lesson planning.

*Theme 1*. Differentiating instruction to respond to student difficulties *in the moment.*

A fundamental challenge for the teachers was further differentiating their instruction when enabling prompts were insufficient. For example, in her survey response, Angela explained:

> I find it very challenging to differentiate the instruction for students who were vulnerable as some enabling prompts worked well for some students but didn't work well for others.

This dilemma was further elaborated in Angela's interview. She found it challenging to provide *in the moment* direction to cater for "the exact need for each child". She noted that some prompts promoted engagement for those who were vulnerable, "the child has really gotten into it" but sometimes "it did not work". Bec highlighted in her interview that it was challenging to provide tasks that were sufficiently scaffolded so "that [students] were able to start the task and able to complete it and understand the concept". Similarly, Deb found it challenging to anticipate (Stein et al., 2008) how students "might go with a task" and whether the enabling prompt might be too difficult for some. In the survey Deb wrote,

> Students' learning is unexpected and if my already differentiated task does not cater to the students learning needs during the lesson, it is hard for me to find ways in which I can further help them.

The challenges for teachers arising from selecting suitable tasks and anticipating the range of student responses for a task is apparent in these data, along with the challenges of (1) planning suitable enabling and extending prompts to differentiate learning prior to a lesson, and (2) adapting tasks and instruction to differentiate learning *in the moment*.

*Theme 2*. Finding time to support students who are mathematically vulnerable.

The interview data for Deb, Bec and Emma highlighted their struggle to find the time and opportunity to assist individual students amidst the complex work of leading inquiry-based learning for a diverse group of students. For example, Bec explained that having time in a lesson to "get to" each one, including the capable students, is challenging, and this challenge was also apparent in Bec's survey response:

> Having the time to work with vulnerable students one on one in an effective way while also catering to the needs of and supervising the other students in the class.

Deb noted in her interview that:

> My other students, they may seem like they're so confident, usually, but if you go up to them, sometimes they're actually not doing so well. It's that remembering I need to go back to everyone … which is something that I did struggle with as well.

This tension was also noted in Emma's survey response.

> There are so many students in a class. It is difficult to spend quality time with vulnerable [students] doing activities that are helping them, whilst still fulfilling the requirements of a proper maths lesson—taking photos of student work samples, reflecting, repeating the process.

Underpinning all these challenges was the desire of the teachers to have the time, opportunity, and expertise to differentiate mathematics instruction for students *in the moment* amidst the complexities and demands of whole class mathematics teaching.

### *Valued Actions of EMU Specialist Teachers*

In the survey, classroom teachers rated (out of 10) the value of ten supports provided by the EMU ST during co-teaching cycles. Table 1 provides a summary of results. The Likert items have been ordered from lowest to highest mean. These data suggest that the teachers valued the EMU ST: (1) observing their teaching and their students for the purpose of providing feedback; (2) co-planning, including anticipating students' solutions and misconceptions, and suggesting manipulatives and representations to support students' learning; (3) modelling the use of questioning and prompts to develop students' thinking; and (4) meeting to discuss the mathematics progress of their vulnerable students.

**Table 1**

*EMU Specialist Teacher Supports for Classroom Teachers*

| Type of Support from the EMU Specialist Teacher | Class Teacher Rating (out of 10) | | | | | Mean |
|---|---|---|---|---|---|---|
| | Amy | Bec | Clare | Deb | Emma | |
| a. Suggests professional readings to enhance my mathematics Knowledge for teaching. | 9 | 2 | 5 | 8 | 3 | 5.4 |
| b. Modelling or demonstrating mathematics lessons. | 8 | 9 | 7 | 9 | 5 | 7.6 |
| c. Modelling the discussion during the summarise phase of the lesson. | 8 | 9 | 8 | 9 | 5 | 7.8 |
| d. Co-teaching (team teaching) mathematics lessons. | 8 | 9 | 8 | 9 | 7 | 8.2 |
| e. Observing my classroom mathematics teaching for the purpose of providing feedback. | 9 | 9 | 7 | 9 | 8 | 8.4 |
| f. Observing my students' learning and providing me with feedback. | 9 | 9 | 7 | 10 | 7 | 8.4 |
| g. Modelling the use of questioning or prompts to develop students' thinking. | 9 | 10 | 7 | 10 | 6 | 8.4 |
| h. Co-planning with me including anticipating student responses, solutions, & misconceptions. | 9 | 10 | 7 | 10 | 7 | 8.6 |
| i. Co-planning with me, including suggesting manipulatives and representations. | 9 | 10 | 7 | 10 | 7 | 8.6 |
| j. Meeting with me to discuss my vulnerable students' progress in mathematics. | 9 | 9 | 7 | 10 | 8 | 8.6 |

Aspects of the actions listed in Table 1 were also apparent in responses for the survey item that invited teachers to describe any *other* supports from the EMU ST that they found valuable. Seven supports were described, but these mostly elaborated the actions listed in Table 1. Illustrative examples follow for Deb's response related to item (i) co-planning, and for Bec's response related to both item (d) co-teaching and item (g) modelling.

> Meeting to co-plan allowed an outside perspective with more ideas on how to extend my students who need to be challenged, as well as differentiate tasks to cater to my more vulnerable students. [Deb]

> It was also very helpful to co-teach with her and watch the way she questioned the students to promote their curiosity and have them articulate their understanding. [Bec]

Analysis of the interview transcripts provided more insight about how the EMU STs assisted the teachers to deepen their understanding of high-quality instruction. These insights are described below.

*1*. Using questions to probe student thinking.

Angela explained that the EMU ST would go further with questioning than she would. For example, "Show me how many 10s are in this number first and then we can look at the 1s," rather than simply asking, "Show me this number using the paddle pop sticks."

Bec also valued the EMU ST modelling how to use questions and provide time for thinking.

> I just learned from Tina [ST], just give her that little bit of time to process and then just wait … and Tina asked questions like, 'Well, why did you pick that shape? What is different from that shape to that shape? Can you tell me what the difference is?' Just drawing out her thinking.

*2*. Materials and representations.

The EMU ST helped Deb learn about the importance of using materials such as counters to represent groups in multiplication, and to "place them on black-coloured paper to make the grouping more visible… just things like that that helped a lot."

*3*. Differentiating instruction.

The EMU ST shared with Deb many ideas about how to differentiate instruction.

> I found it very challenging to find ways that I could help them without just doing the same thing every single time. So, when [the EMU ST] would come up with all these ideas, it was like, you know what? Maybe I can try one of these ways.

*4*. Anticipating solutions for tasks.

When co-planning, the EMU ST helped Bec appreciate the worth of anticipating students' solutions for tasks.

> The importance of working out—particularly where it's an open-ended question. [The ST said,] 'Let's work out some solutions so we can anticipate questions. We can anticipate what they're going to come up with. We can anticipate any confusion', so that was really helpful. Just in terms of clarifying the language … the types of resources and materials that worked best.

*5*. Summarise phase of the lesson.

During the summarise phase of the lesson, the EMU ST and Emma would "*bounce ideas off one another*" and this helped Emma select work samples that would enable students to see and hear different approaches to thinking about the content. For example,

> When there's an extra person in the room you sometimes see different things and you see things differently to be able to then pick a few different work samples and one that you perhaps wouldn't have chosen originally.

Overall, it is apparent that the classroom teachers valued the suggestions offered by the EMU STs throughout the co-teaching cycles, but particularly when co-planning and co-teaching.

## Discussion

The schools participating in this pilot study had insufficient resources to provide an EMU intervention program for all students who were mathematically vulnerable. This situation highlighted the need to provide high-quality instruction in their classrooms for all students. The findings of this pilot study suggest that classroom teachers find two aspects of teaching students who are mathematically vulnerable challenging: (1) differentiating instruction effectively; and (2) having sufficient time and opportunity in a lesson to work with students who experience difficulty.

Previous research has highlighted the value of high-quality instruction that involves problem solving and using challenging tasks with enabling and extending prompts (Russo, et al., 2020). However, generalist classroom teachers need the knowledge and confidence of how to differentiate mathematics instruction within a student-centred inquiry approach in order to engage all students and enable all to learn. This is complex work requiring substantial expertise. Given that EMU STs have this level of expertise for teaching students who are mathematically vulnerable, it makes sense to utilise them in classrooms to support the professional learning of teachers.

Following a period of engagement in co-teaching cycles (Sharratt & Fullan, 2012) led by EMU STs, the classroom teachers in the pilot study described the actions of the specialist teachers that helped them to support students who were mathematically vulnerable. These actions included observing their teaching and their students for the purpose of providing feedback; co-planning, including anticipating students' solutions and misconceptions, and suggesting manipulatives and representations to support students' learning; modelling the use of questions and prompts to develop students' thinking and to differentiate instruction; and discussing the progress of their vulnerable students. Teachers work during the co-teaching cycles was supported by the ENRP Growth Point Framework (Clarke et al., 2002) that assisted the teachers to identify students who were not yet thriving, recognise misunderstandings, and plan for the next step in student's learning (Gervasoni et al., 2021). The EMU STs were able to apply their expert knowledge of the Growth Point Framework to highlight the resources and pedagogical actions that would support the learning of students.

The support of a trusted mathematics leader or coach has been highlighted in previous studies as a way to assist classroom teachers develop effective pedagogical actions to support mathematics learning (Anstey & Clarke, 2010: Sexton & Downton, 2014), and this was confirmed by our findings. The co-teaching cycles also provided teachers with an opportunity to re-conceptualise how they supported the students who they previously felt unable to assist during a lesson due to lack of time or opportunity amidst the complexity of teaching a class.

## Conclusion

The findings from this pilot study suggest that classroom teachers valued the opportunity to work with specialist intervention teachers through a series of co-teaching cycles, and that this opportunity promoted their professional growth. Furthermore, the findings highlight the potential value of utilising the expertise of an EMU ST to support classroom teachers to develop high-quality mathematics instruction that responds to the needs of students who are mathematically vulnerable. Overall, a larger study seems warranted to investigate whether the professional learning approach explored in this study is effective in other school settings, and the impact on mathematics teaching and learning.

## Acknowledgements

We acknowledge with gratitude the expertise and generosity of the teachers, EMU Specialist teachers, principals, and system leaders who supported this research.

## References

Anstey, L., & Clarke, B. (2010). Perceived professional learning needs of numeracy coaches. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education. Proceedings of the 33rd annual conference of the Mathematics Education Group of Austrasia* (pp. 45–52). MERGA.

Bruce, C. D., Esmonde, I., Ross, J., Dookie, L., & Beatty, R. (2010). The effects of sustained classroom embedded teacher professional learning on teacher efficacy and related student achievement. *Teacher and Teacher Education: An International Journal of Research and Studies, 26(*8), 1598–1608.

Bryant, D. P., Bryant, B. R., Gersten, R. M., Scammacca, N. N., Funk, C., Winter, A., Shih, M., & Pool, C. (2008). The effects of tier 2 intervention on the mathematics performance of first-grade students who are at risk for mathematics difficulties. *Learning Disabilities Quarterly, 31*(2), 47–63.

Butler, D. L., Lauscher, H. N., Jarvis-Selinger, S., & Beckingham, B. (2004). Collaboration and self-regulation in teachers' professional development. *Teaching and Teacher Education, 20*, 435–455

Clarke, D. (2013). Understanding, assessing and developing children's mathematical thinking: Task-based interviews as powerful tools for teacher professional learning. In A. M. Lindmeier, & A. Heinze (Eds.), *Proceedings of the 37th conference of the International Group for the Psychology of Mathematics Education: Mathematics learning across the life span* (Vol. 1, pp. 17–30). PME.

Clarke, D. M., Cheeseman, J., Gervasoni, A., Gronn, D., … Rowley, G. (2002). *Early numeracy research project final report*, February 2002. Australian Catholic University.

Cobb, P., & The MIST Team. (2019). Investigating what it takes to improve the quality of mathematics teaching and learning on a large scale. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 1–14). MERGA.

Downton, A., Cheeseman, J., & Roche, A. (2022). Goals and challenges of school mathematics leaders. *Mathematics Teacher Education and Development, 24*(1), 96–115.

Gervasoni, A. (2015*). Extending mathematical understanding: Intervention*. Ballarat, Australia: BHS Publishing.

Gervasoni, A., Roche, A., & Downton, A. (2021). Differentiating instruction for students who fail to thrive in mathematics: The impact of a constructivist-based intervention approach. *Mathematics Teacher Education and Development, 23*(3), 207–233.

Gervasoni, A., Roche, A., Giumelli, K., & McHugh, B. (2019). Insights about the progress of grade 1 children who are mathematically vulnerable and participate in a mathematics intervention program. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 300–307). MERGA.

Nickow, A., Oreopoulos, P. and Quan, V. (2020). *The impressive effects of tutoring on PreK-12 learning: A systematic review and meta-analysis of the experimental evidence. 27476*. National Bureau of Economic Research. http://www.nber.org/papers/w27476.

Roche, A., & Gervasoni, A. (2017). Perceived changes in teachers' knowledge and practice: The impact on classroom teachers from leader participation in whole-school reform of mathematics teaching and learning. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 442–449). MERGA.

Russo, J., Minas, M., Hewish, T., & McCosh, J. (2020). Using prompts to empower learners: Exploring primary students' attitudes towards enabling and extending prompts when learning mathematics through problem solving. *Mathematics Teacher Education Development, 22*(1), 48–67.

Sexton, M. & Downton, A. (2014). School mathematics leaders' perceptions of successes and challenges of their leadership role within a mathematics improvement project. In J. Anderson, M. Cavanagh, & A. Prescott (Eds.), *Curriculum in focus: Research guided practice. Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia* (pp. 581–588). MERGA.

Sharratt, L., & Fullan, M. (2012*). Putting FACES on the data: What great leaders do!* Corwin Press.

Sonnemann, J., & Hunter, J. (2023). *Tackling under-achievement. Why Australia should embed high-quality small-group tuition in schools.* Grattan Institute.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Math. Think. Learn*. 10, 313–340.

Thomson, S., Bortoli, L. D., Underwood, C., & Schmid, M. (2019). *PISA 2018: Reporting Australia's results. Volume I: Student performance*. Melbourne: Australian Council for Educational Research (ACER). https://research.acer.edu.au/ozpisa/35

# Investigating Pre-Service Teachers' Skills in Designing Numeracy Activities Across Curriculum Areas Involving Statistics

Seyum Getenet
*University of Southern Queensland*
seyum.getenet@usq.edu.au

Informed by teaching statistics across curriculum areas strategies, this study investigated the skills of 36 pre-service teachers (PSTs) who designed numeracy activities that focused on the statistics strand of the Australian curriculum. The data were analysed using descriptive statistics. The results showed that the PSTs designed numeracy activities focused on the science curriculum area. The results further showed that PSTs emphasised a few year levels and focused on collecting and recording data. The first stage of teaching statistics (designing effective questions) is often ignored in their activities. The approach adopted in this study can be used to identify PSTs' knowledge gaps and their professional learning requirements.

Numeracy is an essential skill for students, and teachers are expected to provide opportunities for them to use their numeracy knowledge in multiple contexts (ACARA, 2019). Across the globe, various strategies and approaches have been implemented to enhance students' numeracy skills. These approaches include teaching numeracy as a separate discipline or across curriculum areas (Forgasz et al., 2017). In Australia, for example, teachers are encouraged to develop students' numeracy skills by using mathematics confidently across non-mathematics curriculum areas, and numeracy is added as a general capability in the Australian national curriculum (ACARA, 2019). Similar approaches are adopted in other countries, such as England, the United States, and New Zealand, where numeracy is considered cross-curriculum areas (Brown et al., 2002; Ford, 2018; Neill, 2001).

Statistics is an essential component of numeracy, and it is increasingly being taught not only as part of mathematics but also across the curriculum in various countries (Watson & Smith, 2022). The development of statistical literacy is essential in a society that places great emphasis on using data to make decisions and interpretations (Booker et al., 2020). In Australia, statistics is taught as one of the major concepts within the Australian mathematics curriculum and developing students' skills in using statistics and data is necessary to validate decisions and inform conversations across all societal contexts (Watson & Callingham, 2020). However, the design of numeracy activities, including the focus on statistics across curriculum areas, requires competent teachers who can design and implement these activities (Bennison, 2015; Carter et al., 2015; Geiger et al., 2015). Initial teacher education institutions (ITEs) are responsible for developing pre-service teachers (PSTs) competencies, skills, and confidence to teach numeracy in schools across the curriculum areas (Sabbag et al., 2018). PSTs who are competent in teaching numeracy across curriculum areas can help students see the relevance of statistical skills in everyday life.

This study investigates the skills of PSTs who designed numeracy tasks that focus on statistics activities. Guided by the research question of how PSTs used statistics concepts in the design of numeracy activities across curriculum areas and which curriculum areas and year levels are focused, the study uses the concept of teaching statistics across the curriculum areas (Usiskin & Hall, 2015) and the process for teaching statistics (Bargagliotti et al., 2020) to frame the study. The study's approach can be used to identify PSTs' knowledge and skills gaps and their professional learning requirements to design statistics activities across curriculum areas. In addition, identifying these gaps can inform course and professional development design strategies to ensure competent teachers who can develop students' strong foundation in numeracy skills, including statistical literacy, to succeed in various academic disciplines and in their personal and professional lives.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 219–226). Newcastle: MERGA.

## Background

This study focused on teaching statistics across the curriculum areas in the primary school context. As a result, the study delved into the intersections between numeracy and different curriculum domains, using statistics teaching as a key framework. More information on these topics is provided in the subsequent section.

### *Numeracy Across the Curriculum Areas and Statistics*

Numeracy activities designed across non-mathematics curriculum areas are encouraged to enhance students' numeracy skills, critical thinking and transfer of mathematical skills to contexts outside the mathematics classroom (Bennison, 2015; Brown et al., 2002; Mathieson & Homer, 2021; Thornton & Hogan, 2004). This approach has the potential to empower students more than approaches that solely focus on developing numeracy skills through mathematics curriculum areas (ACARA, 2019; Bennison, 2015). The teaching of various mathematical concepts such as numbers, geometry, statistics, and data is integrated into this approach, with teachers often preferring to teach basic statistical concepts across curriculum areas (Gough, 2007; Geiger et al., 2015; Koellner et al., 2009).

Teaching numeracy across the curriculum, specifically statistics, is one of four possibilities discussed by Usiskin and Hall (2015) for teaching statistics as shown in Figure 1.

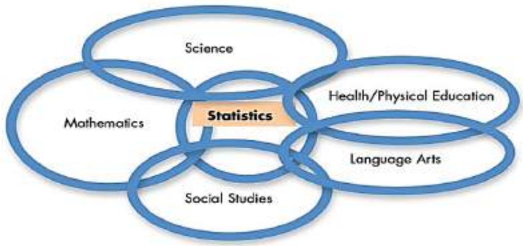



Figure 1. Statistics across the curriculum (Usiskin & Hall, 2015, p. 12).

The other three options are to teach statistics within mathematics, as applied mathematics or as an independent subject. The cross-curriculum approach to numeracy aligns with the expectation of the Australian Curriculum, where numeracy is viewed as one of the seven general capabilities essential across the curriculum. Interpreting statistical information is a key concept highlighted in the description of numeracy general capability (ACARA, 2019). To develop this skill, students need opportunities to solve problems in authentic contexts that involve collecting, recording, displaying, comparing and evaluating the effectiveness of data displays of various types. As a result, the teaching of statistics and data is embedded at all levels across all the curriculum areas to support students’ understanding of other subject areas.

The design of numeracy activities across curriculum areas requires teachers to be competent in designing and implementing these tasks (Bennison, 2015; Carter et al., 2015; Geiger et al., 2015; Goos et al., 2013). ITEs are responsible for developing PSTs’ competencies, skills, and confidence to teach numeracy in schools across the curriculum areas (Bargagliotti et al., 2020; Franklin et al., 2007).

### *Teaching Statistics*

To effectively engage with numeracy aspects beyond the math curriculum, teachers require enhanced knowledge and competencies. ITEs must equip PSTs with the necessary skills, confidence, and understanding of numeracy teaching, including statistics. Therefore, ITEs should provide PSTs with the essential knowledge and skills for teaching statistics as part of their degree program courses.

Studies, such as Bargagliotti et al. (2020) and Franklin et al. (2007), have proposed a four-step process for teaching statistics to school students. For instance, Franklin et al. (2007) suggested the following steps: (1) formulate questions and anticipate variability, (2) design and implement a data collection plan, (3) analyse data using appropriate graphical and numerical methods, and (4) interpret the results and identify any relationships. Similarly, Bargagliotti et al. (2020) presented a comparable four-step process for teaching and solving statistical problems, illustrated in Figure 2.

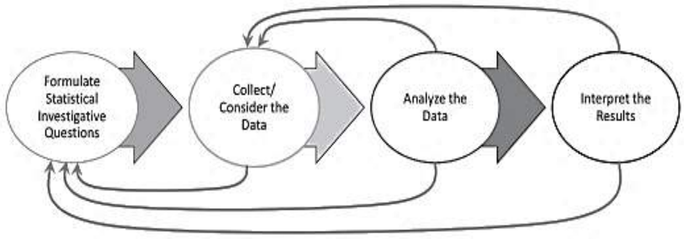



Figure 2. Statistical concept teaching process (Bargagliotti et al., 2020, p. 13).

Bargagliotti et al. (2020) stated that the main objective of statistical teaching is to gather and examine data in order to address statistical inquiry questions. This viewpoint is shared by several authors, including Booker et al. (2021), Reys et al. (2022), and Sabbag et al. (2018), who emphasised the significance of the four stages of teaching statistics and advancing through these stages to enhance primary school students' comprehension of statistical concepts. Each step and its description are provided in Table 1.

**Table 1**

*Summary of Steps for Teaching Statistics Processes and Description*

| Steps | Description |
|---|---|
| Formulating questions | Analyse the situation being investigated and narrow it down to a series of specific questions |
| Collecting and recording data | Begin gathering and organising the data according to the plan developed. Tables or spreadsheets might require alterations to better incorporate the data being collected |
| Organising and representing data | What graphs or charts are needed to display and emphasise patterns or various statistical measures |
| Analysing and interpreting data | What does the data say?' Conclusions may be drawn based on the available data |

According to Bargagliotti et al. (2020), the first step in teaching statistics is to formulate clear questions to be answered with collected data. Stating a good question gives students a reason and motivation for collecting and analysing data. Booker et al. (2021) and Reys et al. (2022) reviewed other studies and suggested that it is beneficial to encourage students to identify their own questions or problems rather than using pre-designed questions. After data is collected, the third step is to organise the information so that the results can be analysed and interpreted. Sabbag et al. (2018) and Reys et al. (2022) suggest that graphs can be used to present or organise data visually. Finally, interpreting data can be done through the use of questioning. Teachers should encourage students to examine their results and discuss questions that may be answered by the data at the final stage of teaching statistics.

The Australian Curriculum Version 8.4 (ACARA, 2019) includes statistics and probability across year levels, forming one of the three central content strands. At the Foundation level, students are expected to collect information to answer yes/no questions and make basic inferences (ACARA, 2019). In Year 2, students are required to collect categorical data on a topic of interest, sort the data, create data displays using tables, picture graphs, and lists, and interpret their data (ACARA, 2019). Similarly, in Year 5, students conduct similar statistical investigations but also examine numerical data and may represent their data using column graphs and dot plots, as well as present their work using digital technologies (ACARA, 2019).

Booker et al. (2021) suggest that these steps can be utilised as a roadmap to enhance students' proficiency and mastery in statistical analysis, allowing them to become active contributors when gathering, analysing, illustrating, and interpreting data. Moreover, it aids students in meaningfully engaging in the study of statistics and the acquisition of knowledge about it. In summary, these steps serve as a useful tool for PSTs seeking to improve students' statistical literacy.

## Method

### *Context*

This research is part of a broader investigation in which PSTs developed numeracy activities across various curriculum areas to determine their knowledge gaps. The research was conducted in the School of Education of an Australian university. Final-year PSTs who were prepared to teach students ranging from Foundation (average age of 5 years) to Year 6 (average age of 11 years) were recruited for this study. As part of their 4-year degree program, primary specialisation PSTs must complete three core mathematics curricula and pedagogy courses. The first two courses concentrate on teaching the Australian curriculum, including data and statistics that include the steps outlined in Table 1. The third mathematics education course emphasises the use of numeracy across curriculum areas. The current study was set in this course. The author of the present study was one of the course designers and has assessed PSTs assignments. In this course, PSTs must design numeracy activities across different curriculum areas, including humanities and social sciences [HASS] (Civics and Citizenship, Economics and Business, Geography and History), English, science, arts (music), technologies (design and technologies, and digital technologies), and health and physical education (HPE), according to version 8.4 of the Australian Curriculum. The general concept of designing numeracy activities was introduced to be used in ways that serve the PSTs' specific choice of curriculum areas and context to design numeracy activities.

The PSTs' design of numeracy activities must meet two requirements. First, they must comprehend the identified mathematical concepts by explaining the mathematical and non-mathematics concepts involved in the designed activities with examples. Furthermore, the PSTs must identify and explain the relevance of the concepts to the Australian Curriculum. Second, PSTs must develop numeracy activities in the specified curriculum area to teach the intended concepts. The numeracy activities must demonstrate how mathematics must be integrated within the identified non-mathematics curriculum area. Furthermore, the activities must be pertinent to the national curriculum outcome.

### *Participants and Data Collection*

Data were part of the larger study gathered from 100 PSTs’ course assignment submissions. The submissions were all from the same group of PSTs. Ethical protocols for collecting the PSTs' archived assignment data were provided by the relevant University and School authorities. The PSTs were given the freedom to choose and design numeracy activities, with the option to select any mathematical concept and any non-mathematics curriculum area ranging from the foundation to Year 6. However, most PSTs focused on statistics (36%) and measurement and geometry strands (25%) of the Australian Curriculum when designing their numeracy activities. The focus of this study was on the 36 PSTs who designed their numeracy activities based on statistics. The participants' year level was all fourth year, and the author of the study had access only to their year level and archived assignment submissions.

### *Analysis*

The analysis of the data was a two-step process. Initially, the assignment submissions were evaluated to meet the course requirements, and notes were taken for the second step of the analysis. In the first step, the numeracy tasks were evaluated based on two criteria and supported by a marking rubric. The first criterion aimed to understand the identified mathematical concepts by explaining the mathematical and non-mathematics concepts involved in the designed activities, with examples. In the second part, PSTs were asked to design numeracy activities in the identified curriculum area to teach the intended concepts. PSTs who designed numeracy activities focused on statistics were identified, and descriptive statistics (mainly frequency and percentage) were utilised to describe the focus of the year levels and curriculum areas using SPSS. Additionally, the focus of the four-step process of teaching statistics was also identified.

## Results and Discussion

The findings are organised into two sections. First, the identified curriculum areas and the dominant mathematics concepts across different year levels are presented. Second, a more detailed analysis of the statistics concepts in relation to the PSTs' focus and how the concepts were taught are presented and discussed. The results showcase how PSTs interpreted and responded to the assignment questions.

### *Focus on Year Levels and Curriculum Areas*

The PSTs had the option to select and design numeracy activities for any year level from Foundation to Year 6 and using any non-mathematics curriculum areas. Table 2 summarises the frequency (N) of occurrence of the various curriculum areas across different year levels.

**Table 2**

*Curriculum Areas Focused and Numeracy Activities Across the Year Levels*

| Year Level | Curriculum area and counts (N) | | | | | Total |
|---|---|---|---|---|---|---|
| | Business | Geography | History | HPE | Science | |
| Year 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| Year 2 | 0 | 0 | 0 | 0 | 1 | 1 |
| Year 3 | 0 | 2 | 1 | 1 | 5 | 9 |
| Year 4 | 0 | 1 | 2 | 3 | 3 | 9 |
| Year 5 | 1 | 2 | 2 | 1 | 6 | 12 |
| Year 6 | 0 | 2 | 0 | 2 | 0 | 4 |
| Total | 1 | 7 | 5 | 7 | 16 | 36 |

Table 2 displays that Science (N=16), Geography (N=7) and HPE (N=7) were the most frequently utilised curriculum areas in the designed numeracy activities. Specifically, Science was most commonly employed in Year 5 (N=6). On the other hand, English and Design and Technologies were not utilised in the designed statistical activities across curriculum areas. This may be due to PSTs' lack of confidence in using numeracy across other curriculum areas such as English and technology, or they may encounter difficulties identifying appropriate statistical ideas to embed in other areas. Similar to these findings, Geiger et al. (2013) and Koellner et al. (2009) reported that teachers were less self-assured in integrating numeracy across English/literacy. The majority of the designed activities focused on Years 3, 4, and 5 (N=30), with less emphasis on Years 1 and 2, as indicated in Table 2.

*Statistical Teaching Steps and Year Levels*

The PSTs were tasked with implementing the teaching principles of statistics in their designed numeracy activities, which involved the four-step process described in Table 1. As indicated in Figure 3, however, only a small number of PSTs (n=7) included all four steps in their activities. Five PSTs only utilised one of the four steps of teaching statistics, which was either analysing and interpreting data or collecting data. For instance, PST1 (with ID 1) incorporated numeracy activities that necessitated collecting, recording, organising, and representing data without a clearly defined question to answer (refer to Figure 2).

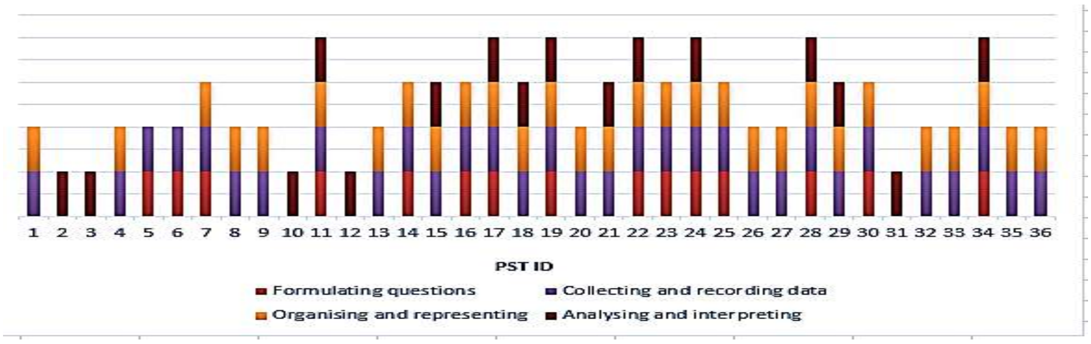



Figure 3. The steps used in the activities across year levels.

Table 3 reveals that a significant number of PSTs (N=29[31.9%]) incorporated the steps of organising and representing data in their designed numeracy activities. However, some critical steps for teaching statistics, such as formulating a good question, were overlooked. Only 15 PSTs included formulating questions in their designed numeracy activities, as indicated in Table 3. More than 50% of the activities did not entail formulating a good question. Nonetheless, stating a good question is a crucial step that provides students with a purpose and motivation for engaging in the other steps of learning/teaching statistics, such as collecting and analysing data (Bargagliotti et al., 2020; Reys et al., 2022; Sabbag et al., 2018). Table 3 provides a detailed breakdown of each step of the teaching statistics process and the frequencies detected in the designed numeracy activities.

**Table 3**

*Frequencies of Statistics Teaching Step Processes in the Numeracy Activities*

| Step | N | % |
|---|---|---|
| Formulating questions | 15 | 16.5% |
| Collecting and recording data | 31 | 34.1% |
| Organising and representing data | 29 | 31.9% |
| Analysing and interpreting data | 16 | 17.6% |
| Total activities designed | 36 | 100% |

Table 3 shows that the majority of PSTs (N=31 [34.1%]) incorporated collecting and recording data in their designed numeracy activities. This finding aligns with Gough's (2007) argument that creating and interpreting data tables and graphs are fundamental mathematical concepts found in many areas of science. However, it is important to note that PSTs must also acquire the necessary skills and strategies for teaching statistics (e.g., Bargagliotti et al., 2020; Franklin et al., 2007) across different curriculum areas. Teacher education programs have a responsibility to cultivate PSTs' competencies, skills, and confidence in teaching numeracy across diverse curriculum areas.

## Conclusion

The importance of statistical literacy has been widely recognised in the literature. According to the Australian Curriculum, Assessment and Reporting Authority (2019), statistical literacy is essential for interpreting and making sense of the world around us. Developing PSTs' skills in designing statistically rich activities across multiple curriculum areas is vital for students' numeracy skill development in schools. This is particularly important given the increasing emphasis on data-driven decision-making in society (Day, 2013).

This study was conducted at an Australian University's School of Education and involved final-year Pre-Service Teachers (PSTs) preparing to teach Foundation to Year 6 students. The study aimed to investigate the PSTs' skills in designing numeracy tasks that focused on the statistics strand of the Australian curriculum. The study's results are consistent with previous research that has found that PSTs tend to focus on a limited range of curriculum areas when designing numeracy activities (Geiger et al., 2015; Koellner et al., 2009). This limited focus can reduce students' confidence and opportunities to use their numeracy skills in complex contexts. Therefore, PSTs need to develop their skills in designing numeracy activities that incorporate a broader range of curriculum areas.

The study also highlights the importance of including all the steps for teaching statistics when designing numeracy activities. As noted by Booker et al. (2021) and Reys et al. (2022), steps such as formulating questions, analysing and interpreting data, and making conclusions are essential in enabling students to proceed to the other steps of solving statistical problems, including collecting and analysing data. Therefore, PSTs need to receive more training and support to develop their skills and confidence in designing numeracy activities that incorporate all the steps for teaching statistics across different curriculum areas.

Moreover, designing numeracy activities across curriculum areas that require students to analyse data and draw conclusions can help students develop critical thinking and problem-solving skills (Sabbag et al., 2018). Therefore, PSTs need to develop their skills in designing numeracy activities that promote these skills.

Although the study's findings are insightful, its small sample size limits the generalizability of the results. Future research could benefit from larger samples and more detailed analyses of PSTs-designed numeracy activities. However, the study's findings provide valuable insights into how PSTs can incorporate statistical concepts into numeracy activities across different curriculum areas and year levels. These findings have relevance for researchers and policymakers interested in developing PSTs' skills in integrating statistics across the curriculum and enhancing students' overall numeracy experiences in schools.

Overall, this study highlights the need for PSTs to develop their skills in designing numeracy activities that incorporate statistical concepts across a broader range of curriculum areas. The study's findings suggest that PSTs need more training and support to develop their confidence and skills in teaching statistics and numeracy across curriculum areas. By doing so, PSTs can help students develop critical thinking and problem-solving skills, which are essential for success in today's data-driven society.

## References

Australian Curriculum, Assessment and Reporting Authority. (2019). *The Australian curriculum*. ACARA. https://www.australiancurriculum.edu.au/

Bargagliotti, A., Franklin, C., Arnold, P., Gould, R., Johnson, S., Perez, L., & Spangler, D. A. (2020). *Pre-K-12 guidelines for assessment and instruction in statistics education II (GAISE II)*. American Statistical Association. https://www.amstat.org/asa/fles/pdfs/ GAISE/GAISEIIPreK-12_Full.pdf

Bennison, A. (2015). Supporting teachers to embed numeracy across the curriculum: A sociocultural approach. *International Journal on Mathematics Education, 47*(4), 561–573. https://doi.org/10.1007/s11858-015-0706-3

Booker, G., Bond, D., Sparrow, L., & Swan, P. (2020). *Teaching primary mathematics (6th ed)*. Pearson Higher Education.

Brown, M., Askew, M., Baker, D., Denvir, H., & Millett, A. (2002). Is the national numeracy strategy research-based? *British Journal of Educational Studies, 46*(4), 362–385. https://doi.org/10.1111/1467-8527.00090

Carter, M. G., Klenowski, V., & Chalmers, C. (2015). Challenges in embedding numeracy throughout the curriculum in three Queensland secondary schools. *The Australian Educational Researcher, 42*(5), 595–611. https://doi.org/10.1007/s13384-015-0188-x

Ford, K. (2018). Persisting gaps: Labor market outcomes and numeracy skill levels of first-generation and multi-generation college graduates in the United States. *Research in Social Stratification and Mobility, 56*(2018), 21–27. https://doi.org/10.1016/j.rssm.2018.06.003.

Forgasz, H. Leder, G., & Hall, J. (2017). Numeracy across the curriculum in Australian schools: Teacher education students' and practising teachers' views and understandings of numeracy. *Numeracy, 10*(2), 1–23. http://doi.org/10.5038/1936-4660.10.2.2

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheafer, R. (2007). *Guidelines for assessment and instruction in statistics education (GAISE) report: A pre-K-12 curriculum framework*. American Statistical Association. http://www.amstat. org/education/gaise/

Geiger, V., Forgasz, H., & Goos, M. (2015). A critical orientation to numeracy across the curriculum. *International Journal on Mathematics Education, 47*(4), 611–624. https://doi.org/10.1007/s11858-014-0648-1

Goos, M., Geiger, V., & Dole, S. (2013). Designing rich numeracy tasks. In C. Margolinas (Ed.), *Task design in mathematics education: Proceedings of ICMI Study 22* (pp. 589–598). Oxford.

Gough, J. (2007). Make your school's numeracy-across-the-curriculum policy. *Australian Mathematics Teacher, 63*(3), 31–39. http://hdl.handle.net/10536/DRO/DU:30007800

Koellner, K., Wallace, F. H., & Swackhamer, L. (2009). Integrating literature to support mathematics learning in middle school. *Middle School Journal, 41*(2), 30–39. https://doi.org/10.1080/00940771.2009.11461710

Mathieson, R., & Homer, M. (2021). "I was told it would help with my Psychology": Do post-16 core maths qualifications in England support other subjects? *Research in Mathematics Education*, 1–19. https://doi.org/10.1080/14794802.2021.1959391

Neill, W. A. (2001). The essentials of numeracy. *Paper presented at the New Zealand Association of Researchers in Education Conference* (pp. 6–9). Available at https://www.nzcer.org.nz/system/files/10604.pdf

Reys, R., Rogers, A., Bragg, L., Cooke, A., Fanshawe, M., & Gronow, M. (2022). *Helping children learn mathematics* (4th ed.). John Wiley & Sons.

Sabbag, A., Garfield, J., & Zieffler, A. (2018). Assessing statistical literacy and statistical reasoning: The REALI instrument. *Statistics Education Research Journal, 17*(2), 141–160. https://doi.org/10.52041/serj.v17i2.163

Thornton, S., & Hogan, J. (2004). Orientations to numeracy: Teachers' confidence and disposition to use mathematics across the curriculum. In M. J. Hoines, & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 315–320). PME.

Usiskin, Z., & Hall, K. (2015). The relationships between statistics and other subjects in the K–12 curriculum. *Chance, 28*(3), 4–18. https://doi.org/10.1080/09332480.2015.1099361

Watson, J., & Callingham, R. (2020). COVID-19 and the need for statistical literacy. *Australian Mathematics Education Journal, 2*(2), 16–21. https://search.informit.org/doi/10.3316/informit.287208637600347

Watson, J., & Smith, C. (2022). Statistics education at a time of global disruption and crises: A growing challenge for the curriculum, classroom and beyond. *Curriculum Perspectives*, *42*(2), 171–179. https://doi.org/10.1007/s41297-022-00167-7

# Challenges to Numeracy Across the Curriculum: Reflections From a Case Study

Vanessa Gorman
*Department for Education, SA*
vanessa.gorman@sa.gov.au

Lisa O'Keeffe
*University of South Australia*
lisa.okeeffe@unisa.edu.au

Amie Albrecht
*University of South Australia*
amie.albrecht@unisa.edu.au

Jarrad McPhee
*Teacher, Department for Education*
jarrad.mcphee183@schools.sa.edu.au

It is well accepted and understood amongst the mathematics education community that numeracy is the responsibility of all teachers, across all levels of schooling. However, the way numeracy is understood and actioned across the Early Childhood, Primary and Secondary contexts is different. This paper reports on a case study of secondary school teachers seeking to be more intentional in embedding numeracy across the curriculum. The findings indicate that dialogue and support to see, and make, numeracy connections result in greater cohesion in terms of understanding numeracy and making numeracy more visible for students.

In Australia, the media message about mathematics, as measured via numeracy assessments such as NAPLAN, PISA and TIMSS has highlighted declining performance (Thomson et al, 2013). The response has had two distinct focuses: (1) teaching and learning practices in mathematics which can improve student experiences and outcomes, and (2) how children transfer and apply their knowledge of mathematics. This transfer includes application of mathematics in mathematical problem solving and application in non-mathematics contexts. In a school setting the application of mathematical knowledge in other learning areas is referred to as *numeracy across the curriculum*. Bennison (2015) highlights the importance of all teachers (1) recognising and understanding the importance of numeracy across the curriculum, and (2) being able to identify opportunities for explicit connection to and development of numeracy ideas. In this study we explore ways in which teachers at one school identified and developed opportunities for numeracy across the curriculum, with a specific focus on resources already in use.

## Numeracy Across the Curriculum

The Australian Curriculum, Assessment and Reporting Authority (ACARA) includes numeracy as one of seven general capabilities in the Australian Curriculum and defines numeracy as "students recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully" (ACARA 2017). The general capabilities are addressed through the content of the eight learning areas, and content descriptions are tagged with one or more relevant general capabilities. ACARA makes it clear that numeracy is both a part of the mathematics curriculum and an essential component of all learning areas across the curriculum. In other words, numeracy development is a responsibility of all teachers. In version 8.4 of the Australian Curriculum, learning areas were ranked based on the proportion of content descriptions tagged with connections to numeracy; see Table 1 for the ranked list. In version 9, all eight learning areas are viewed equally with each having a clearly defined numeracy statement within their curriculum overview.

The difference between numeracy and mathematics must be clearly conveyed to support schools' engagement with cross-disciplinary, interdisciplinary and transdisciplinary approaches to embedding numeracy across the curriculum (Coffey & Sharpe, 2021). Without this clarity, measuring and reporting student numeracy outcomes required to meet the political agenda can promote the idea amongst teachers that numeracy is simply a skill-based pursuit (Goos et al., 2019).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 227–234). Newcastle: MERGA.

**Table 1**

*Learning Areas Ranked by Highest Proportion of Content Descriptions Tagged with Numeracy*

| | | | |
|---|---|---|---|
| 1 | F-6/7 Humanities and Social Sciences | 7 | Science |
| 2 | 7-10 History | 8 | The Arts |
| 3 | 7-10 Geography | 9 | Health and Physical Education |
| 4 | 7-10 Civics and Citizenship | 10 | English |
| 5 | 7-10 Economics and Business | 11 | Languages |
| 6 | Technologies | 12 | Work Studies |

Numeracy requires students to make sense of non-mathematical contexts through a mathematical lens, use critical judgement, and investigate possible solutions to real world problems (Geiger, Goos and Forgasz, 2015). In actioning this, teachers need to provide opportunities where students develop a depth of understanding of the specific application of mathematics within their subject area. This can be achieved either through *numeracy moments* or *numeracy opportunities*. Numeracy moments are encounters (often ad hoc) with mathematics in other learning areas where the learning intention is not reliant on the mathematical connection. In contrast, numeracy opportunities are planned uses of mathematics that are integral to the learning intention; see Goos et al. (2019) for examples of numeracy opportunities.

A teacher's understanding of numeracy depends on the combination of the different types of knowledge held by the teacher and their own personal beliefs (Muir, 2008). Strong content knowledge by itself does not ensure that a teacher can embed numeracy in ways that are meaningful for students (Muir, 2008). Embedding numeracy across the curriculum can be challenging. Teachers need to understand how students learn, appreciate the applications of mathematics, recognise mathematical possibilities in non-mathematics subjects, and be willing to collaborate with colleagues (Goos et al., 2019). An additional complexity, with a directive that all teachers should embed numeracy in their teaching, is that it may cause may anxiety for some teachers, particularly if their own mathematical experiences have not given them the confidence and aptitude to identify or seek out numeracy moments or opportunities.

To capture the complex demands of numeracy across the curriculum, Goos, Geiger and Dole (2010) developed a model designed to reflect the nature of numeracy in the 21st century. The model builds on previous understandings of numeracy. Early work by Goos (and later with colleagues) used the Australian Association of Mathematics Teachers definition of numeracy: "to be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life" (AAMT 1997, p.15). More recent work by Goos et al. (2019) places *context* at the heart of numeracy and acknowledges the key roles of *mathematical knowledge*, *tools*, and *dispositions*. These four dimensions are embedded in a *critical orientation* to using mathematics, highlighting the potential of numeracy connections to promote opportunities for meaningful connections to real-world issues. The model of numeracy in the 21st century can be used as a framework when teachers are planning for numeracy opportunities.

In this paper we share insights from a cohort of teachers seeking to be more intentional in their approaches to numeracy across the curriculum by planning for, enacting and reflecting on numeracy moments in non-mathematics learning areas.

## Research Design

This study employed a case study approach to explore the ways in which a targeted approach to numeracy across the curriculum was received and enacted in a secondary school context. Case study, as discussed by Yin (2009, p.1211), enables an "intense focus on a single phenomenon within its real-life context", and as such can encompass both qualitative and quantitative methods of data collection. The 'case' in this study was an urban secondary school (Years 7–12), in a predominantly middle-class area with more than 1300 enrolled students.

Following the school's own scoping survey, the project team, in collaboration with the Head of Mathematics and some of the mathematics faculty at the school, unpacked the school's own data about perceptions of numeracy. An information session was held for teachers in the mathematics faculty on numeracy and numeracy across the curriculum. This session included discussion around the faculty's strengths and priorities and informed the co-design (with the Head of Mathematics and the project team) of upcoming professional learning workshops.

Goos (2020) infers a three-step approach to developing numeracy across the curriculum that was used to guide the work with the school, with particular emphases on steps 1 and 2.

1. Exposure to exemplar activities—*knowing* about numeracy.
2. Trying out initial ideas—*doing* in relation to numeracy.
3. Continued interaction (developing knowing and doing)—numeracy as part of *being*.

As part of this project, the school identified a focus on Health and Physical Education (HPE) and Design and Technologies (DT) as a starting point for their approach to numeracy across the curriculum. These middle-years teachers participated in a series of professional learning workshops. The workshops focussed on unpacking expectations of, and current practices associated with numeracy, introducing the 21st century numeracy model (Goos et al., 2014), and illustrating ways to embed numeracy ('*knowing*'). The workshops also encouraged teachers to identify and reflect on numeracy moments, and supported teachers to plan for, enact, and reflect on numeracy opportunities (*'doing'*). These workshops were held across three terms and were facilitated by the project team.

The main data collected in this study were teacher surveys and reflections on both numeracy moments and numeracy opportunities to identify teachers' current experiences of numeracy across the curriculum, the main resources they use, and also their confidence in and experiences as users of mathematics. Where feasible, teachers recorded and watched back their own lessons to support their reflections. All teachers completed:

- An initial survey and reflection prior to the start of the project (in term 1),
- A second survey and reflection after their first 'have a go' activity (in term 3), and
- A final survey and reflection after their lesson incorporating a planned numeracy opportunity (in term 4), ideally after reviewing their own recording of the lesson.

The initial survey was anonymous so that teachers would feel comfortable sharing negative perceptions or confidence rankings without fear of being identified. Responses to subsequent surveys were identifiable. In the final survey, 360° cameras were made available so teachers could record their lessons. The research team processed and made the video recordings available, but teachers reviewed them independently to reduce any perceived scrutiny of them. In this paper we present and discuss findings from the first two project stages (up to term 3).

## The Context

In the school's initial preparation for a focus on numeracy, all middle school teachers (Years 7-10) were surveyed and asked about their perceptions of numeracy. Of particular relevance to this project is the variation in responses from teachers regarding their current understanding of the numeracy demands of the subjects they currently teach. The results from an anonymous survey distributed by the Head of Mathematics at the site indicates a variety in teacher perceptions (n=60 middle school teachers) of what numeracy is, as well as a range in teacher self-reported confidence in understanding of the numeracy demands of their learning area. Confidence ranged from very good or good (n=14, n=20), fair (n=17), through to poor or very poor (n=8, n=1); this cohort of 60 teachers includes the mathematics faculty.

When reflecting on the challenges specific to this site, which informed the project, the Head of Mathematics noted:

> From a site perspective, numeracy across the school is important as it can allow students (and teachers) to improve dispositions towards Numeracy and use mathematics effectively and critically in their personal & civic life. There is often a misrepresentation of what numeracy is (the fact NAPLAN—Numeracy is essentially a mathematics exam doesn't help) by teachers and the general public, so being able to change those perceptions is important. Like literacy, it is hoped that we can recognise numeracy as everyone's responsibility and not an "opt-out". It is also hoped that by being able to recognise numeracy opportunities within their learning area, students have multiple exposures (a High Impact Strategy) to different mathematics from a variety of topics. This could potentially have a symbiotic relationship through which other learning areas' numeracy opportunities are mapped, meaning in mathematics, we bring other areas into ours.

The participants in this study were all teachers at the case study site teaching HPE (n=12), Food Technologies (FT; n=2, part of the D&T middle school curriculum teaching team), and DT (n=4) in the middle years (Years 7 to 10). Three of the teachers (HPE) also teach mathematics, one of whom would be considered out-of-field based on their teaching qualification. The last teacher indicated they would have liked to study mathematics as part of their qualification, but it wasn't an option. Three other teachers also indicated that they opted to study some mathematics as part of their teacher training and indicated that they enjoyed maths. Seven of the 18 participants indicated that they don't really enjoy mathematics (5 HPE and 2 FT). The data for the 2 FT teachers is included where relevant in the 'overall' data but excluded from cohort-specific data as one teacher did not complete all parts of the survey.

## Findings: Stage 1

Aside from the three HPE teachers currently teaching mathematics, seven other teachers (three of whom don't particularly like mathematics) indicated they have taught it in the past (two at the case-study school and five at other schools). Of these seven teachers only three said they make pre-planned numeracy connections. The most common response from the participants (8 out of 18) was using a mixture of both planned and ad hoc. A further five said they typically plan, and four indicated they typically make ad hoc in-the-moment connections.

Teachers were asked what they personally wanted to get out of the project. Their responses predominantly indicated pedagogical content knowledge (PCK) related to numeracy (n=8), along with numeracy resources (n=3), improved outcomes for students (n=3), mathematics PCK (n=2), and mathematics content knowledge (n=1). Interestingly, four responses exclusively referenced mathematics rather than numeracy (n=10), and two people did not mention either mathematics or numeracy. Categorised in terms of the five dimensions of the model of numeracy for the 21st century, four respondents mentioned wanting students to enact numeracy in context, and one talked about tools for numeracy.

All participants were asked to rate their confidence (with 10 being the highest) in explaining a range of mathematics topics. Table 2 summarises the responses from all 18 participants (HPE, DT and FT), noting that some did not rate some topics and two participants (1 HPE and 1 DT) did not rate any topics. A breakdown by learning area is also shown for interest, noting that little can be inferred about differences between the two cohorts given the small sample sizes. Fifteen teachers gave rating for 75% or more of the topics listed in Table 2 (with 13 teachers giving ratings for all topics). These average confidence for each teacher was computed and ranged from 2.9 to 8.8 (out of 10).

**Table 2**

*Participants' Mean Confidence Scores (Self-Rated) where 1 is Low and 10 is High*

| How would you rate your confidence in explaining the following aspects of mathematics: | Overall (n=18) | HPE (n=11) | DT (n=3) |
|---|---|---|---|
| Fractions, decimals and percentages | 7.50 | 7.64 | 8.67 |
| Ratios and proportions | 7.20 | 7.09 | 8.33 |
| Measurement: units, instruments and accuracy | 8.13 | 8.00 | 9.33 |
| Perimeter, area, and volume | 7.80 | 7.36 | 9.33 |
| Geometric figures: definitions and properties | 6.40 | 6.27 | 8.00 |
| Geometric figures: symmetry, motions, transformations, congruence, similarity | 5.13 | 4.91 | 7.00 |
| Coordinate geometry | 6.07 | 5.50 | 8.67 |
| Algebraic representation | 6.07 | 5.40 | 8.67 |
| Evaluate and perform operations on algebraic expressions | 5.64 | 5.10 | 8.67 |
| Solving linear equations and inequalities | 5.77 | 5.11 | 7.67 |
| Representation and interpretation of data in graphs, charts, tables | 8.27 | 8.27 | 7.67 |
| Simple probabilities: understanding and calculations | 7.20 | 7.09 | 6.67 |

Table 3 outlines the commonly used resources identified by the teachers—included verbatim. Teachers have been grouped according to their average self-rated confidence from Table 2. The three groupings used are:

- *Low confidence:* the average confidence of each teacher in this group is 3.0 or less
- *Mid confidence:* the average confidence of each teacher in this group is 5.0 to 7.5
- *High confidence:* the average confidence of each teacher in this group is 7.5 or more.

**Table 3**

*Participants' Preferred Resources, by Confidence Grouping*

| Group | The resources you use that you find the most helpful: |
|---|---|
| Low (n=1) | None listed |
| Mid (n=8) | *My own resources:* [identified but no examples given]<br>*Digital resources:* live it up, internet articles; data collection tools<br>*Physical resources:* maps; pre-set out route cards, embed into PowerPoints<br>*Knowledge:* Understanding the different calculations that I'm working through so I can explain them; previous knowledge and colleagues. |
| High (n=6) | *My own resources:* PowerPoint notes highlighting numeracy connections<br>*Physical resources:* Videos, diagrams, diaries and match play booklets; a myriad of resources from books to websites<br>*Digital resources:* Arduino open-source web site, A.C recourses website<br>*Other:* Prior assessments; contextualized numeracy, highlighting the numeracy in my curriculum |

When asked what support was available at their school to make better numeracy connections across the curriculum, one respondent in the low confidence group said that other colleagues were their main support. In the mid confidence groups, there were six responses: one 'not a lot', two 'not sures' including one who said they had neither been offered nor sought out any support, and three who indicated support from across the faculty and/or from mathematics teachers. The high confidence group gave responses such as colleagues in the mathematics faculty (n=2), electronic and Power BI (n=1), not sure (n=4), two of whom indicated they were new to the school and one other stating "The maths faculty are very open with helping when it comes to these numeracy moments". The range in responses suggests that while support is available, it is not visible or accessible equitably among teachers.

## Findings: Stage 2

Ten teachers from Stage 1 participated in Stage 2, 8 HPE and 2 DT (Table 4).

**Table 4**

*Participants' Chosen Numeracy Moments in Stage 2*

| Describe the numeracy connection: | Type | Numeracy Continuum connection: |
|---|---|---|
| DT: Year 7's are building a model of a COLA and had to scale the building and furniture to ensure that the model didn't look "weird". | M* | Using fractions, decimals, percentages, ratios, rates |
| DT: Changing a design sketch into a life size pattern and allowing ease and seam allowance. | O | Estimating and calculating with whole numbers |
| HPE: Design and perform dance movement patterns | M | Recognising and using patterns and relationships |
| HPE: During an inquiry process, the students needed to collect a lot of data ... I spent some time with them describing the importance of identifying themes, comparing the data sources across the variety of sources to recognise patterns to then draw conclusions. | O | Recognising and using patterns and relationships |
| HPE: Teaching Outdoor Ed where students needed to work with time and distance when planning route cards. | M | Estimating and calculating with whole numbers |
| HPE: Stage 1 Physical Education—Biomechanics; we have discussed and explored speed, velocity, displacement, distance and projectile motion. These make connections from theory to a practical setting and how these aspects can be utilised effectively. | O | Using spatial reasoning |
| HPE: Stage 2 PE lesson looking at analysing and evaluating training programs using HR, GPS data and game statistics. | M | Using fractions, decimals, percentages, ratios, rates |
| HPE: Heart rate and training zone; recording your heart rate in beats per minute. Record pulse for 15 seconds then multiply it by 4. | M | Using fractions, decimals, percentages, ratios, rates |
| HPE: Yr. 10 Human Movement, analysing HR data; calculating % by looking at max heart rate then we looked at calculating different % of their heart rates and linking them into which energy system was dominant at different points of the game. | O | Using fractions, decimals, percentages, ratios, rates |
| HPE: We analysed a tennis match plotting depth of shots, first serve percentage and then percentage of points won when the first serve went in. We then had students calculate the number of hours they spend on court training at various intensity levels and used a formula to calculate overall workload for that period of time. | O | Using fractions, decimals, percentages, ratios, rates |

*Note.* M=numeracy *Moment*, O=numeracy *Opportunity.*

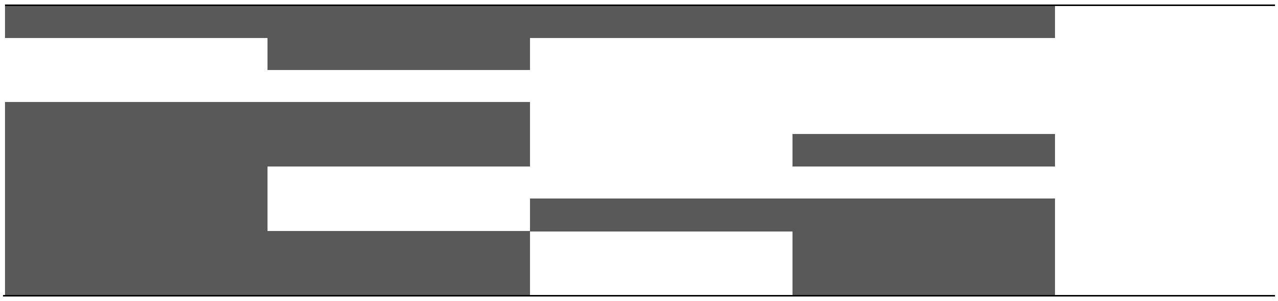
In Stage 2, teachers were asked to identify a significant numeracy moment and plan for an explicit elaboration or discussion in class in relation to the mathematics. Teachers were asked to reflect on the experience. Table 4 summarises their numeracy moments and associated mathematical content (their words). The most common connection to the numeracy continuum was with using *fractions, decimals, percentages, ratios and rates* (n=5, 1 DT and 4 HPE), followed by *estimating and calculating with whole numbers* (n=2, 1 DT and 1 HPE), *recognising and using patterns and relationships* (n=2, 2 HPE) and *using spatial reasoning* (n=1, HPE). No connections were made in the planned numeracy moments to *using measurement* or *interpreting statistical information.*

Teachers were also asked which elements of the numeracy for the 21st century model aligned to their numeracy moment. Table 5 shows a visualisation where each row corresponds to a participant, and grey shading indicating an identified connection. Connections were primarily to knowledge, contexts and tools, with only two to dispositions, and none to critical orientation. This has parallels to work reported by Goos, Geiger and Dole (2014), in which only four of 18 teachers participating in numeracy professional development self-identified critical orientation as part of their trajectory through the numeracy model, and only then at the end point.

**Table 5**

*Visualisation of Connections Each Teacher Made to the Model of Numeracy for the 21st Century*

| Mathematical Knowledge | Contexts | Dispositions | Tools | Critical Orientation |
|---|---|---|---|---|



Eight of the 10 teachers felt their numeracy connection was embedded throughout the lesson. When asked to reflect on how the lesson went and if they noticed anything different about their students, the general feeling was that students seemed to have greater attention to detail and were engaged, as they felt the mathematical connection was “relevant context to their learning” (HPE teacher) and/or “they had a strong interest to make connections” (HPE teacher). This was noted in upper year levels too but more strongly amongst the students already opting for mathematics and physics courses.

No concerns were raised about a lack of student content knowledge other than one comment that students seemed to need a lot of support “analysing a wider range of data and recognising the relationships different pieces of evidence have to each other” (HPE teacher). Only one negative response about student engagement appeared across the ten reflections: in the heart rate lesson, the teacher reflected that some students were really reluctant to "do maths in PE … really avoided it”.

While the reflections indicated that most lessons didn’t go exactly as planned, all teachers indicated they would do their lesson again with modifications that, in general, related to making the mathematical connections clearer. We share one teacher’s reflection (data comparison lesson) below as this signals a shift in the importance given to planning numeracy connections.

> Ok—it wasn't specifically planned, I only noticed there was a need for the discussion. It was only a 5–10-minute teaching moment where I spoke through how to do it, but I didn't have any examples. …. Unfortunately, time got away and I ... forgot to come back to this activity, but I think it could have been quite valuable. [I noticed that] some seemed to be more actively accessing their data and reading with a more critical lens. Rather than just

seeing the data and then drawing some basic information from it that is easy to comprehend (which seemed to be an issue), some definitely reduced this…. This could have [been] great value if more specifically planned with examples to support and an engaging activity, rather than 'chalk and talk' impromptu.

## Summary and Conclusion

In Australia, students with low socio-economic status have experienced declining performance in mathematics (O'Keeffe & Paige, 2021), creating challenges for students when presented with critical numeracy experiences. Educational policy highlights the importance of numeracy across the curriculum, however there is still huge variation in the ways numeracy is defined and understood within the sector. The challenge of including more numeracy across the curriculum is multi-faceted. Whole school approaches to numeracy are reliant on clear and consistent messages about what numeracy is and whose responsibility it is. In this study, we saw better alignment between teachers' understanding of numeracy after targeted workshops aimed at challenging teachers to think differently about numeracy. Through dialogue and support to see and make numeracy connections rather than feel obliged to teach mathematics, these teachers and their students had predominantly positive experiences in making numeracy across the curriculum a priority in their learning areas.

Teachers in this study connected their numeracy moments primarily with mathematical knowledge, contexts and tools which is consistent with what they personally aimed to gain from being in the project. The limited connections to student dispositions and critical orientation suggest that more work needs to be done to support these teachers in developing a richer picture of numeracy beyond transfer of mathematical knowledge to other learning areas.

## References

AAMT (1997). *Policy on numeracy education in schools.* The Australian Association of Mathematics Teachers Inc. Adelaide.

Australian Curriculum, Assessment and Reporting Authority. (n.d.-a). *Foundation to year 10 curriculum: Mathematics* (version 8.4). ACARA. https://www.australiancurriculum.edu.au/f-10-curriculum/general-capabilities/numeracy/

Australian Curriculum, Assessment and Reporting Authority. (n.d.-b). *Foundation to year 10 curriculum: Mathematics* (version 9.0). ACARA. https://v9.australiancurriculum.edu.au/f-10-curriculum/general-capabilities/numeracy/

Bennison, A. (2015), Supporting teachers to embed numeracy across the curriculum: A sociocultural approach. *ZDM Mathematics Education*, *47*, 561–573.

Coffey, P., & Sharpe, R. (2021). An investigation into the teaching of numeracy in subjects other than mathematics across the curriculum. *International journal of mathematical education in science and technology*.

Geiger, V., Goos, M., & Forgasz, H. (2015). A rich interpretation of numeracy for the 21st century: A survey of the state of the field. *ZDM Mathematics Education*, 47(4), 531–548.

Goos, M., Geiger, V., & Dole, S. (2010). Auditing the numeracy demands of the middle years curriculum, in L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 210–217). Fremantle: MERGA.

Goos, M. (2020). Mathematics crossing borders: A comparative analysis of models for integrating mathematics with other disciplines in pre-service teacher education. In Radakovic, N., & Jao, L. (Eds.), *Borders in mathematics pre-service teacher education* (pp. 91 –115). Springer.

Goos, M., Geiger, V., & Dole, S. (2014). Transforming professional practice in numeracy teaching. In Y. Li, E. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 81–102). New York: Springer.

Goos, M., Geiger, V., Dole, S., Forgasz, H., & Bennison, A. (2019). *Numeracy across the curriculum: Research-based strategies for enhancing teaching and learning.* Allen and Unwin.

Muir, T. (2008). Principles of practice and teacher actions: Influences on effective teaching of numeracy. *Mathematics Education Research journal,* 20(3), 78–101.

O'Keeffe, L., & Paige, K. (2021). Re-highlighting the potential of critical numeracy. *Mathematics Education Research Journal, 33*, 285–299.

Thomson, S., De Bortoli, L. & Buckley, S. (2013). *PISA 2012: How Australia measures up*. Melbourne: ACER.

Yin, R. (2009*). Case study Research design and methods* (4th Edition). SAGE.

# The Use of Rubrics to Enhance Mathematical Teaching and Learning Practices when Engaging with Challenging Mathematical Tasks

Alison Hall
*Australian Council for Educational Research*
alison.hall@acer.org

The planned use of challenging mathematical tasks is explored in this paper. These tasks provide the opportunity for students to improve mathematical thinking by working on problems that they do not yet know how to answer. This research involved a heterogeneous class of Year Three students from a Catholic Parish Primary School in the northern suburbs of Melbourne. A rubric was also developed that was used, in conjunction with these tasks, to support discussions with students, broaden their strategies in finding solutions-and thereby improve their conceptual understanding. These pedagogical approaches were found to support the improvement of both students' conceptual understanding in mathematics and teachers' reflective practice.

This paper examines action research that tested a rubric as a method of identifying different levels of thinking by students working on challenging mathematical tasks. The research explored ways of using challenging tasks to increase the sophistication of students' mathematical strategies and explanations. The study aimed to answer the question "To what extent can rubrics be used to support teachers' use of challenging tasks to broaden the sophistication of students' mathematical concepts?"

This study involved the development and use of a rubric as an instrument to support teachers, students, and other stakeholders in knowing where students are in their mathematical learning and "where to next". The use of rubrics may allow teachers to infer gaps between a student's existing learning and the learning objectives. Francisco and Maher (2011) find that observations made by teachers helps them realise the value of providing students with opportunities to explore ideas and make decisions about their own mathematical reasoning and its development. Teachers need to know how to assess student's reasoning, in addition to assessing mathematical skills but they must be deliberate in the choice of tasks that ask students to apply both reasoning and mathematics skills.

Research evidence supports the use of challenging tasks in developing students mathematical reasoning skills. Clark and Clark (2002) recognise four qualities that characterise such tasks: they must enable students to produce different solutions, use different strategies, offer diverse final presentations, and fully engage students. Boaler (2019) argues that number sense, which is fundamentally important for students to learn, includes the learning of mathematical facts along with a deep understanding of numbers and the ways they relate to each other. Cheeseman et al. (2013) reflect on practices teachers use with these tasks and identify that students can engage with important mathematical ideas, be encouraged to explain their strategies, justify their thinking, and extend their knowledge in new ways.

Teachers must also consider a range of pedagogical factors when selecting challenging tasks. Kirschner et al. (2006) advocate for guided instruction being superior to minimal or even no guidance with challenging tasks. However, Marshall and Horton, (2011) alongside Russo and Hopkins (2017) agree that the value of exploring concepts, before any direct instruction, is in realising students' abilities to reason and think critically. Sullivan et al. (2012) also acknowledge the importance of matching tasks to curriculum content when selecting tasks. Cheeseman et al. (2016) discuss the importance of how a teacher introduces a task, including preparing students to have persistence, connecting the task with student experiences, providing manipulatives, and clarifying the task without showing how to reach the solution. This paper explores an approach aligned with the views of Russo and Hopkins (2017).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 235–242). Newcastle: MERGA.

Choosing tasks, structuring lessons around them and then incorporating them successfully into a thematic program of work requires careful thought. Teachers may need support in developing a classroom culture which supports this style of learning (Sullivan et al., 2013). Teachers can, therefore, be disinclined to use challenging tasks (Cheeseman et al., 2013) because they view them as unclear, too demanding or are concerned about low attaining students. Clarke et al. (2014) highlight the significance of a teacher's interest in a task in influencing its success, as well as teacher confidence in the enthusiasm and ability of their students. Kirschner et al. (2006) suggest that minimising instruction may lead to misunderstandings or piecemeal knowledge, so a teacher's approach needs to be balanced against providing too much information to reduce the level of challenge within the task. Jacobs et al. (2014) notes a danger of teachers taking over student thinking, controlling available tools and asking closed questions, removing the agency of students in their learning and development of conceptual understanding. Simon (2017) argues that an understanding of mathematical concepts requires students to learn concepts through mathematical activities in the form of challenging tasks. Rather than students using a sequence of actions already available to them based on their prior knowledge, challenging mathematical tasks support students to build new knowledge. The current study looks at supporting teachers to give sufficient, but not excessive instruction, using an assessment rubric to provide appropriate, timely feedback to students that progresses their conceptual mathematical understanding.

The rubric used for this study was developed from Bloom's Taxonomy (Krathwohl, 2002) and Webb's (1997) Depth of Knowledge Framework. Webb (1997) suggests that "challenge" in learning tasks promotes growth by keeping students engaged and his framework describes the quality of student thinking in various tasks. Krathwohl's update of Bloom's Taxonomy (2002) describes the cognitive level students demonstrate during learning, while the Depth of Knowledge (DoK) focuses more on the context—in this case the challenging task. While Hess et al. (2009) identify some limitations with Bloom's Taxonomy, the current study incorporates both Bloom's and Webb's models into the rubric.

## Project Design

### *Participants*

The participants were members of a heterogeneous Year Three classroom in a Catholic Parish Primary School in the northern part of Melbourne. The researcher was the full-time teacher of the class. A Pre-Service Teacher (PST) was also working full time in the classroom at the time of the study and was involved in the data collection. There was no requirement for a selection process as a convenience sample was being used.

### *Method and Rationale*

This study used an instrumental case study approach alongside action research. The action research aspect addressed the need to improve practices and the instrumental case study approach aligned with the observation of a situation. Such approaches provide teachers with opportunities to apply research methods to their teaching (Mills et al., 2010). They can also improve teachers' understanding of classroom practices and raise awareness of student learning that requires further investigation. Teachers can test approaches that may transfer well to similar classrooms (Yin, 2014) and integrate assessments generated by their own research into practice.

The current study used two tasks designed by Russo (2006, a and b): The Doughnut Tree task (which explores exponential doubling) and The Big (not so) Friendly Giant task (which explores halving). The first task was chosen because the class had been working on multiplication using doubling. Russo (2016a) contends that students working in middle primary classrooms are expected to have developed fluency with their doubles facts and should be exploring doubling as a rule. He argues that students would benefit from exploring exponential doubling at a younger age. The second task was chosen to meet the needs of a diverse group of students both providing accessibility and extension using enabling and extending prompts. Both tasks address the mathematically related skill of doubling and its inverse, halving, supporting students to link these two concepts. These tasks were conducted, and data collected from one class in Term 3 of 2018.

Enabling prompts are an integral aspect of challenging task design as they reduce the level of challenge through simplifying the problem, changing how the problem is represented, helping the students connect the problem to prior learning and/or removing a step in the problem (Sullivan, et al., 2006). Examples include reducing the starting numbers for the tasks, providing concrete materials, reducing the number of steps and altering the task presentation expectations.

Extending prompts can be used to engage students who finish the main challenge and may expose students to an additional task that is more challenging, but still requires them to use similar mathematical reasoning, conceptualisations, and representations as the main task (Sullivan et al., 2006). The appropriate prompt is selected by the teacher in real time, developed from their analysis of the potential task difficulties based on perceived cognitive load.

Russo and Hopkins (2017), reflecting on cognitive load theory (Sweller, 1998), identify seven steps to produce challenging mathematical tasks that aim to optimise the cognitive load for each student. These steps are: identify the primary learning objective, develop the task, look for possible other learning objectives, sort any objectives in line with their cognitive load, redesign the task, develop prompts to optimise the cognitive load and propose a lesson summary.

A launch, explore, discuss model (Stein et al., 2008) was used to deliver the lessons. This facilitated more explicit explanations, scaffolded connections and highlighted big mathematical ideas. In the launch phase, the word "challenging" was discussed with the students and the word was defined to engage the students in characterising an appropriate mindset. Each problem was introduced in a separate lesson, alongside available materials and recording expectations.

In the explore phase, students worked on the task individually or in pairs. Students were supported in solving the problem in whichever way suited them. Enabling prompts were offered and students had access to counters, number lines, 100 squares, notes about doubling and halving and were able to ask clarifying questions. Whilst students were working on the solutions three main questions were asked: How would you describe the problem in your own words? Would it help to create a diagram, draw a picture, or make a table? Could you try it with different numbers?

In the discuss phase, the teacher presented a summary of what had been observed, referring to specific strategies used by students, some of whom shared their thinking with the class. After looking at work from the first task, it was noted that although students were solving the problem, their thinking was not clearly shown. Support to assist this was provided in the launch phase for the second task.

In response to observations and discussion with the PST, a follow up lesson was proposed based on discussing ways of presenting strategies and solutions, answering questions with detail, revising, and editing work and explaining mathematical reasoning. It was felt that the students required more explicit teaching. alongside detailed examples of possible ways of presenting their solutions and reasoning. The lesson was based on a simpler task 'What Else Belongs' involving students identifying connections between three numbers–30, 12, 18. The students were asked to look for relationships between the numbers and why they might be placed in a group and then discover other numbers which could also belong in that group.

### *Student Work*

Student work was assessed against criteria from the rubric rather than being competitively ranked. This approach provides students feedback regarding how to improve rather than how they compare with others. A process of moderation was undertaken in which work was de-identified by the PST and shared between a team of three: the researcher, the Numeracy Leader, and the PST, to provide inter-relater reliability. Any work pieces with which scoring was uncertain were placed together for further consideration by another member of the team. If there was still uncertainty the whole team would look at the work. For the purposes of moderation, each team member selected a sample piece for each level of thinking using the DoK stages, and these were compared. For this study, due to ethical requirements, no student work could be reported or presented.

### *Limitations of the Data Collection*

Student work was collected at the end of each teaching session by the PST to maintain as much anonymity as possible. Apart from recognising some handwriting the researcher was not aware as to who had completed which work samples. According to Fraser (1997) the concept of a teacher as a researcher enables credible educational research to be undertaken, but the ethical predicaments faced could be more challenging than those met by an external researcher.

### *Data Sources and Analysis*

The primary data for analysis was generated by scoring student performances using rubrics and was based on expert teacher evaluation of student responses and explanations of their thinking during challenging tasks and student work (artefacts). Artefacts were grouped in terms of similar approaches to the task. These were rated against the rubrics. Scores were then categorised according to the nature of the students' responses. The final analysis involved a final sample size of 15 students selected randomly from those who had completed both tasks.

The approaches used by students reflected varying degrees of sophistication in their application of mathematical strategies. Approaches included the use of drawings, number lines, repeated addition and subtraction, formal algorithms, partitioning numbers to make doubling/halving simple, as well as the direct use of multiplication and division. The use of drawings was the most common strategy followed by partitioning numbers. The use of number lines and repeated addition and subtraction occurred with equal frequency. The direct use of multiplication facts and division was rare in both tasks.

### *Analysis*

The collected work was assessed against the developed rubric, as shown in Table 1. Responses were scored from 0 to 4 according to the levels on the rubric. Table 2 shows the types of thinking—Problem Solving, Reasoning, Representation and Connection—with a Zone of Proximal Development (ZPD) (Vygotsky & Cole) for each task. This ZPD enables the teacher to see areas of growth and where further support is needed.

**Table 1**

*Rubric For Levels of Thinking Used in Exploring Challenging Mathematical Tasks*

| Level of Thinking | Problem Solving | Reasoning | Representation | Connection |
|---|---|---|---|---|
| Level 1<br>Recall and reproduction | Did not understand the task.<br>What did the student appear to interpret the task as entailing? | Mathematical thinking is incorrect. | Used no mathematical language and/or notation.<br>Diagrams not relating to task. | Did not make any connections to the task or the numbers in the task. |
| Level 2<br>Basic application of skills and concepts | Understood part of the task.<br>Needed help to understand the entire task. Strategy works for part of the task. | Some mathematical thinking or explanation is correct.<br>Needed help to explain the task. | Used some mathematical language and/or notation.<br>Some diagrams were used to represent the task. | Tried to make connections to previous learning related to the task. |
| Level 3<br>Strategic thinking | Understood the task and the strategy they used works. | Mathematical thinking and explanation correct.<br>Some thinking systematic. | Used clear mathematical language and/or notation<br>Diagrams related to task. | Made some mathematical connections to previous |
| Level 4<br>Extended Thinking | Understood the task.<br>Used an efficient strategy.<br>Extension activities were completed. | Detailed and accurate explanation of the strategy used to solve the task.<br>Mathematical thinking was correct and systematic. | Used specific math language and/or notation throughout their work.<br>Diagrams related directly to task and explained student thinking.<br>Extension activities were completed. | Recorded mathematical connections to mathematical big ideas and strategies previously used.<br>Extension<br>activities were completed. |

It appears that in Task 1 students had more problems with Representation whereas in Task 2 Connections was more of an issue. Scores for three areas—Problem Solving, Reasoning and Representation—increased quite considerably, whereas the Connection score remained very similar. It appears this was an area with which many students struggled and where future explicit teaching needs to be focused. Problem Solving scores had the greatest increase. This could be attributed to greater familiarity with the type of tasks, students acting on discussions and feedback from the first task and/or students finding the second task easier to solve.

**Table 2**

*Conditional Formatting of Rubric Elements for Analysed Work Samples to Create A ZPD*

| | Task 1. The Doughnut Tree | | | | Task 2. The Big (not so) Friendly Giant | | | |
|---|---|---|---|---|---|---|---|---|
| | Problem Solving | Reasoning | Representation | Connection | Problem Solving | Reasoning | Representation | Connection |
| 22 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 1 |
| 19 | 1 | 2 | 1 | 1 | 2 | 2 | 2 | 1 |

| | Task 1. The Doughnut Tree | | | | Task 2. The Big (not so) Friendly Giant | | | |
|---|---|---|---|---|---|---|---|---|
| | Problem Solving | Reasoning | Representation | Connection | Problem Solving | Reasoning | Representation | Connection |
| 23 | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 1 |
| 21 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 1 |
| 5 | 1 | 1 | 1 | 2 | 3 | 3 | 2 | 1 |
| 27 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 |
| 12 | 2 | 2 | 1 | 1 | 3 | 3 | 3 | 1 |
| 6 | 2 | 2 | 1 | 1 | 4 | 3 | 2 | 2 |
| 10 | 2 | 2 | 2 | 2 | 4 | 3 | 2 | 1 |
| 8 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 2 |
| 26 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 2 |
| 28 | 3 | 2 | 2 | 2 | 3 | 3 | 3 | 3 |
| 11 | 3 | 2 | 2 | 2 | 4 | 3 | 4 | 4 |
| 24 | 3 | 3 | 2 | 1 | 4 | 4 | 4 | 3 |

## Results and Evaluation

The data analysis and results were related back to the initial research question proposed: "To what extent can rubrics be used to support teachers' use of challenging tasks to broaden the sophistication of students' mathematical concepts?" Concepts analysed included Problem solving, Representation and Reasoning. Results from the ZPD analysis showed several interesting findings. Nearly all students used a broader range of strategies in the second task compared with the first. Students who used broader strategies used them at a more sophisticated level. Most students diversified the way they represented their thinking. About half of the students showed limited development in Connection, indicating an area for targeting teaching strategies. That students were least successful with Connection confirmed the teachers' view that previous tasks these students have experienced have focused on solving the problem rather than conceptual understanding. The element of Connection scored low in both tasks. This may have been because these tasks were different from other mathematics tasks being undertaken and students viewed them in isolation from their everyday mathematics lessons.

## Conclusion

This research was undertaken hoping to identify different levels of thinking by students working on challenging tasks. With all classroom-based research, uncertainty in data is likely. There are limitations of this being a small study as it was conducted in one classroom of one school, over a short period of time. For these reasons any claims cannot be generalised. However, the findings suggest there are many types of thinking demonstrated by students working on challenging tasks. This study suggests that it is possible to assess the depth of thinking that students engage in when solving challenging tasks. It appears that teachers can support students in refining their thinking and their ability to record strategies and reasoning.

There are implications for the classroom. These include the potential of teacher developed rubrics that support observation and timely feedback. Feedback may move students on from their current level of conceptual understanding, broaden the range of strategies they are comfortable in using and encourage clear explanations of thinking. Teachers could develop a progression of strategies towards a conceptual understanding of multiplication and division incorporating what they observe in their students.

This small study could be used to inform a larger study around using evidence informed practice and formative assessments to improve teaching and learning for a range of different mathematical concepts. Further research could include refining the descriptive sections of the rubric and developing challenging tasks for use in other areas of mathematics. Consideration would need to be taken of other elements involved in the successful implementation of challenging tasks including encouraging students to persist, fostering students in the skills of listening to others and teacher reflections on their own and student experiences. Another study, published after this research was completed, produced an Assessing Mathematical Reasoning Rubric (Loong et al., 2018). This work creates a rubric which teachers found helpful to support their understanding of how to develop reasoning in students and in reporting student progress. As with the study by Loong et al. (2018) further research is needed to ensure that such rubrics are pragmatic, time efficient and provide appropriate information.

## References


American Psychological Association. (2020). *Publication manual of the American Psychological Association* (7th

Boaler, J. (2019). Developing mathematical mindsets: The need to interact with numbers flexibly and conceptually. *American Educator, 42*(4), 28–33.

Cheeseman, J., Clarke, D., Roche, A., & Wilson, K. (2013). Teachers' views of the challenging elements of a task. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow. Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia* (pp. 154–161). MERGA.

Cheeseman, J., Clarke, D., Roche, A., & Walker, N. (2016). Introducing challenging tasks: Inviting and clarifying without explaining and demonstrating. *Australian Primary Mathematics Classroom, 21*(3), 3.

Clarke, D., & Clarke, B. (2002). Challenging and effective teaching in junior primary mathematics: What does it look like? In M. Goos, & T. Spencer (Eds.), *Mathematics: Making waves. Proceedings of the 19th biennial conference of the Australian Association of Mathematics Teachers* (pp. 309–318). AAMT.

Clarke, D., Roche, A., Sullivan, P., & Cheeseman, J. (2014). Creating a classroom culture which encourages students to persist on cognitively demanding tasks. In K. Karp (Ed.), *Annual perspectives in mathematics education* (pp. 67–76). National Council of Teachers of Mathematics.

Francisco, J. M., & Maher, C. A. (2011). Teachers attending to students' mathematical reasoning: Lessons from an after-school research program. *Journal of Mathematics Teacher Education, 14*(1), 49–66.

Fraser, D. M. (1997). Ethical dilemmas and practical problems for the practitioner researcher. *Educational Action Research, 5* (1), 161–171.

Hess, K. K., Jones, B. S., Carlock, D., & Walkup, J. R. (2009). Cognitive rigor: Blending the strengths of Bloom's taxonomy and Webb's depth of knowledge to enhance classroom-level processes. https://eric.ed.gov/?id=ED517804

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist, 41*(2), 75–86. https://doi.org/10.1207/s15326985ep4102_1

Krathwohl, D. R. (2002). A revision of Bloom's taxonomy: An overview. *Theory into Practice, 41*(4), 212-218. https://doi.org/10.1207/s15430421tip4104_2

Loong, E., Vale, C., Widjaja, W., Herbert, S., Bragg, L. A., & Davidson, A. (2018). Developing a rubric for assessing mathematical reasoning: A design-based research study in primary classrooms. In J. Hunter, P. Perger, & L. Darragh (Eds.), *Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 503–510). Auckland: MERGA.

Marshall, J. C., & Horton, R. M. (2011). The relationship of teacher-facilitated, inquiry-based instruction to student higher-order thinking. School Science and Mathematics, 111(3), 93–101. https://doi.org/10.1111/ j.1949-8594.2010.00066

Mills, A. J., Durepos, G., & Wiebe, E. (Eds.) (2010). *Encyclopedia of case study research*. Sage.

Jacobs, V. R., Martin, H. A., Ambrose, R. C., & Philipp, R. A. (2014). Warning signs! *Teaching Children Mathematics, 21*(2), 107-113. https://doi.org/10.5951/teacchilmath.21.2.0107

Russo, J. (2016a). Exploring exponential growth: Through repeated doubles patterns. *Prime Number, 31*(1), 18.

Russo, J. (2016b). Teaching mathematics in primary schools with challenging tasks: The big (not so) friendly giant. *Australian Primary Mathematics Classroom, 21*(3), 8.

Russo, J., & Hopkins, S. (2017). Class challenging tasks: Using cognitive load theory to inform the design of challenging mathematical tasks. *Australian Primary Mathematics Classroom, 22* (1), 21.

Schön, D. A. (2017). *The reflective practitioner: How professionals think in action*. Routledge.
Stake, R. E. (1995). *The art of case study research*. SAGE Publications.
Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10* (4), 313–340. https://doi.org/10.1080/10986060802229675
Simon, M. A. (2017). Explicating mathematical concept and mathematical conception as theoretical constructs for mathematics education research. *Educ Stud Math, 94,* 117–137. https://doi.org/10.1007/s10649-016-9728-1
Sullivan, P. A., Aulert, A., Lehmann, A., Hislop, B., Shepherd, O., & Stubbs, A. (2013). Classroom culture, challenging mathematical tasks and student persistence. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow. Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia* (pp. 618–625). MERGA.
Sullivan, P., Clarke, D., & Clarke, B. (2012). *Teaching with tasks for effective mathematics learning* (Vol. 9). Springer.
Sullivan, P., Mousley, J., & Zevenbergen, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. *International Journal of Science and Mathematics Education, 4*(1), 117–143–143. https://doi.org/10.1007/s10763-005-9002-y
Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science, 12*(2), 257–285. https://doi.org/10.1016/0364-0213(88)90023
Vygotsky, L. S., & Cole, M. (n.d.). Mind in society: The development of higher psychological processes.
Webb, N. L. (1997). Criteria for alignment of expectations and assessments in mathematics and science education. *Research Monograph* No. 6. https://files.eric.ed.gov/fulltext/ED414305.pdf
Yin, R. K. (2014). *Case study research* (5th ed). Sage.

# Sketching as a Spatial Tool: A Qualitative Study of Grade Three Students' Representation of Reflection

Danielle Harris
*University of Canberra*
danielle.harris@canberra.edu.au

Tracy Logan
*University of Canberra*
tracy.logan@canberra.edu.au

Tom Lowrie
*University of Canberra*
thomas.lowrie@canberra.edu.au

The concept of symmetry is intrinsic to spatial reasoning and mathematics. Yet many students struggle with the concept, as well as its wide-reaching affordances for mathematics. Through a study of students' drawings of reflection we were able to identify some characteristics of students' representation of bilateral symmetry, and where errors commonly occur. We propose suggestions for how these findings can be linked more broadly to mathematics instruction.

Spatial reasoning is considered a powerful tool in the teaching and learning of mathematics (Ishikawa & Newcombe, 2021). Countries around the world have incorporated spatial reasoning into core curriculum standards (e.g., Canada, Singapore) and the notion of space forms a content strand of the new Australian curriculum (ACARA, 2022a). At an individual skill level, spatial visualisation is deemed crucial to supporting mathematics achievement (Hawes et al., 2022). However, much of what we know about spatial visualisation has emerged from psychological studies based on testing (Lowrie et al., 2020), or inferred when a task is considered spatial (Patahuddin et al., 2020). Furthermore, spatial visualisation is often used as a catch-all phrase for complex mental manoeuvres with little regard for how this skill is implemented in mathematical practice (Ramful et al., 2015). This study was designed in response to the high level of interest in spatial visualisation for supporting mathematics understanding, and the need for further research on the nature of spatial reasoning beyond performance on spatial tests (Ishikawa & Newcombe, 2021).

One component of spatial visualisation that is contained both within the psychological construct (Linn & Petersen, 1985; Lowrie & Logan, 2018) and as a central idea in mathematics (ACARA, 2022a; Ng & Sinclair, 2015) is symmetry. Symmetry is often discussed in terms of folding over a line (Leikin et al., 2000), and in fact, that is how the construct is often measured (i.e., Paper Folding Test; Ekstrom et al., 1976). However, focusing on the line of symmetry has been shown to impact the ability to recognise symmetry in different ways (Clements & Battista, 1992; Mulligan et al., 2020). In this study, we analysed student drawings of reflections, with a focus on spatial representation of symmetry as a proposed precursor to mathematical language.

## Theoretical Framing of Spatial Visualisation

Spatial visualisation (SV) is a spatial skill, distinct from visualisation (Vs). The former (SV) is primarily a psychological construct used to predict or train skills (Hawes et al., 2022), and is often operationalised by tests requiring complex, multistep spatial manoeuvres, rather than a clearly defined skill (Linn & Petersen, 1985). By contrast, visualization (Vs) is a term more dominant in mathematics education and concerns the creation and interpretation of visual representations, both in the mind's eye and in concrete form (Arcavi, 2003; Gutiérrez, 1996). These two constructs are woven together in work by Hegarty and Kozhevnikov (1999) and Presmeg (1986), where the ability to visualise spatially was closely related to performance on mathematics tasks, while the ability to visualise tended to result in representations that were lacking the critical information to support successful problem-solving. In fact, Kozhevnikov et al. (2010) found a trade-off between one's

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 243–250). Newcastle: MERGA.

ability to visualize objects and visualize spatially. The magnitude of this difference increases with age, where as one type of skill develops, the other often wanes, particularly in the science and visual arts fields.

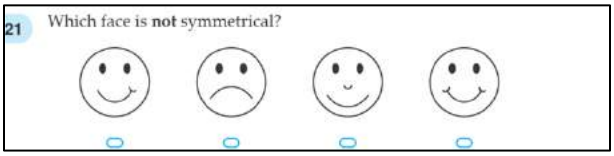
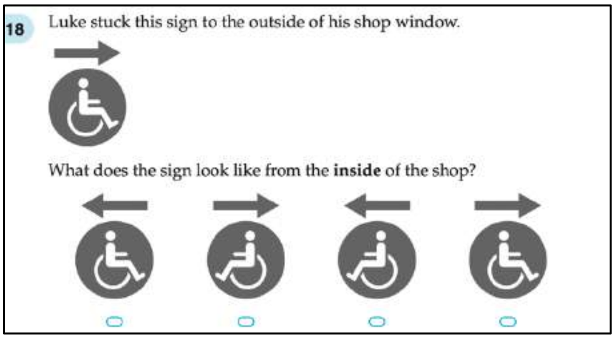
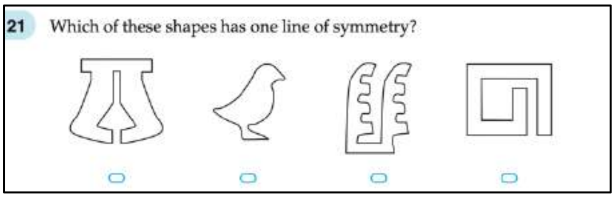
### *Symmetry/Reflection*

In this paper we are focused on spatial visualisation in the form of primary students' ability to represent bilateral symmetry through reflections. The new Australian mathematics curriculum introduces symmetry in year 4 (as opposed to year 3 in the previous curriculum) through identifying environmental symmetry, line and rotational symmetry, and creating symmetrical patterns (ACARA, 2022a). It is noteworthy that this curriculum includes symmetry in movement and shape from year 3 within Health and Physical Education with links to mathematics learning areas. This reflects the idea that symmetry has implications and opportunities for development across a range of areas.

Students tend to struggle with the concept of symmetry (Clements & Battista, 1992; Sarama & Clements, 2008). Mulligan et al. (2020) found that a common error amongst students results from focusing on lines of symmetry of the bounding shape, not the spatial relations within. It is possible that students are struggling more with the mathematical language around symmetry than lacking spatial competence. To examine this, we consider performance on three publicly available symmetry items from the National Assessment Program—Literacy and Numeracy (NAPLAN; ACARA, 2022b) as a means of understanding broad understanding and challenges surrounding symmetry comprehension (see Table 1).

**Table 1**

*NAPLAN Year 3 Symmetry Items (*ACARA, *2022b)*

| Item | | Year | N | Success | Errors |
|---|---|---|---|---|---|
| 1 |   | 2013 | 9847 | A: 50% | B: 34%<br>C: 11%<br>D: 4% |
| 2 |   | 2012 | 3828 | C: 59% | A: 7%<br>B: 8%<br>D: 26% |
| 3 |   | 2012 | 3799 | A: 54% | B: 14%<br>C: 11%<br>D: 20% |

In Table 1 success rates are above chance, but the errors provide insights into student difficulties. For item 1, half the students had both the conceptual knowledge and ability to discriminate between symmetrical and non-symmetrical images. Of those that were incorrect, the largest proportion chose the image depicting a negative emotion, perhaps reflecting the negative wording of the question.

Item 2 measured students' ability to visualise and could be completed using a perspective-taking strategy. The greatest proportion of errors indicate spatial difficulties, but the item had the highest success rate, coinciding with no language around symmetry. For Item 3, just over half the students identified a line of symmetry but the distribution across the alternate responses suggests that some students had trouble with the question demands. This may indicate a lack of exposure to explicit mathematical language such as the term *symmetry*, rather than students' spatial knowledge of the concept of symmetrical.

Some have argued that young children's capacity for understanding symmetry is much greater than curriculum guidelines (and assessment) would suggest, and that early introduction to symmetry concepts, through engagement with concrete materials, can provide a strong foundation for later problem-solving (Ng & Sinclair, 2015; Sarama & Clements, 2008). Through intentional teaching, exposure to spatial tasks helps students build understanding of spatial transformations like rotation and reflection (Mulligan et al., 2020).

## Sketching Representations in Mathematics

Visualisation, in the form of pictorial representation, is an important component of mathematics (Arcavi, 2003; Gutiérrez, 1996; Way, 2021). Presmeg and Balderas-Cañas (2001) cite two purposes of visualisation in mathematics: to make sense and to solve. For both functions, technical drawing knowledge and drawing quality are not the most critical elements, rather an ability to extract and represent key information (Rellensmann et al., 2021). The language used to differentiate realistic depictions from simplistic representations that embody critical relations varies between studies (e.g., Hegarty & Kozhevnikov, 1999; Presmeg, 1986), but these abstractions support the shift from visual to symbolic representation (Lowrie, 2020). However, incorporating drawing into mathematical representation is not a natural phenomenon, even amongst young children, and therefore may require teacher help to incorporate drawing into mathematical practice (Bakar at al., 2016).

## Context of this Study

Children can differentiate between symmetrical and asymmetrical; however, 1) this is a static process involving visual discrimination, not an understand of the relationships between the components (Ng & Sinclair, 2015); and 2) in advanced mathematics, even after years of instruction, many still struggle to identify lines of symmetry (Clements & Battista, 1992). Therefore, although young children can understand the concept of symmetry, the mathematical language can be a hurdle to demonstrating this knowledge. In this study we focus on student drawings as a means of understanding their spatial representations (Lowrie & Logan, 2018; Way, 2021). We sought to combine two related, yet previously isolated, lines of enquiry:

- SV instruction is beneficial for mathematics outcomes (Hawes et al., 2022). Yet much of the intervention work remains focused on spatial skills training, instead of activities aligned to mathematics that promoting spatial reasoning (Lowrie et al., 2020).
- Sketching is helpful for understanding students' mental processes (Way, 2021) and supports learning (Bakar et al., 2016). However, the way students demonstrate their spatial visualization skills using pictorial representations has not been examined.

*Research questions*. We focus on students' sketched reflections to extend what we know about SV from studies based on testing. Although young children can perform symmetry tasks earlier than curriculum suggests, they may not have the language or capability to express their spatial thinking. Here we present an intentional spatial lesson designed to help students think spatially and apply their SV skills through demonstration of reflection to answer the questions:

- How do grade 3 students use self-generated drawings to encode their SV skills?
- What impact does type of representation (i.e., detailed or structured) have on accuracy?

# Method

## *Participants and Method*

The participants were Grade 3 students from three NSW primary schools. Students were given a series of photographs to reflect, or the option to generate their own. The photographs were images of children participating in physical activities (see examples in Figure 1).

Figure 1. Sample images provided to students for reflection activity.

Students were provided with mirrors, drawing implements, and paper or whiteboards. The open nature of the task meant that students were free to complete the task as they wished with minimal constraints. Drawings were photographed for later analysis. Here we present a sample of data from students who chose a soccer player ($N = 8$), ballerina ($N = 10$), or free drawn houses ($N = 6$). Ethics approval was granted by the authors' university and relevant state educational jurisdictions. Parental consent was sought from all students before participation.

## *Data Analysis*

We coded student drawings using the constant comparison methodology reported by Corbin and Strauss (2008). In this way, we drew meaning and classifications from the drawings through similarities, differences, and errors. student drawings were categorised as detailed (i.e., containing multiple/complex elements that add to visual detail but are extraneous to subject structure) or structured (i.e., containing key structural components only e.g., stick figures or minimal geometric shapes). The coding rubric we used for errors is presented in Table 2.

**Table 2**

*Classification of Student Errors*

| Errors | Coding/Description |
|---|---|
| E1 | Failure to reflect large components (e.g., legs, arms) |
| E2 | Failure to reflect small components (e.g., facial features) |
| E3 | Direction of large components (e.g., body—limbs pointing the wrong way would be E1) |
| E4 | Direction of small components (e.g., hair, patterns) |
| E5 | Missing detail in reflection (e.g., patterns or objects) |
| E6 | Errors in spatial relations between components |

Although proportionality is highly studied in student drawings (Quane et al., 2021), the focus in our coding was on the representation of attributes and spatial relations in line with our conceptual distinction between Vs and SV.

## Results

Just over half of the drawings contained a line of symmetry (14/24). Most students chose to recreate the photos as well as the reflected version (16/18). This required students to decode the photo and extract critical information, this representation became the source of their reflection. Student recreations of the photos were the basis of our analysis, not the accuracy relative to the original. In terms of errors in the reflected drawings, we categorised these based on the subject and the nature of the depiction (i.e., detailed versus structured; see Table 3).

**Table 3**

*Error Types and Frequencies for Detailed and Structured Drawings*

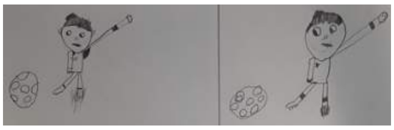
| | | Detailed | | | | | | | Structural | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Subject | N | E1 | E2 | E3 | E4 | E5 | E6 | N | E1 | E2 | E3 | E4 | E5 | E6 |
| Ballerina | 7 | 3 | 1 | 2 | 1 | 4 | 2 | 3 | 1 | 2 | 1 | 1 | 5 | 1 |
| Soccer | 2 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 0 | 1 | 0 | 0 | 0 | 2 |
| House* | 4 | 0 | 2 | 0 | 0 | 1 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |

*Freeform—no photo provided.

The ballerina drawings included a higher proportion of detailed drawings compared with the soccer drawings. This difference is not merely a result of the photo content, as both images include details on the clothing and shoes of the subjects as well as the unusual body and face orientation. We present examples of detailed and structured drawings for each of the subject categories with varying degrees of error (see Table 4). These drawings were typical of other representations that we observed.

**Table 4**

*Sample Drawings in the Different Categories*

| | Ballerina | Soccer | House |
|---|---|---|---|
| Detailed |  Some elements reflected such as the direction of the face and the position of the bun. However, the legs are translated rather than reflected (E1). |  Explicit detail about boy and ball, but little indication that the student understood the term symmetry as both images are positioned the same way (E1, E2). |  Main form is reflected across line of symmetry but components (i.e., windows) not reflected in terms of number or position (E5, E6) |
| Structured |  Subject structure reflected to match original but missing details (E6) and small errors (E2, E4). |  Boy and soccer ball show minimal detail and fairly accurate reflected images, down to the grass. |  Another structural drawing with minimal detail but well-matched reflected components. |

From these drawings it is apparent that the detailed drawings did leave more opportunities for error, on both large (E1; see Ballerina legs in the detailed drawing) and small (E4; see Ballerina hair in detailed drawing) scales. By contrast, the detailed house drawing errors presented in Table 3 demonstrate primarily missing details in the reflected images compared with the structural drawings which contained no errors.

## Discussion

In this paper we explored student representation of bilateral symmetry and found that even at a young age, students were able to accurately generate reflected drawings of complex images. The use of detailed or structured drawings was associated with the subject, and the nature of the errors varied by subject as well. That is, geometric drawings of houses and the representations of a soccer player which used a structured approach contained fewer errors. By contrast, the types of errors varied in proportion between detailed and structured representations for the ballerina drawings, both contained missing parts, but more large components were not reflected in the detailed drawings, compared with small components in the structured drawings. It is noteworthy that even the structured ballerina drawings contained a greater amount of detail on the central subject than the soccer drawings (see Table 4).

The accuracy with which reflections were depicted for structured drawings is consistent with existing work in mathematics where representations focused on key structural relations were less error-prone than detailed images (Bakar et al., 2016; Hegarty & Kozhevnikov, 1999). However, as there was no mathematical content to extract in this task it is interesting that this pattern still holds. The drawings generated by students to represent symmetry provide insights into their internal spatial representations (Way, 2021). Even in the absence of mathematical content, similar patterns emerged in students' representations in terms of their use of detail or structure to represent symmetry. For most students, some measure of reflection was represented, indicating they understood the concept of symmetry. However, the nature of the encoded representations, the subject matter, and their use of the line of symmetry impacted their ability to demonstrate this knowledge.

### *Efficient Encoding*

There were a smaller proportion of errors for the house drawings, compared with the photos, in fact none for the structured house drawings contained errors. There are two possibilities for this: 1) the geometric structure of the houses made it easier for one-to-one encoding, or 2) there was no need to decode an original image, thereby reducing the amount of cognitive work required in the task. However, it is less likely to be (2) as most students generated their own images for reflection even for the photos. Therefore, when supporting students to use drawings the content matters in how accurately they can encode the representation. This aligns with Lowrie's (2020) analysis of a word problem requiring analysis of an array of chairs, the subject was simple, but the more components students included in their drawings, the more opportunity there was for error in solving the task.

### *Line of Reflection*

Just over half the students chose to include a line of symmetry, however, even in the absence of the line students demonstrated their ability to conceptualise bilateral symmetry. Despite the provision of tools such as mirrors, it seems that students treated each drawing as independent and the line of symmetry between each side was not a defining characteristic of the drawing. This has implications for how we conceptualise symmetry. Much of the curriculum-related symmetry content exists within bounded geometric objects (ACARA, 2022a; Mulligan et al., 2020). However, symmetry exists in many forms within mathematics and the world more broadly (Ng & Sinclair, 2015), as reflected by the new Australian curriculum (ACARA, 2022). Therefore students' conceptualisation of symmetry and the classroom learning around the construct needs to be broader too.

### *Implications of our Study for Teaching and Learning*

At a surface level this task was about students' ways of representing reflection, however the rich, underlying mathematical opportunities in a task such as this are important for future work (Ng & Sinclair, 2015). Symmetry tasks promote relational thinking in terms of spatial organisation that have implications when understanding the construction of coordinated reference systems such as the cartesian plane (Clements & Battista, 1992).

Reflection and symmetry are important for higher level mathematics (Clements & Battista, 1992; Leikin et al., 2000; Ng & Sinclair, 2015). For example, in geometry, lines of symmetry in geometrical figures are explicit. However, there is also an opportunity to use symmetry to help students with number-based problems. Consider the balance between sides of an equal sign, rather than a one-directional operation, reframing conceptual understanding to include the notion of equivalence may help students with more complex mathematical content (Kieran, 1981; Patahuddin et al., 2020). Currently the Australian Curriculum refers to symmetry in dynamic terms such as flip or fold. If other language was incorporated to reflect balance and equivalence, not just the end result of an operation, we may see better transfer to number-based work providing foundations for algebraic reasoning.

### *Limitations and Future Directions*

Many students naturally use graphical representations in their play, although there is still some reluctance in mathematics problem-solving (Bakar et al., 2016). If students are supported to encode representations using structural relations they will have greater opportunities for success. However, it is evident from this and other work (Hegarty & Kozhevnikov, 1999) that some students don't naturally identify the critical components and therefore errors in representations occur. This has implications for future work in that 1) students need support to focus on key structural components for spatial representations, and 2) effective spatial representations need to be linked to mathematical content in a way that builds conceptual understanding. We did not connect students' spatial representations with their mathematical competence in this study, that is an avenue for future work.

There is a possibility that gender had an impact on the choice of subject matter and the use of detailed or structured representations. We did not collect gender information in this lesson, but future work could explore the role of gender in the use of graphic representations.

### *Conclusion*

Students represent space and relations in different ways and although we know these skills are important, we need to 1) foster their development in the context of mathematics classrooms, and 2) not reduce children's spatial thinking down to performance on tests. There is a wealth of literature to draw on to build thoughtfully designed spatial tasks that take advantage of students' skills and move beyond the drill and practice of spatial tests to support mathematical understanding.

## References

Australian Curriculum, Assessment and Reporting Authority. (2022a). *The Australian curriculum: Mathematics*. ACARA. https://v9.australiancurriculum.edu.au/

Australian Curriculum, Assessment and Reporting Authority. (2022b). *NAPLAN sample tests.* ACARA. https://www.nap.edu.au/nap-sample-assessments

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52,* 215–241.

Bakar, K. A., Way, J., & Bobis, J. (2016). Young children's drawings in problem solving. In B. White, M. Chinnappan, & S. Trenholm (Eds.), *Opening up mathematics education research. Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia* (pp. 86–93). Adelaide: MERGA.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). Macmillan.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research (3rd ed.): Techniques and procedures for developing grounded theory*. SAGE Publications.

Ekstrom, R. B., French, J. W., Harman, H. H., & Dermen, D. (1976). *Kit of factor-referenced cognitive tests*. Educational Testing Service.

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. In L. Puig, & A. Gutiérrez (Eds.), *Proceedings of the 20th international conference for the Psychology of Mathematics Education* (Vol. 1, pp 3–19). Valencia, Spain.

Hawes, Z., Gilligan-Lee, K. A., & Mix, K. S. (2022). Effects of spatial training on mathematics performance: A meta-analysis. *Developmental Psychology, 58*(1), 112–137.

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4), 684–689.

Ishikawa, T., & Newcombe, N. S. (2021). Why spatial is special in education, learning, and everyday activities. *Cognitive Research: Principles and Implications, 6*(20).

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317–326.

Kozhevnikov, M., Blazhenkova, O., & Becker, M. (2010). Trade-off in object versus spatial visualization abilities: Restriction in the development of visual-processing resources. *Psychonomic Bulletin & Review, 17,* 29–35.

Leikin, R., Berman, A., & Zaslavsky, O. (2000). Learning through teaching: The case of symmetry. *Mathematics Education Research Journal, 12*(1), 18–36.

Linn, M. C., & Petersen, A. C. (1985). Emergence and characterization of sex differences in spatial ability: A meta-analysis. *Child Development, 56,* 1479–1498.

Lowrie, T. (2020). The utility of diagrams in elementary problem-solving. *Cognitive Development, 55*(July-September).

Lowrie, T., & Logan, T. (2018). The interaction between spatial reasoning constructs and mathematics understandings in elementary classrooms. In K. S. Mix, & M. T. Battista (Eds.), *Visualizing mathematics: The role of spatial reasoning in mathematical thought* (pp. 253–276). Springer.

Lowrie, T., Resnick, I., Harris, D., & Logan, T. (2020). In search of the mechanisms that enable transfer from spatial reasoning to mathematics understanding. *Mathematics Education Research Journal, 32*(2), 175–188.

Mulligan, J., Oslington, G., & English, L. (2020). Supporting early mathematical development through a 'pattern and structure' intervention program. *ZDM Mathematics Education, 52,* 663–676.

Ng, O., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM Mathematics Education, 47*, 421–434.

Patahuddin, S. M., Rokhmah, S., & Ramful, A. (2020). What does teaching of spatial visualisation skills incur: An exploration through the visualise-predict-check heuristic. *Mathematics Education Research Journal, 32*(2), 307–329.

Presmeg, N. (1986). Visualization in high school mathematics. *For the Learning of Mathematics, 6*(3), 42–46.

Presmeg, N., & Balderas-Cañas, P. E. (2001). Visualization and affect in nonroutine problem solving. *Mathematical Thinking and Learning, 3*(4), 289–313.

Quane, K., Chinnappan, M., & Trenholm, S. (2021). Draw yourself doing mathematics: Developing an analytical tool to investigate the nature of young children's attitudes towards mathematics. *Mathematics Education Research Journal.* https://doi.org/10.1007/s13394-021-00399-2

Ramful, A., Ho, S. Y., & Lowrie, T. (2015). Visual and analytical strategies in spatial visualisation: Perspectives from bilateral symmetry and reflection. *Mathematics Education Research Journal, 27*, 443–470.

Rellensmann, J., Schukajlow, S., Blomberg, J., & Leopold, C. (2021). Does strategic knowledge matter? Effects of strategic knowledge about drawing on students' modeling competencies in the domain of geometry. *Mathematical Thinking and Learning*, https://doi.org/10.1080/10986065.2021.2012741

Sarama, J., & Clements, D. H. (2008). Mathematics in early childhood. In O. N. Saracho, & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 67–94). Age Publishing Inc.

Way, J. (2021). Issues and affordances in studying children's drawings with a mathematical eye. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in mathematics education: Foundations and pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (p. 109). Singapore: MERGA.

# Preservice Secondary Mathematics Teachers' Perceptions of Teacher Knowledge and its Sources

Vesife Hatisaru
*Edith Cowan University*
v.hatisaru@ecu.edu.au

Julia Collins
*Edith Cowan University*
julia.collins@ecu.edu.au

Preservice secondary mathematics teachers' perceptions of teacher knowledge and of possible sources of that knowledge is investigated through examining their responses to an open-ended questionnaire. Participants place greatest emphasis on mathematics content knowledge and mathematics pedagogical knowledge and expect to gain most of their knowledge through formal preparation within the professional learning system rather than through self-study or through interactions with peers. This emphasises how important it is for schools and professional associations to provide regular formal learning opportunities, because future teachers may otherwise not be self-motivated to continue improving their skills independently of this.

PISA results from 2003 to 2018 show a consistent decline in the mathematics literacy of Australian students over the past 15 years, together with growing mathematics anxiety, particularly among girls (Thomson et al., 2019). A growing proportion of mathematics teachers are teaching out-of-field (Thomson et al, 2021), meaning students are entering university degrees with a less strong mathematical background than cohorts from previous generations. Significant attention has been given to both teacher quality standards and professional growth. What is less well known is how teachers, including preservice secondary mathematics teachers (PSMTs), perceive the notions of teacher knowledge and sources of that knowledge. We direct our focus to this area guided by the research questions:

- What are PSMTs' perceptions of teacher professional knowledge and of sources of that knowledge?
- To what extent do PSMTs recognise teacher knowledge domains and possible sources of teacher knowledge?

Considering that teachers' perceptions can influence their actions (e.g., Richardson, 1996), we believe that eliciting teachers' views is always something that we, as mathematicians and mathematics educators, should keep in mind. These elicitations can inform mathematics teacher educators in enacting a vision of teacher education and development and plausibly can improve the design of teacher education and development programmes.

## *Teacher Knowledge*

For decades, what teachers should know and understand to teach the content effectively has been a focus of interest for researchers and teacher educators. Various teacher knowledge frameworks have been developed. For example, according to Shulman (1987), dimensions of knowledge necessary for teachers include content knowledge, general pedagogical knowledge (e.g., strategies of classroom management), curriculum knowledge (e.g., the materials and programs), pedagogical content knowledge (PCK), knowledge of learners and their characteristics, knowledge of educational contexts (e.g., the governance and financing of school districts), and knowledge of educational ends, purposes, and values. To Grossman (1990), the core of professional knowledge for teaching are general pedagogical knowledge, subject matter knowledge, pedagogical content knowledge, and knowledge of context. Ernest (1989) proposes a model where both theoretical knowledge (e.g., knowledge of mathematics and of other subject matter) and practical knowledge (e.g., knowledge of organisation and management for mathematics teaching and knowledge of the context of teaching mathematics such as knowledge of school and students taught) for teaching are identified. From Fennema and Franke's (1992) point of view, teacher knowledge includes

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 251–258). Newcastle: MERGA.

knowledge of content (the concept, procedures, and problem-solving process in mathematics), knowledge of pedagogy (teaching procedures, planning, management, and motivation), knowledge of students' cognition (students' thinking, learning, and difficulties), and beliefs of teachers.

PCK—the special domain of teacher knowledge that links content, students, and pedagogy (Shulman, 1987) has received significant attention in informing teacher education and development initiatives. Based on Shulman's (1987) conceptualisation, Grossman (1990) identifies four components of teacher knowledge: teachers' "overarching conception of the purposes for teaching particular subject matter; knowledge of pupils' understanding and potential misunderstanding of a subject area; knowledge of curriculum and curricular materials; and knowledge of strategies and representations for teaching particular topics" (p. 40). Similarly, Marks (1990) proposes four components of PCK: "subject matter for instructional purposes, students' understanding of the subject matter (student learning process, typical understandings, common errors, difficulties), media for instruction in the subject matter (i.e., texts, materials), and instructional processes for the subject matter" (p. 4). An et al. (2004) define PCK as the knowledge of effective teaching comprising three components: knowledge of content, knowledge of curriculum (selecting and using appropriate textbooks and materials, understanding the goals of textbooks and curricula), and knowledge of teaching (knowing students' thinking, planning instruction, understanding the modes of presenting instruction). In this model, knowledge of teaching is the main component of PCK. Knowing students' thinking includes addressing students' misconceptions, engaging them in mathematics learning, building on their mathematical ideas, and promoting their mathematical thinking.

Another regarded teacher knowledge model is the Mathematical Knowledge for Teaching (MKT; Ball et al., 2008). MKT distinguishes between subject matter knowledge (SMK) and PCK. It consists of three domains of SMK: common content knowledge (the mathematical knowledge and skills used in settings other than teaching), specialised content knowledge (the mathematical knowledge and skills unique to teaching), and horizon content knowledge (how mathematical subjects are related in the continuum of mathematics included in the curriculum). The PCK domain includes knowledge of content and students (knowledge of how students think about, know, or learn a particular content), knowledge of content and teaching (knowledge of a mathematical content, idea or procedure and knowing pedagogical principles for teaching that content), and knowledge of content and curriculum. Within the Teacher Education and Development Study in Mathematics (TEDS-M), teacher knowledge has been differentiated according to Mathematics Content Knowledge-MCK (knowledge of subject matter), Mathematics Pedagogical Content Knowledge-MPCK (e.g., curricular knowledge, knowledge of planning for mathematics teaching and learning, analysing and diagnosing students' questions), and General Pedagogical Knowledge-GPK (Tattoo et al., 2012).

### *Sources of Knowledge for Teachers*

There have been comprehensive approaches in teacher education and development fields to identifying sources of knowledge for teachers to improve the quality of teaching. Shulman (1987) defined sources of knowledge as "the domains of scholarship and experience from which teachers may draw their understanding" (p. 5) and identifies at least four main sources of knowledge for teachers. (1) Scholarship in subject areas: i.e., the content knowledge, understanding and skills that teachers teach. This knowledge comes mainly from the accumulated studies in the relevant subject area. (2) Educational materials and institutional contexts: i.e., the materials and structures that are created for teaching and learning including curricula, textbooks, tests and testing materials. (3) Research on schooling: i.e., the existing body of academic literature on understanding the process of teaching and learning such as empirical research findings in the fields of teaching, learning, and foundations of education. (4) The wisdom of practice: i.e., the knowledge that can be gathered from the pedagogical principles that guide and are used by exemplary teachers.

In their investigation into teachers' views about good mathematics teaching and how good teaching develops, Wilson et al. (2005) have found that the participating teachers commonly thought that good teaching "requires a sound knowledge of mathematics, promotes mathematical understanding, engages, motivates students, and requires effective management skills" and it "is developed from experience, [formal] education, personal reading and reflection, and interaction with colleagues" (p. 83). Buehl and Fives (2009) identify several major sources of knowledge for teachers based on Shulman's (1987) model:

- Formal education (college coursework, workshops, conferences, subject area classes).
- Formal bodies of knowledge: information stores (books, literature, the internet); accumulated findings (educational research).
- Observational learning (formal or informal observations of good or bad teaching).
- Collaboration or interactions: meaning construction (co-construction of knowledge through sharing and collaborating); learning from others (e.g., experts, parents, peers, and colleagues).
- Enactive experiences: personal experiences (time spent in schools as a student, the way the individual was taught); professional experiences (on-the-job, actual teaching practice, listening to students).

### *Prior Research*

Most relevant to this investigation, the participating teachers in Mosvold and Fauskanger's (2013) study generally believed in the importance of mathematical definitions as an aspect of MKT, while the teachers did not think they needed to know definitions. The teachers in Mosvold and Fauskanger's (2014) investigation did not necessarily view the horizon content knowledge (i.e., how mathematical subjects are related in the continuum of mathematics included in the curriculum; Ball et al., 2008) as an important part of their teaching knowledge. Beginning teachers in Leong's (2014) study thought that having a sound content knowledge, classroom management skills, and motivation are the key characteristics of good teaching. Perry (2007) described a group of Australian teachers' beliefs about effective mathematics teaching and learning within a ZDM Special Issue that focused on a cross-cultural study based on interviews with participating teachers from Australia, Hong Kong SAR, Mainland China, and the USA. In the same Issue, Bryan et al. (2007) reported similarities and differences in teachers' beliefs from these four nations, while Kaiser and Vollstedt (2007) compared teachers' beliefs from the four nations reported by Bryan et al. with teachers' beliefs in Europe. According to the participating teachers from the four countries, there were commonalities in some of the attributes of effective mathematics teachers. These included competence in mathematics and necessity of in-depth understanding of the curriculum and textbooks. The latter quality was especially emphasised by teachers from Mainland China. Teachers from the US and Australia valued teachers' ability to listen to students and getting them to express themselves. Classroom management skills manifested themselves in the US teachers' responses, while this was not expressed by other teachers. All the teachers emphasised that teachers need to know their students and understand the educational needs of student. Hatisaru (submitted) investigated the extent to which a sample of in-service secondary mathematics teachers recognised the professional knowledge needed for teaching mathematics and their perceptions of its sources. Data were generated through the same questionnaire used in the current investigation. The teachers recognised the knowledge domains needed for mathematics teachers (e.g., knowledge of content, knowledge of mathematics teaching), but mostly a single knowledge dimension (e.g., knowledge of content and/or knowledge of mathematics teaching) was emphasised by the teachers as opposed to the multidimensionality of teacher knowledge. Several sources of teacher knowledge were evident in teachers' responses including both formal (workshops, conferences) and informal (peer interactions or collaboration) sources, while university coursework and educational research sources were absent.

# Method

The informants for this study were PSMTs who expressed interest in participating in a research project aiming to enhance their representational competence. The research was advertised in two mathematics education units, in the Bachelor of Education (BEd; 19 enrolled) and Master of Teaching (MTeach; 11 enrolled) programmes, at the authors' university. Within this research, interested PSMTs completed an open-ended questionnaire containing eight items sourced from the literature. Relevant to this paper are Items 1 to 4 (adapted from An et al., 2004) where the aim is to access the PSMTs' perceptions of the types of professional knowledge that teachers of mathematics should have (Item 1) and how that knowledge is developed (Item 3), with a specific focus on how knowledge of students is gained (Item 4):

> Item #1: What type of professional knowledge should a teacher of mathematics have?
>
> Item #2: How important is it for teachers to have this knowledge?
>
> Item #3: How do teachers continue to enhance their professional knowledge?
>
> Item #4: How do teachers know about their students' strategies and understanding of a particular mathematical content?

From the entire population enrolled in two units, six second-year BEd (four male and two female) and six first-year MTeach (three male and three female) PSMTs ($n$ = 12) voluntarily completed the questionnaire. Nine of them were majoring in mathematics while the other three were studying a minor in mathematics. Participants are coded as PSMT 1, PSMT 2, PSMT 3, etc. to protect their identity.

The data analysis was conducted by the first author and coding was reviewed by the second author. The examination of the words and language used by the respondents was the key aspect of data analysis. We therefore content analysed the PSMTs' responses to four questionnaire items to discover and describe their perceptions, informed by the categories of teacher knowledge, and its sources, identified in the relevant research literature presented in the second section. We were also *open* to adding more knowledge dimensions if they should arise within the data. Some responses were coded in more than one category to not lose the richness of the data. Once the coding was completed, the frequency of teacher knowledge dimensions and sources of knowledge manifested in the PSMTs' responses were counted and presented in Tables 1 and 2. We present the findings below under two sections. Using quotes from the PSMTs' responses, we aim to capture the perceptions of the PSMTs and to describe the trends in their descriptions. Within these quotes, knowledge dimension coding is given in brackets, and italics are added by the authors.

# Findings

## *Perceptions of Teacher Knowledge*

Table 1 captures the teacher knowledge dimensions which manifested themselves in the PSMTs' responses to relevant questionnaire items. The PSMTs put greater value on teacher professional knowledge (Item #1) and mostly indicated the importance of understanding mathematical content (17 occurrences), knowledge of mathematics pedagogy (9 occurrences), and knowledge of general pedagogy (6 occurrences) (Items #1 and #2). Typical examples included, in responding to Item #1:

> A maths teacher should firstly have *a basic understanding of and knowledge of mathematics* [K of content], and in my opinion, should have done maths ATAR in high school as a minimum. In addition, I think it's important that a maths teacher *has some knowledge of instructional practice when it comes to teaching maths* to know how students may learn best [K of maths pedagogy]. (PSMT 10)
>
> Probably a decent grasp on *what is being taught* [K of content]. As well as an understanding of *the Australian/WA curriculum* [K of curriculum]—*OLNA, NAPLAN, etc*. [K of context] (PSMT 7)

**Table 1**

*Dimensions of Teacher Knowledge Revealed in PSMTs' Responses*

| Item #1 | Item #2 |
|---|---|
| Knowledge of content (11) | Knowledge of content (6) |
| Knowledge of mathematics pedagogy (5) | Knowledge of mathematics pedagogy (4) |
| Knowledge of curriculum (3) | Knowledge of pedagogy (4) |
| Knowledge of context (2) | Knowledge of students (1) |
| Knowledge of pedagogy (2) | |
| Knowledge of students (1) | |

*Note.* Numbers in brackets indicate the number of occurrences.

Additionally, three participants foregrounded *content-free* mathematical competencies and applications of particular mathematics concepts, or real-life applications of mathematics:

> How to *use mathematics to solve* simple but useful *problem*s, a little bit of the philosophy of mathematics (e.g., what is it? what sort of things do mathematicians fight over? what's the difference between a conjecture and a theorem?) and *the ability to do basic mathematical proofs* … [K of content] (PSMT 8)

> A maths teacher must fully *understand a concept*, *different ways this concept can be taught* as students don't all learn the same way and must ideally include *relevant applications of this concept* [K of content; K of maths pedagogy] (PSMT 6)

In responding to Item #2, where the PSMTs were prompted about the importance for teachers to have professional knowledge needed to teach mathematics, seven of them reinforced what teachers would know or understand, as in these example responses:

> A stronger *knowledge of the content* [K of content] as well as *different styles about how to convey that information* [K of maths pedagogy] means they can assist a larger number of students. (PSMT 4)

> It's important that *teachers understand the concepts of maths* [K of content] and *different ways of teaching maths* to be able to effectively convey that knowledge to students [K of maths pedagogy]. (PSMT 10)

Out of twelve, five PSMTs underlined the importance of teachers having knowledge of pedagogy. The ways they described this kind of knowledge can be understood as knowing how to engage students in mathematics learning and knowledge of different teaching strategies, as described (for example) by Fennema and Franke (1992). Whilst three of the PSMTs' responses referred to the knowledge of how to motivate students towards learning (PSMT 2, Item #1; PSMT 5, Item #2; and PSMT 12, Item #2), the other responses addressed to knowing "how to teach" (PSMT 2, Item #2), "as well as [knowing] different teaching styles" (PSMT 4, Item #1).

> I believe this grounding [teacher professional learning] is essential, as it provides the teacher with the foundational understandings that students require. Learning *how to teach maths*, *using manipulatives*, *different representations*, *games* [K of maths pedagogy; K of content] and *stimulating student interest* [K of pedagogy] will enable maths teachers to be more effective. (PSMT 12, Item #2)

## *Perceptions of Sources of Teacher Knowledge*

PSMTs' suggestions of possible knowledge sources for teachers are presented in Table 2. In responding to Item #3, almost all PSMTs consider formal preparation as one of the major sources of teacher knowledge. This includes professional development opportunities such as seminars, workshops, professional learning days, online MOOCs, as well as postgraduate studies and university graduate certificates. Some representative examples are:

> By doing *professional learning* (may it be on PD days [Formal preparation], *seeking assistance from colleagues* [Interactions or collaboration] or *prior reading regarding a topic* [Formal bodies of knowledge]) (PSMT 6)

> Teachers enhance their knowledge by completing *online professional learning courses*, completing professional learning *workshops*, *completing graduate certificates in particular subjects*, [Formal preparation], *joining professional associations* [Formal bodies of knowledge], such as … (PSMT 12)

**Table 2**

*Sources of Teacher Knowledge Revealed in PSMTs' Responses*

| Item #3 | Item #4 |
|---|---|
| Formal preparation (18) | Professional experiences (18) |
| • professional learning days, seminars, courses, activities (12); postgraduate studies (5); external studies (1) | • conducting (formative) assessment (7); conversations with students (5); observations (4); on-the-job (2) |
| Formal bodies of information (5) | Interactions or collaboration (3) |
| • books (2); research (1); curriculum (1); professional associations (1) | • communication for growth and understanding others (1); cooperation with other teachers (1); interacting in the school (1) |
| Self-study/reflection (5) | Formal preparation (2) |
| • self-study (4); self-reflection (1) | • studying at university (1); learnings at university (1) |
| Interactions or collaboration (3) | Formal bodies of information (2) |
| • from others (1); from peers (1); communicating with colleagues (1) | • checklists (1); handbooks showing students' misconceptions (1) |
| Professional experiences (1) | Self-study/reflection (2) |
| • interacting with students | • educate themselves (1); be receptive to feedback (1) |

*Note.* Numbers in brackets indicate the number of occurrences.

Whilst the PSMTs predominantly regarded formal preparation as a way of gaining professional knowledge in general (Item #3), they perceived professional experiences as one of the main sources of ways for acquiring knowledge of students' strategies and understanding regarding a particular mathematical content (Item #4), like participants in Buehl and Fives (2009). Many responses within this group were grouped into conducting (mostly formative) assessment, listening to, or observing students. Perhaps these are best reflected in the following words:

> *Talk to them*, *look at their working*, *look over their shoulder* at the work. (PSMT 8)
>
> Diagnostic, formative and summative assessment. Asking students not only to answer questions but to give justification for their answers. (PSMT 9)
>
> Continuously checking student understanding in the classroom through *formative assessments* is important to be able to gauge student understanding of a particular topic. (PSMT 10)

Fewer PSMTs perceived formal preparation as a source of acquiring knowledge of student thinking and understanding. PSMT 7 wrote:

> Perhaps from *their own experiences* [on-the-job] as well as those they have taught, and the information they have learned from university [Formal preparation].

## Discussion and Conclusions

PSMTs place greatest emphasis on mathematics content knowledge and mathematics pedagogical knowledge, two components of teacher knowledge that have been generally considered as key factors for effective mathematics teaching and students' mathematics learning (e.g., An et., 2004). This is perhaps to be expected in a subject such as mathematics, which faces unique challenges among school disciplines in terms of student engagement, student anxiety, the wide range of applications, and cultural expectations. It is therefore expected that PSMTs would highly value content knowledge as well as specialised knowledge on how to engage students with mathematics. However, we do not know preservice teachers' values of different knowledge domains across different subject disciplines and whether mathematics is an outlier. The PSMTs expect to gain most of their knowledge through formal preparation within the professional learning system rather than through self-study or through interactions with peers. This emphasises how important it is for schools and professional associations to provide regular formal learning opportunities, because teachers may otherwise not (usually) be self-motivated to continue improving their skills independently of this. Saying that, we would like to see a study done of the professional development opportunities in Australia for mathematics teachers, to determine whether the range of topics being taught matches with what preservice, and practising, mathematics teachers feel are the most important areas to know. What gaps, if any, are there in professional development opportunities, and are all the knowledge domains being adequately explored by the professional service providers?

Our research has looked into the expectations of knowledge of PSMTs, before they have yet had to put their knowledge into practice. PSMTs put a large emphasis on content knowledge in their training, however, as noted by Buehl and Fives (2009), the content knowledge of mathematics tends to remain stable and consistent over time, in comparison to other subjects such as social studies which may change more rapidly. We would expect, then, that as participants progress from preservice to in-service, they begin to rate pedagogical knowledge and context more highly than content knowledge as observed in Hatisaru (submitted). Future work could follow these students as they progress to becoming in-service teachers and look at how their answers to our questionnaire items, or to those similar ones, would change with their professional practice.

The results from only twelve PSMTs, in one state of Australia within one single university may not apply to different contexts. The educational background of the participants was not taken into account—both their prior level of knowledge starting university and the country where they were educated. In a larger study we may also have considered whether gender influences the answers of the PSMTs, together with their mathematics training background and differences between BEd and MTeach students. Furthermore, findings are limited to PMSTs' responses to four questionnaire items; we do not have, for example, interview data which may expand on their written responses.

On a final note, teacher knowledge is a multi-dimensional construct, including several knowledge components, and all these components overlap. For example, knowledge about the applications of mathematics spans the categories of content knowledge, pedagogical content knowledge, and knowledge of context. Furthermore, different mathematics educators or researchers define these components differently. The knowledge of context category has typically been considered as educational context by previous authors but could be expanded to include mathematical context, as in Ernest (1989), to encapsulate those knowledge domains regarding the purpose and philosophy of mathematics. Researchers aiming to understand teachers' beliefs about teacher knowledge need to be prepared for the complexity of the construct of teacher knowledge.

## References

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education*, *7*, 145–172.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal for Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Buehl, M. M., & Fives, H. (2009). Exploring teachers' beliefs about teaching knowledge: Where does it come from? Does it change? *Journal of Experimental Education*, *77*, 367–408.

Bryan, C. A., Wang, T., Perry, B., Wong, N-Y., & Cai, J. (2007). Comparison and contrast: Similarities and differences of teachers' views of effective mathematics teaching and learning from four regions. *ZDM Mathematics Education*, *39*(4), 329–340. https://doi.org/10.1007/s11858-007-0035-2

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching*, *15*(1), 13–33.
Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, (pp. 147–164). Macmillan Publishing Company.
Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education*. Teachers College Press.
Hatisaru, V. (submitted). Secondary teachers' perceptions of teacher knowledge and its sources. *Quadrante*.
Kaiser, G., & Vollstedt, M. (2007). Teachers' views on effective mathematics teaching: Commentaries from a European perspective. *ZDM Mathematics Education*, *39*, 341–348.
Leong, K. E. (2014). What are the important attributes of good mathematics teaching? *Asia-Pacific Educational Research*, *24*(1), 211–223. https://doi.org/10.1007/s40299-014-0173-6
Marks, R. (1990). Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Teacher Education*, *41*(3), 3–11.
Mosvold, R., & Fauskanger, J. (2013). Teachers' beliefs about mathematical knowledge for teaching definitions. *International Electronic Journal of Mathematics Education*, *8*(2–3), 43–61.
Mosvold, R., & Fauskanger, J. (2014). Teachers' beliefs about mathematical horizon content knowledge. *International Journal for mathematics teaching and learning*, 1–16.
Perry, B. (2007). Australian teachers' views of effective mathematics teaching and learning. *ZDM Mathematics Education*, *39*(4), 271–286. https://doi.org/10.1007/s11858-007-0032-5
Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of research on teacher education* (pp. 102–119). Macmillian.
Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1–22.
Tatto M. T., Schwille J., Senk S. L., Ingvarson L., Rowley G., Peck R., . . . Reckase M. (2012). *Policy, Practice, and Readiness to Teach Primary and Secondary Mathematics in 17 Countries: Findings from the IEA teacher education and development study in mathematics (TEDS-M)*. IEA.
Thomson, S., De Bortoli, L., Underwood, C., & Schmid, M. (2019). *PISA 2018: Reporting Australia's results. Volume I: Student performance*. Melbourne: Australian Council for Educational Research (ACER). https://research.acer.edu.au/ozpisa/35
Thomson, S., Wernert, N., Buckley, S., Rodrigues, S., O'Grady, E., & Schmid, M. (2021). *TIMSS 2019 Australia. Volume II: School and classroom contexts for learning*. Melbourne: Australian Council for Educational Research (ACER). https://doi.org/10.37517/978-1-74286-615-4
Wilson, P. S., Cooney, T. J., & Stinson, D. W. (2005). What constitutes good mathematics teaching and how it develops: Nine school teachers' perspectives. *Journal of Mathematics Teacher Education, 8*, 83–111.

# High School Learners’ Mathematical Dispositions and The Influences of Mathematics Clubs

Wellington Hokonya
*Rhodes University*
w.hokonya@ru.ac.za

Mathematics clubs support development of learners’ dispositions through engagement with challenging problems and supporting students to communicate ideas with others in fun ways. This paper reports on the mathematics dispositions of high school learners who participated in a mathematics club in primary school. Data are drawn on their narrative stories that they wrote in high school to reflect their mathematics journeys. Data were analysed qualitatively. Learners report that the promoted learning dispositions in the clubs such as resilience, affinity for mathematics, perception of self as doers of mathematics and enjoyment in working with mathematics supported different relationships with mathematics. The paper shows that there is potential in afterschool clubs to support learners in navigating productive mathematical dispositions that extend beyond the time frames of club participation.

## Introduction

South African mathematics education is widely acknowledged to be in a state of crisis (Fleisch, 2008) where mathematics achievement is abnormally low, as evidenced by sustained poor performance across international, regional, and national assessments (Department of Basic Education [DBE], 2017; Spaull & Kotze, 2015). The Eastern Cape province in South Africa, where I carried out the broader study, was the worst performing province in the National Senior Certificate [NSC] Mathematics examinations in 2020, with an average pass rate of 39.7% (DBE, 2020). The general trend of Mathematics results shows that as learners progress to higher grades their performance deteriorates, and they choose not to pursue Mathematics post-compulsory education (Mutodi & Ngirande, 2014).

To mitigate the challenges, the South African Numeracy Chair Project at Rhodes University established afterschool mathematics clubs to provide extra-curricular activities focused on developing a supportive learning community where learners’ active mathematical participation, engagement, enjoyment, resilience, and sense making are the focus (Graven, 2011). The six learners in this study participated in one of the clubs for periods ranging from two to five years each while they were in primary school. The stories that are thematically analysed in this study are part of data from a larger study (Hokonya, 2021).

According to Kilpatrick et al. (2001, p. 131), “if students are to develop conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning abilities, they must believe that mathematics is understandable, not arbitrary; that with diligent effort, it can be learned and used; and that they are capable of figuring it out.” Developing such productive dispositions will improve the learners usual learning mood and temperament and develop habitual inclination to learning and the tendency to acquire knowledge. Interestingly, one of the salient goals of mathematics education is to help learners to develop positive dispositions toward mathematics so that they may become persistent, agentic, and confident in the subject (Kilpatrick et.al, 2001). These traits are the cornerstone of powerful and productive mathematics dispositions that help learners to handle frustrations and struggles not only in mathematics but also in all areas of learning and life in general (Bishop, 2012).

## Literature Review

Scholars agree that for learners to develop a productive disposition, they require frequent opportunities to make sense of mathematics, to recognise the benefits of perseverance, and to

(2023). In B. Reid-O’Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 259–265). Newcastle: MERGA.

experience the rewards of sense making in mathematics (Dweck, 2008; Kilpatrick et al., 2001). They need to develop conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning abilities. They must believe that mathematics is understandable and not arbitrary, but that, with effort and resilience, it can be learned. Kilpatrick et al. (2001) argue that for learners to develop a productive disposition, they require frequent opportunities to make sense of mathematics, to recognise the benefits of perseverance, and to experience the rewards of sense making in mathematics. All these can be purposefully developed in the mathematical practices that they take part in, for example in the classroom or afterschool mathematics clubs. Ally (2011) notes that a high frequency of opportunities for developing productive dispositions were mostly linked to the inclusion of real world or out of class situations.

Mathematical dispositions are considered to increase learners' ownership of their learning, their opportunities to understand and value other people's ways of knowing, and their capacities to work collaboratively with colleagues in developing their mathematics practice. Black et al. (2004) further report that positive dispositions make learners connect with and appreciate mathematics, which is a tendency to think and act positively. Further, Maxwell (2001) connects learner engagement and enjoyment in mathematics directly with the activities the teacher uses in the classroom arguing that activities need to motivate the students and "make mathematics worthwhile" (p. 35).

Unfortunately, activities that motivate learners are rare in many South African classrooms as research show that many teachers foreground ritual participation (including chanting), passive listening and little sense-making, resulting in many failing to progress beyond inefficient one-to-one counting methods, even by Grade 7 (Heyd-Metzuyanim & Graven, 2016; Hoadley, 2012; Venkat & Naidoo, 2012).

However, the afterschool mathematics clubs the learners in this study attended aimed to redress this apartheid legacy of compliant, passive, and dependent learners that work counter to developing critical, creative, and active participation in mathematics problem solving as envisioned in the South African curriculum (Graven, 2015). In the clubs, learners work in smaller groups focusing on specific learner needs informed by their competence level and learning disposition (Graven, 2015). In addition, club sessions and the take-home activities, games and worksheets provided extend the time learners engage with mathematics and might promote continued mathematical learning outside of clubs and classrooms. This likely increases learners' mathematical dispositions.

In this paper I address the question: How do mathematics clubs influence learners' mathematical dispositions? I do not seek the relationship between dispositions and mathematics performance in terms of high or low marks in assessment, but to illuminate how learners' reflections show how attitude influence engagement in mathematics.

## Methodology and Results

Six indicators of mathematical productive dispositions drawn from Kilpatrick et al. (2001) and Carr and Claxton (2002) were used to thematically analyse data that were gathered in the form of learners' stories written in high school about their mathematical dispositions over time. The chronological trajectory of the stories was apparent as the prompt to write the narrative story explicitly required participants to reflect on their mathematics stories from primary school to high school. The six indicators are used to thematically analyse the six stories of the learners' mathematical dispositions over time. I tabulate the six indicators of mathematical dispositions from Kilpatrick et al. (2001), and Carr and Claxton (2002) below.

My reading of the indicators of Kilpatrick et al. (2001) and Carr and Claxton (2002) revealed that there were two similar indicators (*belief that steady effort in learning maths pays off (resilience)* and *showing tendencies of resilience, persisting in difficulty respectively*). For this paper I merged them to come up with belief that steady effort in learning maths pays off (resilience) K3&CC3.

**Table 1**

*Codes for Indicators of Mathematical Dispositions*

| Authors | Code | Description |
|---|---|---|
| Kilpatrick et al. (2001) | K1 | Seeing sense in mathematics |
| | K2 | Perceiving mathematics as useful and worthwhile |
| | K3 | Belief that steady effort (resilience) in learning mathematics pays off |
| | K4 | Seeing self as an effective learner and doer of mathematics |
| Carr and Claxton (2002) | CC1 | Showing tendencies of playfulness |
| | CC2 | Showing tendencies of reciprocity (willingness to engage or taking another point of view) |
| | CC3 | Showing tendencies of resilience (persisting in difficulty) |

To analyse the narrative stories written by learner participants, different utterances were allocated a code that was nearest to it in meaning. The utterances were designated as sentences or consecutive sentences or phrases that conveyed meaningful learners' mathematics dispositions. The six indicators were further counted to determine whether the different mathematical dispositions were attributed to primary school, high school or whether they were directly attributed to the after-school mathematics clubs. The six indicators of mathematical dispositions are expressed as either positive or negative, where the negative dispositions carry codes that have a negative sign for example, -K1. Positive indicators are utterances that show that learners' participation in mathematical activities was not coerced but done willingly and happily, for example, "Since then, I loved maths more and more, it became my favourite subject" (Unathi K4), "It's when I started maths club, in the maths club, in the maths club we had an opportunity to count using cards and we used very exciting maths activities" (Emihle K4). Negative indicators are utterances where leaners reveal anguish and discomfort in their participation in the subject for example, "I was confused when it comes to be using the column when multiplying". (Siya -K4), "In grade 7, I used to struggle because it was a little bit hard (-K1) because it was my first-time seeing x's and I couldn't understand how to solve them" (Kamva -K4).

**Table 2**

*Indicators of Mathematical Dispositions*

| Level of School | Indicator of disposition | Number of participants who mention the disposition | Positive disposition | Negative disposition |
|---|---|---|---|---|
| Primary | K1 | 5 | 4 | 6 |
| | K2 | 0 | 0 | 0 |
| | K3&CC3 | 4 | 8 | 0 |
| | K4 | 6 | 22 | 16 |
| | CC1 | 3 | 4 | 0 |
| | CC2 | 3 | 4 | 0 |
| High School | K1 | 6 | 5 | 4 |
| | K2 | 1 | 1 | 0 |

| Level of School | Indicator of disposition | Number of participants who mention the disposition | Positive disposition | Negative disposition |
|---|---|---|---|---|
| | K3 | 4 | 7 | 0 |
| | K4 | 5 | 10 | 5 |
| | CC1 | 4 | 2 | 0 |
| | CC2 | 5 | 9 | 1 |
| Clubs | K1 | 4 | 5 | 3 |
| | K2 | 1 | 1 | 0 |
| | K3&CC3 | 5 | 7 | 0 |
| | K4 | 6 | 16 | 5 |
| | CC1 | 3 | 9 | 1 |
| | CC2 | 5 | 8 | 0 |

Column one in the table shows the school level in which the indicator of disposition was mentioned, and the second column shows the indicators of all the dispositions. The third column shows the total number of participants who mentioned the given disposition and the last two columns show the total number of positive and negative responses (these are opposite statement made in relation to an indicator) to dispositions, respectively. This table thus summarises six learners' mathematical dispositions using their presence and frequency in the learners' stories. In the following section I unpack the table with examples from learner stories starting with those occurring the most to the least that is K4 (74), K1 (27), K3 & CC3 (24), CC2 (21), CC1 (15) and K2 (2).

*Seeing Themselves as Effective Learners and Doers of Mathematics (K4)*

This mathematical disposition indicator had the highest number of both positive (48) and negative (26) utterances from the learners. They described themselves as confident and effective doers of mathematics in their stories. The highest number of positive utterances was recorded in relation to their primary school (22) followed by linkages to the afterschool mathematics clubs (16). One learner explained that: "I passed it with flying colours and got a level 7 & that gave me power to love and to continue doing it till I loved it" (Sethu). Another learner (Zozo) describes herself as being "good at mathematics" in primary. From the data it seems the after-school mathematics clubs played a significant role in the development of positive mathematical dispositions in the six participants as represented in the following utterances:

> "After joining the club, I saw that my results are changing, and they are increasing" (Tando)
>
> "I loved the maths club because we had maths tricks that we used while doing on(sic) activity" (Sihle)
>
> "The maths club really helped improve my maths skills and I will be always grateful for that" (Silakhe)
>
> "Our maths club teacher was Debbie who taught us nothing but good about maths and taught us tricks about maths which we used to teach other learners in class" (Sethu)."But when I joined the maths club it made me realise that maths isn't that hard and it is not about getting the right answer, but it was about understanding it, making everything fun and coming up with great new ideas on how to solve maths problems" (Kimi)

All the learners' descriptions of the maths club as having helped in developing their perception of themselves as effective doers of mathematics cohere with Graven's (2015) argument that club activities are designed to create more engaging and confident learners. She continues that the environment in the clubs disrupt passive teacher-dependent ways of being and create an environment where learners can increase their opportunities to re-author themselves as mathematical producers, questioners, and explorers.

Some learners, however, did not perceive themselves as effective learners of mathematics in several time frames across their schooling. They narrated negative mathematics dispositions where they considered quitting the subject. For example, Sethu writes that, "I wasn't excelling in it or understanding it. I chose to give up and quit school if they were to teach me maths." However, the time in the club helped her to develop a more positive mathematical disposition. She concluded her story with an emphatic declaration that, "I am a good maths learner and I'm very proud of myself" (Sethu).

### *Seeing Sense in Mathematics (K1)*

According to Kilpatrick et al. (2001), opportunities to see sense in mathematics and to experience the rewards of sense making in mathematics are critical aspects in the development of positive mathematics disposition in learners. All the learners in this paper reveal that they sometimes saw sense in mathematics (14 utterances) where it was easy and fun to do and they could use it in their daily contexts, but they sometimes experienced challenges (13 utterances) where the subject was difficult, and this decreased the development of positive mathematical dispositions.

Although Kimi never saw any sense in it in the first place, he writes that "I really appreciate everything that [club facilitators] did for me, because they (sic) did not start the clubs I would be hating and failing maths." On the same spectrum of enjoyment, Sihle explains that "in grade 9 my maths was fun, it is challenging. It is very exciting because I compete with my friends and help each other with our maths problems." The learners had probably developed the sense of seeing sense and enjoyment in mathematics when they engaged in their afterschool mathematics clubs in primary school.

On the other hand, the learners came across challenging times where mathematics was difficult and not making much sense to them as represented in these excepts from Tando and Sethu: "when I started in the intermediate phase, I was struggling again because mathematics was not the same, we were working with big numbers and I was confused when it comes to be using the column when multiplying." (Tando) and "<u>In grade 7 maths became more harder (sic)</u> and there was a lot of work done, there came solving x variables, geometry and so on" (Sihle).

### *Beliefs That Steady Effort (Resilience) in Learning Pays Off (K3 & CC3)*

This dispositional indicator produced 22 positive utterances from four learners, and they were evenly distributed in primary, high school and after school mathematics clubs. According to Carr and Claxton (2002) key indicators of resilience include sticking with a difficult learning task, having relatively high tolerance for frustration without getting upset and being able to recover from setback or disappointment relatively quickly. According to Sihle, mathematics "is fun, but you need to focus, that's the rule I told myself while I was going to high school and if you practise maths, then you will achieve." She stated that it is crucial to 'focus' and endure the difficult tasks that one may come across in their mathematical journey. Another learner stated high levels of tolerance and agency in mathematics practice when she explained how she approached the teacher when the subject was difficult in high school and requested that "can't the maths teacher get us some people to help us with mathematics. Then she got some learners from (Thuthuka High School) to help us." Zozo expresses her understanding that you cannot instantly be good at the subject, but it takes time and endurance and constant practice to excel.

Acknowledging that one could not do mathematics alone and asking for assistance maybe traced back to their club ethos which valued and encouraged collaboration and agency in learning the subject.

### *Showing Tendencies of Reciprocity (CC2)*

The dispositional tendency of reciprocity is a valuable resource in learning mathematics. According to Carr and Claxton (2002), learners who lack reciprocity, that is the ability to articulate their own learning processes and problems, the ability to communicate these to others or the inclination or the courage to do so are inevitably handicapped learners. Most of the learners in this paper show their inclinations towards the disposition of reciprocity and how they used it to succeed in their mathematics practice. For example, Zozo explains that after receiving low marks and feeling bad about himself, “I asked my mom who is really good in maths to help me and s little while later, I was slowly getting there.” The interaction between the learners and their parents produced positive results. The courage to articulate or share their handicaps in mathematics resulted in them getting invaluable assistance which further developed their mathematical dispositions as evidenced by their descriptions of themselves as capable and confident doers of mathematics.

### *Showing Tendencies of Playfulness (CC1)*

Being playful includes the learners’ readiness, willingness and being more creative in reacting to problems and challenges in engaging in mathematics (Carr & Claxton, 2002). In this paper, Tando showed readiness and willingness to engage with mathematics regardless of challenges when she writes that “I didn’t want to go to maths club because after school I was getting hungry and, in the club, we were doing some work for an hour. I made my final decision to attend the club because I really need help.” Similarly, Lise reflects that, “In grade 7, I used to struggle because it was a little bit hard because it was my first-time seeing x’s and I couldn’t understand how to solve them. I talked to Pam about the issue about how to solve x’s, she helped me.”

The learners’ disposition of playfulness pushed them to endure and find creative ways without being coerced to help them to overcome some of the mathematics difficulties that they faced in their journey from primary to high school.

### *Perceiving Mathematics as Useful and Worthwhile (K2)*

Only one learner in the paper explicitly perceived mathematics as useful and worthwhile. Sethu seemed to agree with the club coordinators on the idea that passing the subject “could get us far” in terms of life after school. This however does not mean that the other participants did not see mathematics as useful and worthwhile, they were just not explicit, but their narrative stories reveal that they saw value in the subject, hence their perseverance, resilience, and affinity regardless of the challenges and difficulties in their journeys from primary to high school.

## Concluding Remarks

Positive mathematical dispositions are considered to increase learners’ chances to collaborate with others and develop affinity and resilience in maths practice. All the learners in this paper illuminated how their participation in the maths club contributed towards the development of their positive mathematical dispositions in primary and high school. In South African schools, learners usually have little agency and are generally passive, and teaching is generally procedural, allowing learners little agency and requiring them to mechanically follow rules like little robots (Umugiraneza et al., 2017).

The findings from the learners’ stories showed above show that learners developed positive and productive mathematical dispositions as indicated by them seeing themselves as effective learners and doers of maths and stating their resilience. The stories also reveal productive dispositional indicators that steady effort and working collaboratively and creatively help in developing their knowledge in maths. The paper shows that there is more potential in the maths clubs to support learners in navigating productive mathematical dispositions throughout their schooling.

## Acknowledgements

The financial assistance of the National Research Foundation (NRF) towards this research (Grant No. 74658) is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the authors and are not to be attributed to the NRF.


## References


Ally, N. (2011). *The promotion of mathematical proficiency in Grade 6 mathematics classes from the Umgungundlovu district in KwaZulu-Natal*. [Master's Thesis]. KwaZulu-Natal: KwaZulu-Natal University.

Bishop, J. P. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom. *Journal for Research in Mathematics Education, 43*(1), 34–74. doi:10.5951/jresematheduc.43.1.0034

Black, P. H., Lee, C. C., Marshall, B., & Wiliam, D. (2004). Working inside the box: Assessment for learning in the classroom. *Phi Delta Kappan, 86*(1), 8–21. doi:3A10.1177/003172170408600105

Carr, M., & Claxton, G. (2002). Tracking the development of learning dispositions assessment in education. *Principles, Policy & Practice*, *9*(1), 9–37. doi:10.1080/09695940220119148

Department of Basic Education. (2020). *National senior certificate schools subject report.* Pretoria: DBE. http://www.education.gov.za/Resources/Reports.aspx

Dweck, C. S. (2008). *Mindset: The new psychology of success*. New York: Ballantine Books. https://www.worldcat.org/title/mindset-the-new-psychology-of-success/oclc/188040712

Fleisch, B. (2008). *Primary education in crisis: Why South African school children underachieve in reading and mathematics*. Johannesburg: Juta & Co.

Graven, M. (2015). Strengthening maths learning dispositions through 'math clubs'. *South African Journal of Childhood Educati*on, 1–7. doi:10.21070/acopen.7.2022.4064

Heyd-Metzuyanim, E., & Graven, M. (2015). Between people-pleasing and mathematizing—South African learners' struggle for numeracy. *Educational Studies in Mathematics*. doi:10.1007/s10649-015-9637-8

Hokonya, W. M. (2021). *An exploration of the mathematical learner identities of high school learners who participated in afterschool mathematics clubs in primary school* [Doctoral thesis, Grahamstown: Rhodes University].

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington DC: National Academy Press. doi:10.17226/9822

Maxwell, K. (2001). Positive learning dispositions in mathematics. ACE papers. *ACE Papers, 11*, 30-39.

Mutodi, P., & Ngirande, H. (2014). The influence of students' perceptions on mathematics performance. A case of a selected high school in South Africa. *Mediterranean Journal of Social Sciences*, *5*(3), 431–445. doi:10.5901/mjss.2014.v5n3p431

SACMEQ IV. (2017). *Project in South Africa: A study of the conditions of schooling and the quality of education*. Pretoria: Ministry of Basic Education. http://www.sacmeq.org/sites/default/files/sacmeq/publications/sacmeq_iv_project_in_south_africa_report.pdf

Spaull, N., & Kotze, J. (2015). Starting behind and staying behind in South Africa: The case of insurmountable learning deficits in mathematics. *International Journal of Educational Development*, *41*, 13–24. doi:10.1016/j.ijedudev.2015.01.02

Umugiraneza, O., Bansilal, S., & North, D. (2017). Exploring teachers' practices in teaching mathematics and statistics in KwaZulu-Natal schools. *South African Journal of Education*, *37*(2). doi:10.15700/saje.v37n2a1306

Venkat, H., & Naidoo, D. (2012). Analyzing coherence for conceptual learning in a grade 2 numeracy lesson. *Education as Change, 16*(1), 21–33. doi:10.1080/16823206.2012.691686

# Students' Strategies for Addition and Subtraction Within 20

Sarah Hopkins
*Monash University*
s.hopkins@monash.edu

Karina Wilkie
*Monash University*
karina.wilkie@monash.edu

Anne Roche
*Monash University*
anne.roche@monash.edu

In this study we individually interviewed 127 students in Years 3 and 4 to investigate their strategies for solving addition and subtraction problems within 20. The problems that students were asked to solve were carefully planned to represent particular problem sets, and strategy use was coded in detail to delineate retrieval from different decomposition strategies. The findings highlight the prevalent use of decomposition strategies for single-digit problems that sum to between 10 and 20 and the use of counting strategies for corresponding subtraction problems.

The expectation that by a particular year level, students are able to recall answers to addition and subtraction problems (i.e., know answers from memory) is often made explicit in mathematics curricula. 'Recall', commonly termed retrieval in the research literature, refers to the direct retrieval of an answer from a store of facts held in long term memory (Ashcraft, 1995). Back-up procedures (encompassing any procedure other than retrieval) include Decomposition Strategies, which involve partitioning and regrouping operands to make use of known facts (e.g., the bridging-10 strategy and the near-doubles strategy) and Counting Strategies (e.g., the count-on from-larger strategy, also called the min-counting strategy). There is strong evidence in the cognitive sciences that retrieving an answer from memory represents a qualitatively different action than computing an answer using a highly automated back-up procedure (e.g., Polspoel et al., 2017) and that retrieval develops as problem-answer associations strengthen in memory through the correct application of back-up procedures (Siegler, 1996).

While the importance of students learning to retrieve certain facts for solving addition and subtraction problems is clear, there is surprisingly little research to illuminate exactly what problems they do come to solve using retrieval (i.e., just know) and what problems they continue to solve using Decomposition Strategies. The aim of the current study was to investigate the strategies students use in Years 3 and 4 to solve addition and subtraction problems within 20. (In this paper we refer to specific procedures as strategies, capitalise the name of strategy types, and use lower case when referring to named strategies.)

## Curriculum Context

In Australia, the written mathematics curriculum sets out what students are expected to be able to demonstrate by the end of each year level in the achievement standards. In 2022, changes to the mathematics curriculum were made to reflect "a stronger focus on students mastering the essential mathematical facts, skills, concepts and processes, and being introduced to these at the right time" and include "lifting standards for mathematics in Year 1 in relation to addition and subtraction" (Australian Curriculum Assessment and Reporting Authority [ACARA], n.d.). By the end of Year 2, students are now expected to recall and demonstrate proficiency with addition and subtraction facts within 20 (ACARA, version 9, n.d.); whereas previously, students were expected to recall addition facts for single-digit numbers by the end of Year 3 (ACARA, version 8.4, n.d.). The change in year level brings this particular standard to be more in line with curricula in other countries. For example, in the US, students are expected to "know from memory all sums of two one-digit numbers" by the end of second grade (Council of Chief State School Officers [CCSSO], 2010). In

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 267–274). Newcastle: MERGA.

England, Year 2 students are to be taught to recall and fluently use addition and subtraction facts to 20 (Department of Education, 2013).

Expanding the Australian curriculum standard to encompass (i) the operation of subtraction as well as addition, and (ii) sums within 20 rather than sums produced by single-digit operands, represents a marked increase in the number of problems explicitly referred to in the standard (as illustrated in Figure 1). Excluding problems with an operand of 0 and 1, and limiting problems to have two operands only, there are 21 single-digit addition problems with sums below 10 (Problem Set 1), 7 single-digit addition problems with sums equal to 10 (Problem Set 2), and 36 single-digit addition problems with sums between 10 and 20 (Problem Set 3). Changing the wording of the standard to include sums within 20 adds another 88 single-digit/double-digit problems (Problem Set 4). Including corresponding subtraction problems further increases the number of problems two-fold. Thus, the change in curriculum wording translates to a substantial increase in the number of problems explicitly referred to in the standard, from 64 problems to 304 problems.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2+2=4 | 4-2=2 | | | | | | KEY: | Set 1 | sum < 10 | | |
| 2+3=5 | 5-3=2 | | | | | | | | | | |
| 3+2=5 | 5-2=3 | 3+3=6 | 6-3=4 | | | | | Set 2 | sum = 10 | | |
| 2+4=6 | 6-4=2 | 3+4=7 | 7-4=3 | | | | | | | | |
| 4+2=6 | 6-2=4 | 4+3=7 | 7-3=4 | 4+4=8 | 8-4=4 | | | Set 3 | 10 < sum < 20 both single digit | | |
| 2+5=7 | 7-5=2 | 3+5=8 | 8-5=3 | 4+5=9 | 9-5=4 | | | | | | |
| 5+2=7 | 7-2=5 | 5+3=8 | 8-3=5 | 5+4=9 | 9-4=5 | 5+5=10 | 10-5=5 | Set 4 | 10 < sum ≤ 20 one single digit | | |
| 2+6=8 | 8-6=2 | 3+6=9 | 9-6=3 | 4+6=10 | 10-6=4 | 5+6=11 | 11-6=5 | | | | |
| 6+2=8 | 8-2=6 | 6+3=9 | 9-3=6 | 6+4=10 | 10-4=6 | 6+5=11 | 11-5=6 | 6+6=12 | 12-6=6 | | |
| 2+7=9 | 9-7=2 | 3+7=10 | 10-7=3 | 4+7=11 | 11-7=4 | 5+7=12 | 12-7=5 | 6+7=13 | 13-7=6 | | |
| 7+2=9 | 9-2=7 | 7+3=10 | 10-3=7 | 7+4=11 | 11-4=7 | 7+5=12 | 12-5=7 | 7+6=13 | 13-6=7 | 7+7=14 | 14-7=7 |
| 2+8=10 | 10-8=2 | 3+8=11 | 11-8=3 | 4+8=12 | 12-8=4 | 5+8=13 | 13-8=5 | 6+8=14 | 14-8=6 | 7+8=15 | 15-8=7 |
| 8+2=10 | 10-2=8 | 8+3=11 | 11-3=8 | 8+4=12 | 12-4=8 | 8+5=13 | 13-5=8 | 8+6=14 | 14-6=8 | 8+7=15 | 15-7=8 |
| 2+9=11 | 11-9=2 | 3+9=12 | 12-9=3 | 4+9=13 | 13-9=4 | 5+9=14 | 14-9=5 | 6+9=15 | 15-9=6 | 7+9=16 | 16-9=7 |
| 9+2=11 | 11-2=9 | 9+3=12 | 12-3=9 | 9+4=13 | 13-4=9 | 9+5=14 | 14-5=9 | 9+6=15 | 15-6=9 | 9+7=16 | 16-7=9 |
| 2+10=12 | 12-10=2 | 3+10=13 | 13-10=3 | 4+10=14 | 14-10=4 | 5+10=15 | 15-10=5 | 6+10=16 | 16-10=6 | 7+10=17 | 17-10=7 |
| 10+2=12 | 12-2=10 | 10+3=13 | 13-3=10 | 10+4=14 | 14-4=10 | 10+5=15 | 15-5=10 | 10+6=16 | 16-6=10 | 10+7=17 | 17-7=10 |
| 2+11=13 | 13-11=2 | 3+11=14 | 14-11=3 | 4+11=15 | 15-11=4 | 5+11=16 | 16-11=5 | 6+11=17 | 17-11=6 | 7+11=18 | 18-11=7 |
| 11+2=13 | 13-2=11 | 11+3=14 | 14-3=11 | 11+4=15 | 15-4=11 | 11+5=16 | 16-5=11 | 11+6=17 | 17-6=11 | 11+7=18 | 18-7=11 |
| 2+12=14 | 14-12=2 | 3+12=15 | 15-12=3 | 4+12=16 | 16-12=4 | 5+12=17 | 17-12=5 | 6+12=18 | 18-12=6 | 7+12=19 | 19-12=7 |
| 12+2=14 | 14-2=12 | 12+3=15 | 15-3=12 | 12+4=16 | 16-4=12 | 12+5=17 | 17-5=12 | 12+6=18 | 18-6=12 | 12+7=19 | 19-7=12 |
| 2+13=15 | 15-13=2 | 3+13=16 | 16-13=3 | 4+13=17 | 17-13=4 | 5+13=18 | 18-13=5 | 6+13=19 | 19-13=6 | 7+13=20 | 20-13=7 |
| 13+2=15 | 15-2=13 | 13+3=16 | 16-3=13 | 13+4=17 | 17-4=13 | 13+5=18 | 18-5=13 | 13+6=19 | 19-6=13 | 13+7=20 | 20-7=13 |
| 2+14=16 | 16-14=2 | 3+14=17 | 17-14=3 | 4+14=18 | 18-14=4 | 5+14=19 | 19-14=5 | 6+14=20 | 20-14=6 | | |
| 14+2=16 | 16-2=14 | 14+3=17 | 17-3=14 | 14+4=18 | 18-4=14 | 14+5=19 | 19-5=14 | 14+6=20 | 20-6=14 | | |
| 2+15=17 | 17-15=2 | 3+15=18 | 18-15=3 | 4+15=19 | 19-15=4 | 5+15=20 | 20-15=5 | | | | |
| 15+2=17 | 17-2=15 | 15+3=18 | 18-3=15 | 15+4=19 | 19-4=15 | 15+5=20 | 20-5=15 | | | | |
| 2+16=18 | 1816=2 | 3+16=19 | 19-16=3 | 4+16=20 | 20-16=4 | | | 8+8=16 | 16-8=8 | | |
| 16+2=18 | 18-2=16 | 16+3=19 | 19-3=16 | 16+4=20 | 20-4=16 | | | 8+9=17 | 17-9=8 | | |
| 2+17=19 | 19-17=2 | 3+17=20 | 20-17=3 | | | | | 9+8=17 | 17-8=9 | 9+9=18 | 18-9=9 |
| 17+2=19 | 19-2=17 | 17+3=20 | 20-3=17 | | | | | 8+10=18 | 18-10=8 | 9+10=19 | 19-10=9 |
| 2+18=20 | 20-18=2 | | | | | | | 10+8=18 | 18-10=8 | 10+9=19 | 19-9=10 |
| 18+2=20 | 20-2=18 | | | | | | | 8+11=19 | 19-11=8 | 9+11=20 | 20-11=9 |
| | | | | | | | | 11+8=19 | 19-8=11 | 11+9=20 | 20-9=11 |
| | | | | | | | | 8+12=20 | 20-12=8 | 10+10=20 | 20-10=10 |
| | | | | | | | | 12+8=20 | 20-8=20 | | |

Figure 1. Addition problems with two operands that have sums within 20 and corresponding subtraction problems (excluding problems with operands of zero and one) divided into problem sets.

While the new wording of the Australian standard has increased the number of problems explicitly mentioned in the standard, it does not specify precisely which of these problems students are expected to solve using retrieval (i.e., just know) by the end of Year 2. The US standard makes it clear that by the end of second grade, students are expected to retrieve answers to addition problems in Problems Set 1, 2 and 3, but is silent about Problem Set 4 and the strategies that students are expected to use to solve corresponding subtraction problems. The UK standard suggests that students are taught to use a combination of retrieval and Decomposition Strategies for solving both addition and subtraction problems in Problem Sets 1 – 4, but does not specify any particular facts that should be retrieved by this stage. The Australian standard is similar to the UK standard but is even less precise in that it suggests that students are expected to use retrieval and other 'proficient' strategies—meaning, possibly, fast and accurate Decomposition Strategies but also Counting Strategies (if the addend counted on, or the subtrahend counted down, is small). It seems straightforward to surmise that students should learn to retrieve answers to addition problems that add to 10, and to problems that involve a doubles fact, given these facts are commonly used in Decomposition Strategies. But beyond these particular problems, what other facts should students come to 'just know'?

## Literature Review

The importance of students learning to retrieve certain facts is clear in the literature; what is not clear is exactly what facts they should be able to retrieve. As a corollary, it is not clear if solving some problems within 20 using Decomposition Strategies is just as 'good' (i.e., acceptable in terms of standards) as solving them using retrieval, if these strategies are executed with speed and accuracy. There are at least three reasons why this is so.

First, students' retrieval use has not always been distinguished from their use of Decomposition Strategies. Retrieval and Decomposition Strategies have often been combined in studies and called *Retrieval-Based Strategies* (Canobi, 2009) *or Memory-Based Strategies*. For example, Geary (2011) examined students' use of Memory-Based Strategies to solve simple addition problems (problems with single-digit operands that sum to 10 or less) and complex addition problems (problems with one single-digit operand and one double-digit operand like 17 + 6). Geary found US students at the beginning of Grade 1 used Memory-Based Strategies to solve 19% of simple addition problems and 7% complex addition problems. The main finding was that students' use of Memory-Based Strategies (i.e., the number of problems solved using these strategies) predicted their mathematics achievement in Grade 1, as well as their growth in mathematics achievement assessed at the end of fifth grade - after controlling for domain-general factors (e.g., IQ, processing speed and working memory capacity). What is not clear from this study is what facts were retrieved and what facts were derived using Decomposition Strategies. Similarly, in our own research, we provided evidence indicating the importance of students using Retrieval-Based Strategies to solve single-digit addition problems by Year 3, but have not expanded on which facts they should be able to retrieve (Hopkins et al., 2022).

Second, when students' use of retrieval has been recorded separately, their use of different back-up strategies has not been delineated. For example, Barrouillet et al. (2008) examined the subtraction strategies used by Grade 3 French students. They coded students' strategies as (i) retrieval, (ii) applying the corresponding known addition fact, or (iii) using an algorithmic procedure. Findings indicated that students used retrieval on 19% of problems, applied the corresponding known addition fact on 28% of problems and applied an algorithmic procedure on 53% of problems. Barrouillet et al. described algorithmic procedures as being Counting Strategies, but the methods they described did not allow them to discern students' use of Decomposition Strategies.

Third, very few studies have considered students' use of addition *and* subtraction strategies in the same study (i.e., with the same participants). This is a noteworthy omission since it restricts the depth of investigation possible. For example, if students do not solve subtraction problems using corresponding addition facts (i.e., using their knowledge of fact families), it could be because they do not know these addition facts or because they do not apply these facts by making use of the *complementary relationship* between addition and subtraction (Canobi, 2009). If strategies for both operations are investigated together, then the first conjecture could be researched. Furthermore, in studies that have concentrated on one operation and not the other, problem groups across the operations have been treated differently so that synthesising findings becomes a complicated endeavour. For example, Geary (2011) referred to simple addition and complex addition where problems were akin to those included in Problem Sets 1 and 2, and Problem Set 3 respectively (shown in Figure 1). Like many researchers studying students' addition strategies, Geary did not include problems from Problem Set 4. In other studies that have focused on students' subtraction strategies, problems in Problem Set 4 have been included (e.g., Robinson, 2001), along with problems in other sets, but these have been grouped differently to how addition strategies have been grouped. For example, Barrouillet et al. (2008) divided problems into three sets that included subtraction problems with, (i) a minuend less than 10, (ii) a minuend greater than 10, and (iii) a minuend of 10. These sets do not correspond to how addition problems sets are generally constructed. Furthermore, it means that problems like 16 – 12 = 4 are categorised differently to a problem like 16 – 4 = 12; yet these two problems are related to the same fact family.

## The Current Study

The aim of the current study was to investigate the strategies students use in Years 3 and 4 to solve addition and subtraction problems within 20. To our knowledge, this is the first study where students' addition *and* subtraction strategies for solving problems that represent *all* problem sets shown in Figure 1 were investigated, and where strategy use was recorded in detail to distinguish between the different types of Decomposition Strategies (e.g., bridging-10 strategy and a near-doubles strategy). The research questions addressed were:

- How often do students use each strategy type to solve addition and subtraction problems within 20?
- How efficient (in terms of accuracy and speed) are students' use of particular back-up strategies?

Due to space limitations, initial findings addressing only the first research question are presented here. The study is framed by a social cognitive perspective of students' strategy development, which posits that individual factors (e.g., experience), as well as actions of significant others (e.g., teachers), and environmental factors (e.g., intended curriculum), influence the strategies students utilise to solve mathematical problems. Thus, investigating the strategies students use provides insights into factors that have influenced their learning.

## Methods

Participants included 127 Year 3 or 4 students from four independent schools in metropolitan Melbourne, ranging from average socioeconomic status (SES) (47th percentile; one school) to high SES (82-95th percentile; three schools). National assessments of Numeracy in Year 3 showed all schools to be achieving as expected for SES. Year 3 was selected given that the standard relating to retrieval targeted students in this year up until most recently. Year 4 was also included as students were organised into composite Year 3/4. Data were collected toward the end of the year. Given there were too many problems to ask each student to solve in one interview (around 20 minutes long), systematic sampling was used to select the problem set each participant would solve: 43 students solved all the problems in Problem Set 1, 43 students solved the same selection of 44 problems in Problem Set 3, and 41 solved the same selection of 44 problems in Problem Set 4. Selected problems represented the range of problems in the set and included fact families (e.g., 6 + 8 = 14, 8 + 6 = 14, 14 – 8 = 6 and 14 – 6 = 8). Participants also solved all 14 problems in Problem Set 2. Thus, in total participants solved 57 or 58 problems.

Students were individually withdrawn from their classroom to work with the research assistant (RA) in a quiet room nearby. Problems were presented on a computer screen one at a time, but in a random order: addition problems were presented first, then, after a short break, subtraction problems were presented. After solving each problem, the student called out their answer and the RA immediately pressed the space bar thereby removing the problem and stopping a timer, which recorded the time between problem presentation and answer. After each problem, the student was asked to explain the strategy they had used. The strategy was coded in situ by the RA if the self-report was consistent with what she had observed. If it wasn't (which was rare), she prompted the student further and coded what the student then reported. Tests of validity were conducted using reaction time analysis. The approach taken here to identify strategy use on a problem-by-problem basis, utilising self-report combined with observation, has been shown to provide valid data representing students' solution strategies for addition (Siegler, 1987) and subtraction (Robinson, 2001).

For addition problems within 20, 14 strategies were coded and labelled, and categorised into four main strategy types (see Table 1 for details). Given previous studies have generally not included addition problems from Problem Set 4, new codes/labels were needed to indicate that an operand had been split (i.e., partitioned into standard place value units) and the unit digits considered first. (See examples of the *split-retrieval, split-counting*, and *split-decomposition* strategies in Table 1). We also used a separate code for the decomposition strategy that made use of the known fact 5 + 5 and labelled this the *fives strategy.* (Since 5 + 5 = 10 can be considered a doubles fact or an add-to-ten fact, its use in a decomposition strategy could be labelled a near-doubles strategy or a bridging-10 strategy. Thus, depending on how the fact is represented, the frequency of one strategy over the other strategy would be inflated, and so it was coded separately.)

**Table 1**

*Addition Within 20: Strategy Types and Strategies*

| Strategy type | Examples | Self-report | Strategies |
|---|---|---|---|
| Retrieval | 3 + 2 = 5 | Just knew it | |
| Decomposition | 13 + 6 = 19 | (3 + 6 knew it) + 10 | Split-retrieval |
| | 8 + 5 = 12 | (8 + 2 knew it) + 3 | Bridging-10 |
| | 6 + 7 = 14 | (7 + 7 knew it) – 1 | Near-doubles (overshoot) |
| | 5 + 8 = 13 | (5 + 5 knew it) + 3 | Fives |
| | 4 + 9 = 13 | (4 + 10 knew it) – 1 | Add-10 (overshoot) |
| | 13 + 4 = 17 | (3 + 3 knew it + 1) + 10 | Split-decomposition |
| Counting | 5 + 9 = 14 | 9; 10,11,12,13,14 | Count-on-from larger |
| | 13 + 6 = 19 | (3; 4, 5, 6, 7, 8, 9) + 10 | Split-count |
| Other | 8 + 9 = 17 | (8 – 1) + (9 + 1) = 7 + 10 = 17 | Equalise |

*Note.* Other Strategies included multiplication, skip-counting, counted all, and counted-on-from-smaller strategies; '*don't know*' was also coded as well as strategies not-elsewhere recorded.

For subtraction problems within 20, 14 strategies were coded and labelled, and categorised into the same four strategy types. When coding the Decomposition Strategies students used for subtraction, we were interested to know whether students applied these strategies to take away the subtrahend (labelled 'down') or if they used indirect addition and started at the subtrahend (labelled 'up'). Solving subtraction problems using indirect addition represents an application of the complementary relationship between addition and subtraction, and is a very efficient way to solve multi-digit subtraction problems (e.g., 75 – 59 = 1 + 15) (Van Der Auwera et al., 2022). To date, subtraction by indirect addition has not been thoroughly investigated for problems within 20.

**Table 2**

*Subtraction Within 20: Strategy Types and Strategies*

| Strategy type | Examples | Self-report | Strategies |
|---|---|---|---|
| Retrieval | 10 – 6 = 4 | Just knew it | |
| Known addition | 10 – 6 = 4 | Knew 6 + 4 = 10 | |
| Decomposition | 14 – 3 = 11 | (4 – 3 knew it) + 10 | Split-retrieval |
| | 15 – 7 = 8 | (15 – 5 knew it) – 2 | Bridging-10 (down) |
| | 12 – 7 = 8 | (12 – 6 knew it) – 1 | Near-doubles (down) |
| | 15 – 7 = 8 | $7 + \boxed{3 + 5}$ | Bridging-10 (up) |
| | 17 – 8 = 9 | $8 + \boxed{8 + 1}$ | Near-doubles (up) |
| | 12 – 8 = 4 | (12 – 10 knew it) + 2 | Subtract-10 (overshoot) |
| | 15 – 11 = 4 | (15 – 10 knew it) – 1 | Subtract-10 |
| Counting | 10 – 6 = 4 | 10; 9, 8, 7, 6, 5, 4 | Count-by-ones (down) |
| | 10 – 6 = 4 | 6; 7, 8, 9, 10 | Count-by-ones (up) |
| | 18 – 3 = 12 | 10 + (8; 7, 6, 5) | Split-count |
| Other | 10 – 6 = 4 | 10 fingers-dropped 6 and counted or looked to see 4 remaining | Model-all |
| | 11 – 6 = 5 | (11 – 1) – (6 – 1) = 10 – 5 = 5 | Equalise |

*Note.* Other codes included don't know and not-elsewhere recorded.

## Initial Results and Discussion

The frequency of students' strategies for addition and subtraction problems within 20 are shown in Figures 2 and 3 respectively. Percentages presented in each figure are based on all trials (correct and incorrect), where a trial represents each time a problem was solved. These encompassed 903 trials for Problem Set 1 (43 students x 21 problems), 889 trials for Problem Set 2 (127 students x 7 problems), 946 trials for Problem Set 3 (43 students x 22 problems), and 902 trials for Problem Set 1 (41 students x 22 problems).

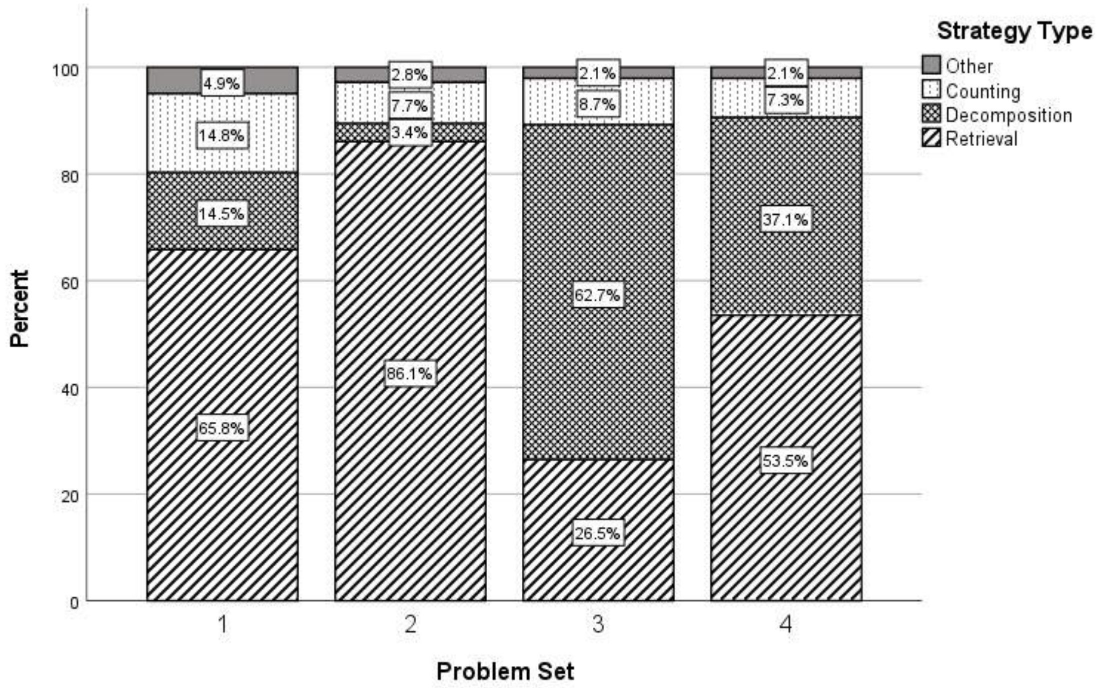



Figure 2. Addition strategies by type and problem set.

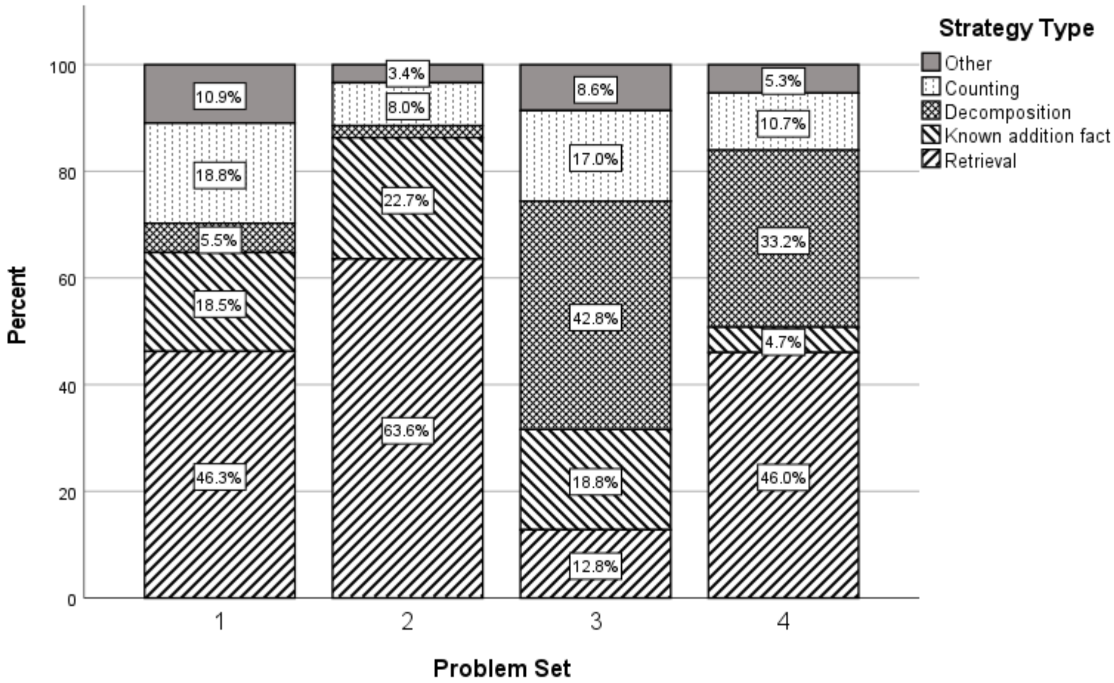



Figure 3. Subtraction strategies by type and problem set.

Students' prevalent use of Counting Strategies for addition (see Figure 2) is noteworthy given the age (year level) of participants; however, the frequency of counting recorded here is somewhat lower than that documented for similar aged students (Hopkins et al., 2022). This difference may reflect the fact that participating schools represented communities of relatively high SES. The prevalent use of Decomposition Strategies shown in Figure 2 is also noteworthy. It is surprising to see students using Decomposition Strategies at all on problems with sums less than ten (14.5% of trials). Another noteworthy finding was that participants used Decomposition Strategies most commonly to solve addition problems in Problem Set 3 (62.7% of trials). This finding highlights how it is unreasonable to expect students to recall addition facts for all single-digit numbers, as previously suggested in the Australian Curriculum (v. 8.4; ACARA, n.d.) and currently suggested in the US curriculum (CCSSO, 2010).

Comparing the frequency of students' addition strategies with their subtraction strategies reveals rates of retrieval for addition problems are similar to rates of retrieval, combined with applying a known addition fact, for corresponding subtraction across all problems sets. This finding suggests that students who solve addition problems using retrieval solve the corresponding subtraction problems using either direct retrieval or by applying the known addition fact. Comparing addition and subtraction strategies further suggests that for addition problems that are solved using Decomposition Strategies, corresponding subtraction problems are solved using Decomposition Strategies or Counting Strategies. Clearly, there are opportunities to interrogate these data further. We expect our findings will highlight the need for greater clarity around expectations to ensure students learn to retrieve the most important facts and develop efficient Decomposition Strategies for addition and subtraction within 20.

## References

Australian Curriculum, Assessment and Reporting Authority (n.d.). *What's changed in the new Australian curriculum.* https://v9.australiancurriculum.edu.au/resources/stories/curriculum-changes

Australian Curriculum, Assessment and Reporting Authority (n.d.). *The Australian curriculum: Mathematics (version 8.4).* https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Australian Curriculum Assessment and Reporting Authority (n.d.). *Australian curriculum: Mathematics (v 9.0).* Retrieved from: https://v9.australiancurriculum.edu.au/

Barrouillet, P., Mignon, M., & Thevenot, C. (2008). Strategies in subtraction problem solving in students. *Journal of Experimental Child Psychology, 99*(4), 233–251. https://doi.org/10.1016/j.jecp.2007.12.001

Canobi, K. H. (2009). Concept–procedure interactions in children's addition and subtraction. *Journal of Experimental Child Psychology, 102*(2), 131–149. https://doi.org/10.1016/j.jecp.2008.07.008

Council of Chief State School Officers (2010). *Common core state standards for mathematics.* https://learning.ccsso.org/wp-content/uploads/2022/11/ADA-Compliant-Math-Standards.pdf

Department for Education. (2013). *The national curriculum (United Kingdom)*. https://www.gov.uk/government/collections/national-curriculum

Geary, D. C. (2011). Cognitive predictors of achievement growth in mathematics: A 5-year longitudinal study. *Developmental Psychology, 47*(6), 1539–1552. https://doi.org/10.1037/a0025510

Hopkins, S., Russo, J., & Siegler, R. (2022). Is counting hindering learning? An investigation into children's proficiency with simple addition and their flexibility with mental computation strategies. *Mathematical Thinking and Learning, 24*(1), 52–69.

Polspoel, B., Peters, L., Vandermosten, M., & De Smedt, B. (2017). Strategy over operation: Neural activation in subtraction and multiplication during fact retrieval and procedural strategy use in children. *Human Brain Mapping, 38*(9), 4657–4670. https://doi.org/10.1002/hbm.23691

Robinson, K. M. (2001). The validity of verbal reports in children's subtraction*. Journal of Educational Psychology, 93*(1), 211. https://doi.org/10.1037//0022-0663.93.1.211

Siegler, R. S. (1987). The perils of averaging data over strategies: An example from children's addition. *Journal of Experimental Psychology: General, 116*(3), 250–264. https://doi.org/10.1037/0096-3445.116.3.250

Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking*. University Press.

Van Der Auwera, S., Torbeyns, J., De Smedt, B., Verguts, G., & Verschaffel, L. (2022). The remarkably frequent, efficient, and adaptive use of the subtraction by addition strategy. *Learning and Individual Differences*, 93, 102107. https://doi.org/10.1016/j.lindif.2021.102107

# Mathematical Competence Exhibited by Year 2 Students When Learning Through Sequences of Challenging Tasks

Jane Hubbard
*Monash University*
Jane.Hubbard@monash.edu

Developing reliable and effective processes to monitor and interpret student progress during problem-solving tasks is an ongoing challenge in mathematics education. This study utilised qualitative data sources inclusive of observations, work samples and interview transcripts from six Year 2 students to investigate exhibited competence in classroom settings. The analysis showed that students demonstrated growth in both cognitive and dispositional elements of competence when learning through sequences of challenging tasks. These elements, consistently identified across different class settings and lesson topics, have the potential to broaden interpretations of mathematical competence within both practice and research domains.

An ongoing challenge within mathematics education is being able to identify what it means to become mathematically competent. Interpretations of competence are heavily context-dependent, leaving little consensus as to how the construct should be defined (Ropohl et al., 2018). Within the discipline of mathematics, definitions of competence become particularly relevant when considering the most effective ways to teach. For example, those supporting traditional approaches to mathematics learning emphasising the mastery of skills and procedures before problem-solving (e.g., Kirschner et al., 2006), are likely to consider competence as the demonstration of such processes. Alternatively, those aligned with reformist orientations, recognising the collaborative nature of problem-solving pedagogies, lean towards more holistic interpretations of competence (Blömeke et al., 2015). From this perspective, value is placed on the simultaneous development of students' conceptual understanding and positive dispositions within mathematics learning. However, a prominent barrier to the latter interpretation of competence is being able to accurately identify and measure the complex way such elements of students' learning manifest and develop.

The research presented in this paper aligns with the interpretation of competence presented by Blömeke et al. (2015), positioning mathematics learning within a challenging task approach (Sullivan et al., 2015). Learning mathematics through a challenging task approach creates opportunities for students of all abilities to engage with cognitively demanding, non-routine problems by: utilising prior knowledge; exploring multiple solutions; and working collaboratively to deepen conceptual understanding (Sullivan et al., 2020). Previous research on challenging tasks has focused on teacher professional development (e.g., Ingram et al., 2020; Sullivan et al., 2015) and student achievement in the middle years (e.g., Sullivan et al., 2016) with limited focus on students in the Early Years (e.g., Hubbard et al., 2022; Russo & Hopkins, 2017). Therefore, identifying how Year 2 students demonstrate and develop mathematical competence when learning through challenging tasks will contribute to the literature. Specifically, this paper aims to address the following research question:

- How do Year 2 students demonstrate and develop mathematical competence when learning through sequences of connected, cumulative and challenging tasks?

## Literature Review

Measures of student achievement in mathematics have traditionally relied on tests and written assessments to evaluate student learning. A constraint of these practices is that the fundamentals of mathematics learning are often misrepresented (Clarke, 2011). Broadening the scope of assessment practices to more accurately reflect students' mathematics learning experiences continue to receive

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 275–282). Newcastle: MERGA.

attention within the literature. Examples of alternative forms of mathematics assessment designed to address this gap have included: multi-mode interviews (Kuzle, 2017); comparative judgement of rich assessment tasks (Jones et al., 2015); and the development of marking keys for interpreting student responses to sequences of non-routine tasks (Hubbard et al., 2022). These studies reported on processes deemed effective in evaluating student competence more comprehensively than traditional tests by enabling the demonstration of both core skills and higher-order thinking. However, the corresponding assessment tasks were disconnected from students' actual classroom experiences, continuing to frame achievement as individual cognitive performance.

Observations of students' mathematical learning in classrooms have the potential to provide insights into competence that cannot be ascertained otherwise, yet the abundance of qualitative sources can become problematic in clarifying the construct of competence. Ropohl et al. (2018) identified one of these challenges as being able to make sense of the complexities within the data. Schlesinger and Jentsch (2016) conducted a systematic literature review focusing on the methodologies used to carry out such classroom observations. The authors reported that little consistency exists in identifying specific aspects of competence being studied and that the unique classroom settings pose challenges in making generalisations from the findings. For example, while Özdemir and Pape (2012) conducted a four-month study revealing that specific classroom practices enhanced students' ability to self-regulate their learning in mathematics, the findings were specific to one Year 6 class, restricting the transferability to other contexts. Another concern raised by Schlesinger and Jentsch (2016) was validity issues that occur when data collection relies too heavily on the recollections of participants. Boesen et al. (2014) encountered this issue when working with over 200 teachers for 12 months in a professional learning capacity. Their study found that despite clearly articulating diverse notions of mathematical competence throughout their program, the participants' final reflections of student achievement were dominated by traditional perceptions of ability such as the accurate completion of procedural tasks. The tendency to preference cognitive achievement over dispositional elements of student learning is not uncommon when defining mathematical competence and one that Beyers (2011) attributed to limited research on the influence dispositions have on mathematical thinking.

Blömeke et al. (2015) suggested adopting analytical processes that recognise related observable behaviour and cognitive abilities in preference to approaches that isolate discrete elements of competence. In doing so, "the successful deployment of capabilities in engagement with mathematical problems and the language and tools of mathematics" (Ropohl et al., 2018, p. 17) can be identified and evaluated. Chan and Clarke (2017) demonstrated this approach in an analysis of video data and student work samples of two Year 7 classes working collaboratively on problem-solving tasks. Their findings showed that the observed negotiations and interactions of students, while rich and complex, could be analysed within several themes comprising both mathematical and social, enabling consistent analysis across the different classes. Similarly, Groth (2017) introduced a proficiency protocol to guide the observation of prospective teachers across a series of lessons based on multiple Year 7 mathematics classes. The provision of the protocol supported a focus on the dispositional aspects of student learning, leading to a better collective understanding of the nuanced happenings occurring across the different mathematics lessons. Adopting similar processes could provide further insights into the ways students holistically develop mathematical competence when learning through sequences of challenging tasks in the Early Years.

## Methodology

This study was conducted within a larger research project entitled *Exploring Mathematical Sequences of Connected, Cumulative and Challenging Tasks* (EMC$^3$) (Sullivan et al., 2020). Building upon previous work on challenging tasks (see Sullivan et al., 2015), EMC$^3$ focused on the ways sequences of challenging tasks and the associated pedagogies support mathematics learning for students in Foundation to Year 2 (5- to 8-years old). One outcome of the EMC$^3$ project to date was the development of an instructional model supporting the implementation of tasks and ensuring adequate provision of agency and inclusion for students (Sullivan et al., 2021). The model, based on the work of Smith and Stein (2011), encourages teachers to anticipate student responses before planning lessons within three structured phases: *Launch, Explore* and *Summarise/Review*.

The focus of this study was to investigate, the ways Year 2 students demonstrate and develop mathematical competence when learning through sequences of challenging tasks. Six focus students (two students across three classes) were selected from a total of 59 Year 2 students at one of the participating EMC$^3$ project schools. Students were selected by identifying prior mathematical achievement with the intention that the focus students would represent the diversity of the overall Year 2 cohort. Fred, Jess, and Tim demonstrated moderate levels of achievement while Zara, Evie, and Annie (all pseudonyms) represented students with lower mathematical levels of achievement. Qualitative data were collected to create *learning portfolios* intended to track changes in student learning as they participated in the EMC$^3$ project. Lesson notes were collected using an observation protocol based on the phases of the EMC$^3$ instructional model. These notes along with student work samples and post lesson interview transcripts were collated from the first three consecutive lessons of the study (Portfolio 1) and compared to the same data sources collected from three consecutive lessons nine-months later (Portfolio 2).

## *Data Analysis*

Lesson artefacts were created to identify the cognitive and dispositional behaviours students demonstrated for each lesson. Figure 1 presents two of Annie's lesson artefacts taken from Portfolio 1 and 2 respectively.

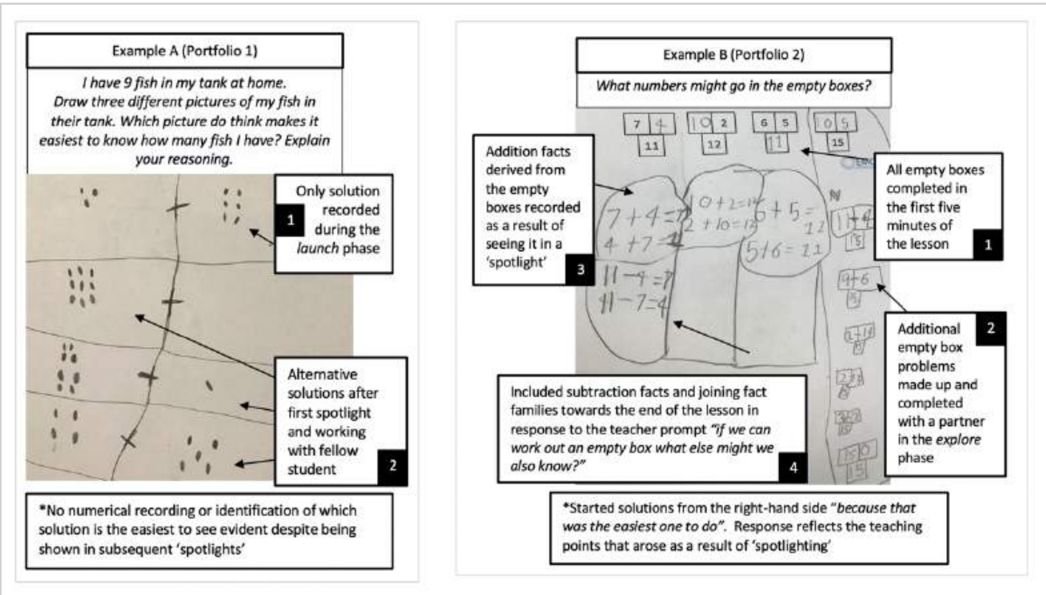



Figure 1. Comparative lesson artefacts (Fish Tank task & Empty Boxes tasks).

The process of annotating work samples with time bound lesson observations was based on a similar method documented by Schoenfeld (2016). This holistic interpretation of student learning highlighted cognitive and dispositional behaviours students exhibited throughout the study, enabling competence elements that aligned with the three EMC$^3$ lesson phases to be generated. The presence of each competence element evident in the artefacts was then coded according to the following: (√) *Most of the time* (three examples, one from each lesson); (♦) *some of the time* (1 to 2 examples, over any lesson); and (-) *not evident* (no example). The coding enabled comparisons to be made between the extent that these elements were present in Portfolios 1 and 2 and were reported in the results.

# Results

The results report on the coded competence elements according to the three EMC$^3$ lesson phases: *Launch, Explore, Summarise.* The first line of coding for each competence element corresponds to Portfolio 1 whereas the second line of coding represents the presence throughout Portfolio 2 (see Tables 1, 2 & 3).

## *The Launch Phase*

The competence elements identified within the *Launch* phase are presented in Table 1. As part of this phase, the task is posed to students without explicit instruction and students are provided with approximately five minutes to engage with the task independently.

**Table 1**

*Competence Elements Demonstrated Throughout the Launch Phase of the Lesson*

| Competence elements *Launch* phase | Fred | Jess | Tim | Zara | Evie | Annie |
|---|---|---|---|---|---|---|
| Demonstrates a willingness to independently read and attempt task without further instruction | ♦ | ♦ | √ | √ | √ | ♦ |
| | √ | √ | √ | √ | √ | √ |
| Records mathematics related to the task | ♦ | ♦ | √ | ♦ | ♦ | ♦ |
| | √ | √ | √ | √ | √ | √ |
| Provides more than one correct solution while working independently | ♦ | ♦ | ♦ | - | - | - |
| | √ | √ | √ | √ | √ | √ |
| Demonstrates an awareness of connections to prior learning | ♦ | ♦ | ♦ | ♦ | ♦ | ♦ |
| | √ | √ | √ | √ | √ | √ |

Table 1 shows throughout Portfolio 2 data, students consistently demonstrated each of the necessary elements that comprise competence, regardless of their prior mathematical achievement status. On its own, the first element, *a willingness to independently read and attempt the task without further instruction*, could simply be representative of compliant and well-behaved students. However, when taken with the presence of the additional three elements (*demonstrating initial mathematics thinking; representing multiple solutions; connecting knowledge to previous experiences*) their willingness to engage independently is likely to be indicative of productive dispositions. The interconnectedness of these elements can be illustrated through the lesson artefacts in Figure 1. Annotations in Example B show that in the first five minutes of the lesson, Annie was able to provide multiple solutions to the four empty box tasks and these solutions were reflective of mathematical thinking (Figure 1, Example B, annotation 1). Observation notes recorded at the bottom of Example B shows that an additional factor was the utilisation of prior knowledge, as Annie started with the empty boxes on the right-hand side *'because it was the easiest one to do'* (Annie, interview transcript). Contrast these elements with the annotations from Example A detailing less evidence of mathematical thinking or the utilisation of prior learning, and the way Annie willingly and independently engaged with the tasks in the *Launch* phase by Portfolio 2 becomes apparent.

## *The Explore Phase*

The *Explore* phase of the lesson opens up the learning experience to encourage students to share their initial thinking and begin collaborating with peers. In this study, teachers selected specific work samples to show the class initiating student-centred discussions. This technique was referred to as 'spotlighting'. Table 2 presents the competence elements identified throughout the *Explore* phase.

**Table 2**

*Competence Elements Demonstrated Throughout the Explore Phase of the Lesson*

| Competence elements *Explore* phase | Fred | Jess | Tim | Zara | Evie | Annie |
|---|---|---|---|---|---|---|
| Attentive during spotlighting<br>(i.e., paying attention to speaker, focused on work being shared) | ♦<br>♦ | ♦<br>♦ | ♦<br>♦ | ♦<br>√ | ♦<br>√ | ♦<br>√ |
| Responsive to spotlighting<br>(i.e., changing strategy or working out after seeing other solutions) | ♦<br>√ | ♦<br>√ | ♦<br>√ | ♦<br>√ | ♦<br>√ | ♦<br>√ |
| Collaborates with peers | ♦<br>√ | ♦<br>√ | ♦<br>√ | -<br>√ | -<br>√ | -<br>√ |
| Responds productively to feedback | -<br>√ | -<br>√ | -<br>√ | -<br>√ | -<br>√ | -<br>√ |
| Work sample demonstrates multiple solutions or solution pathways | ♦<br>√ | ♦<br>√ | ♦<br>√ | ♦<br>√ | ♦<br>√ | ♦<br>√ |

The changes presented in Table 2 show an overall increase in teacher and peer interaction by the end of the study. Warranting further attention is the degree to which some of these elements improved in comparison to others. Extensive development is evident in the way students responded to feedback. The initial lesson artefacts from Portfolio 1 show a range of unproductive habits such as students rubbing out their work or covering their page when asked a question. The following excerpt from an interaction documented at the start of the study demonstrates an example of a less productive response to feedback:

Teacher: Can you tell me what you have done here? (Pointing to 6 dots arranged in 2 rows of 3)

Fred: This is a solution to how the fish would be arranged.

T: I am wondering if this shows all the fish (and points back to the written task where 9 fish is recorded).

F: Oh, it is a mistake (student proceeds to get the rubber and rub out the solution).

In this exchange, Fred's actions suggested that incorrect solutions should not be included within a work sample and needed to be rubbed out and removed. The interaction suggests that Fred was not used to correcting existing solutions (which would have been possible by adding another row of dots) implying that he considered the teachers' role was to correct work rather than support the development of his learning. Lesson observations from the end of the study show more productive responses to feedback opportunities. For example, one student volunteered that *'this part of the page shows my thinking at the start'* (Evie) and another articulated *'I have added in these labels to show what the diagram means'* (Tim). These shifts suggest that the use of guiding and clarifying questions to facilitate interactions between teachers and students likely changed the perception that feedback was intended to support their learning, rather than evaluate it.

The role of 'spotlighting' became a critical element in supporting students to develop competence within the *Explore* phase. How students responded to 'spotlighting' was evident in the ways students explained their written responses. For example, Jess indicated that *'I have done the same strategy but set it out in a table so I will just keep going my way'* when asked if a previous 'spotlight' helped her with the solution. That students can discern if it is necessary to make changes to their thinking is an encouraging observation, even when they were ostensibly not attending to the spotlight discussion. This suggests students have developed an understanding that the purpose of a 'spotlight' is exposure to alternative thinking, not necessarily to showcase a preferred solution or procedure.

*The Summarise Phase*

The competence elements identified throughout the final phase of the lesson are presented in Table 3. The *Summarise* phase provides an opportunity for the whole class to review the mathematical focus of the lesson and discuss their learning experiences.

**Table 3**

*Competence Elements Demonstrated Throughout the Summarise Phase of the Lesson*

| Competence elements Summarise phase | Fred | Jess | Tim | Zara | Evie | Annie |
|---|---|---|---|---|---|---|
| Demonstrates attentive behaviours during class discussion | ♦ | ♦ | ♦ | ♦ | ♦ | ♦ |
| | ♦ | ♦ | ♦ | √ | √ | √ |
| Willingness to contribute to class discussions | ♦ | ♦ | ♦ | - | - | - |
| | √ | ♦ | √ | ♦ | √ | √ |
| Written response reflects thinking trajectory over the whole lesson | ♦ | ♦ | ♦ | - | - | ♦ |
| | √ | √ | √ | √ | √ | √ |

Table 3 shows that by the end of the study, students demonstrated competence to varying degrees throughout the *Summarise* phase. The element that represented the greatest increase was students' written responses, better reflecting student thinking over the whole lesson. The lesson artefacts from the end of the study not only provide broader examples of strategies and solutions but also indicate more flexibility and connectivity of mathematical concepts. Being able to produce a written response in this manner shows students recognised their work reflects learning and understanding rather than an outcome derived from mimicking given processes.

## Discussion and Conclusion

The findings of this study support the claim of Ropohl et al. (2018) that observations within classroom environments afford greater insights into the holistic notion of mathematical competence than can be determined from a written assessment or single work sample analysis. Triangulating the qualitative data sources to create lesson artefacts representative of both cognitive and dispositional behaviours as recommended by Blömeke et al. (2015), extended the interpretation of work samples beyond traditional outcomes and better represented the mathematics problem-solving experiences of students (Clarke, 2011). Using the three lesson phases from the EMC$^3$ project to structure observations proved beneficial in terms of ensuring the nuanced differences in student learning, both within a complete lesson and over a series of lessons, could be accurately identified and compared. Similar to the findings from Chan and Clarke (2017), being able to analyse classroom observations through such schema enabled consistency in the interpretation of student competence across multiple classes. Moreover, the competence elements identified in the different lesson phases showed transferability across class contexts and different lesson topics, reinforcing the notion that competence is broader than lesson-specific content and procedures.

As well as emphasising the interconnectedness of cognitive and dispositional components of mathematics learning, the findings reported various ways sequences of challenging tasks support Early Years students to experience success with their learning. Being able to specify when and how students access their prior knowledge, engage with alternative solutions and respond to feedback encapsulated the experiences of mathematics learning as intended through the EMC$^3$ approach (Sullivan et al., 2020). Furthermore, the growth demonstrated by the focus students, regardless of their prior achievement status suggests that a broad range of students can effectively develop mathematical competence when learning through the EMC$^3$ instructional model (Sullivan et al., 2021). Particularly insightful was the impact that communication, both between peers and student/teacher interactions had on strengthening student competence, and mirrored the findings from Chan and Clarke (2017) that social components are central to successful mathematics learning. Recognising that these elements are critical in mathematics development, even for students in the Early Years, may help teachers to shift the emphasis away from traditional measures of competence and more accurately target suitable areas for future learning. While limitations of this study include a small sample size within a single school setting, the competence elements identified through this investigation may provide a useful structure to guide further research in this area.

## References


Beyers, J. E. R. (2011). Student dispositions with respect to mathematics: What current literature says. In D. J. Brahier, & W. R. Speer (Eds.), *Motivation and disposition: Pathways to learning, 73rd NCTM Yearbook.*

Blömeke, S., Gustafsson, J. E., & Shavelson, R. J. (2015). Beyond dichotomies: Competence viewed as a continuum. *Zeitschrift für Psychologie, 223,* 3–13. https://doi.org/10.1027/2151-2604/a000194

Boesen, J., Helenius, O., Bergqvist, E., Bergqvist, T., Lithner, J., Palm, T., & Palmberg. B. (2014). Developing mathematical competence: From the intended to the enacted curriculum. *Journal of Mathematical Behaviour, 33*, 72–87. http://dx.doi.org/10.1016/j.jmathb.2013.10.001

Chan, M. C. E., & Clarke, D. (2017). Structured affordances in the use of open-ended tasks to facilitate collaborative problem-solving. *ZDM Mathematics Education, 49*, 951–963. http://doi.org/10.1007/s11858-017-0876-2

Clarke, D. J. (2011). Open-ended tasks and assessment: The nettle or the rose? In B. Kaur, & K. Wong (Eds.), *Assessment in the mathematics classroom: Yearbook 2011 association of mathematics educators* (pp.131–163). World Scientific. https://doi.org/10.1142/9789814360999_0007

Groth, R. E. (2017). Classroom data analysis with the five strands of mathematical proficiency. *The Clearing House: A Journal of Educational Strategies Issues and Ideas, 93*(3), 103–109. https://doi.org/10.1080/00098655.2017.1301155

Hubbard, J., Russo, J., & Livy, S. (2022). Assessing mathematical competence through challenging tasks. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 282–289). Launceston: MERGA.

Ingram, N., Holmes, M., Linsell, C., Livy, S., McCormick, M., & Sullivan, P. (2020). Exploring an innovative approach to teaching mathematics through the use of challenging tasks: A New Zealand perspective. *Mathematics Education Research Journal, 32*(3), 497–522. https://doi.org/10.1007/s13394-019-00266-1

Jones, I., Swan, M., & Pollitt, A. (2015). Assessing mathematical problem-solving using comparative judgement. *International Journal of Science and Mathematics Education, 13*, 151–177. https://doi.org/10.1007/s10763-013-9497-6

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist 41*(2), 75–86.

Kuzle, A. (2017). Assessing metacognition of grade 2 and grade 4 students using an adaption of a multi-method interview approach during mathematics problem-solving. *Mathematics Education Research Journal, 30,* 185–207. https://doi.org/10.1007/s13394-017-0227-1

Özdemir, I. E. Y., & Pape, S. J. (2012). Supporting students' strategic competence: A case of a sixth-grade mathematics classroom. *Mathematics Education Research Journal, 24*, 153–168. https://doi.org/10.1007/s13394-012-0033-8

Ropohl, M., Nielsen, J. A., Olley, C., Rönnebeck, S., & Stables, K. (2018). The concept of competence and its relevance for science, technology and mathematics education. In J. Dolin, & R. Evans (Eds.), *Transforming assessment,* Contributions from Science Education Research 4. https://doi.org10.1007/978-3-319-63248-3_1

Russo, J., & Hopkins, S. (2017). Student reflections on learning with challenging tasks: 'I think the worksheets were just for practice, and the challenges were for maths'. *Mathematics Education Research Journal, 29*(3), 283–311. https://doi.org/10.1007/s13394-017-0197-3

Schlesinger, L., & Jentsch, A. (2016). Theoretical and methodological challenges in measuring instructional quality in mathematics education using classroom observations. *ZDM Mathematics Education, 48*, 29–40. https://doi.org/10.1007/s11858-016-0765-0

Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, meta-cognition, and sense making in mathematics (reprint). *Journal of Education 196*(2), 1–38. https://doi.org/10.1177/002205741619600202

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematical discussions.* National Council of Teachers of Mathematics.

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A., Roche, A., & Walker, N. (2015). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education, 18*(2), 123–140. https://doi.org/10.1007/s10857-014-9279-2

Sullivan, P., Bobis, J., Downton, A., Hughes, S., Livy, S., McCormick, M., & Russo, J. (2020). Ways that relentless consistency and task variation contribute to teacher and student mathematics learning. In A. Coles (Ed.), *For the learning of mathematics: Proceedings of a symposium on learning in honour of Laurinda Brown: Monograph 1* (1st ed., pp. 32–37). FLM Publishing Association.

Sullivan, P., Bobis, J., Downton, A., Feng, M., Hughes, S., Livy, S., McCormick, M., & Russo, J. (2021). An instructional model to support planning and teaching student centred structured inquiry lessons. *Australian Primary Mathematics Classroom, 26*(1), 9–12.

Sullivan, P., Borcek, C., Walker, N., & Rennie, M. (2016). Exploring a structure for mathematics lessons that initiate learning by activating cognition on challenging tasks. *Journal of Mathematical Behaviour, 41,* 159–170. http://dx.doi.org/10.1016/j.jmathb.2015.12.002

# Importance and Centrality of Various Beliefs Held by High School Mathematics Teachers

Mairaj Jafri
*The University of Waikato*
mairajjafri@gmail.com

The central beliefs of mathematics teachers play a crucial role in shaping their teaching practices and determining how their students learn mathematics. This study presents the relative importance of various beliefs held by mathematics teachers. The study surveyed over 300 high school mathematics teachers in Pakistan (using a Likert scale of 1 to 5), exploring which beliefs are most important and how central beliefs impact their teaching. The results show that beliefs about exploring problems to discover patterns, textbooks, memorization and mathematical signs, notations, and symbols have many connections and correlations with other beliefs. These beliefs may potentially serve as the central beliefs in the teachers' belief system.

## Introduction

The central beliefs of mathematics teachers play a crucial role in shaping their teaching practices and determining how their students learn mathematics. Positive beliefs about mathematics teaching and learning create an environment of support and exploration (Maasepp & Bobis, 2014). They encourage students to understand mathematics concepts and theories rather than merely memorizing facts and equations (Säljö, 2010). Several studies indicate various beliefs that mathematics teachers hold (Amirali & Halai, 2021; Beswick, 2007; Lloyd, 2002; Pajares, 1992; Schoenfeld, 2011; Yurekli et al., 2020). However, not all beliefs are equally important because "beliefs vary along the central-peripheral dimension" (Rokeach, 1968, p. 3). Only the central beliefs, as they have more connections with other beliefs in our belief system, are important (Rokeach, 1968). Based on any given knowledge, if an individual holds a particular belief, then it is likely that they may hold certain further related ideas and attitudes (Barton & Parsons, 1977). That is, there is a relationship between beliefs, and certain (central) beliefs may cause changes in other beliefs and behaviour (Muijs & Reynolds, 2015). Thus, to understand teachers' beliefs, one has to understand their central beliefs. Examining the information about central beliefs could provide an understanding of how teachers perceive and approach mathematics teaching and learning and may locate areas for professional development or ways to support teachers in enhancing their teaching practices.

## Purpose of the Study

The purpose of the paper is to present quantitative findings that determined the relative importance or centrality of various beliefs held by mathematics teachers in Pakistan. In particular, the study aimed to explore what beliefs are most important to teachers and how central beliefs influence other beliefs about teaching and learning mathematics. The study focused on investigating the beliefs held by teachers concerning the nature of mathematics, their teaching practices, and the use of resources in teaching and learning mathematics.

## Theoretical Framework

There has been a plethora of literature on understanding teachers' beliefs in educational psychology (Ernest, 1989; Pajares, 1992; Schoenfeld, 2011). The previous studies provide understanding why teachers do what they do and how they behave in response to different pedagogical situations. Philipp (2007, p 259) define beliefs as "psychologically held understandings, premises or propositions about the world that are thought to be true." Considering beliefs as a psychological construct, Philipp (2007) emphasized on a continuous need to investigate and understand teacher system of beliefs. In relation to mathematics, Jafri (2022) articulates mathematics

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 283–290). Newcastle: MERGA.

teachers' beliefs as opinions, dispositions, (pre)-conceptions, and philosophies they hold about the nature of mathematics and its teaching as a whole. Jafri argues that a teacher could develop a "self-confirming bias" about a particular teaching practice or a resource because that is how he or she learned when they were a student. Such biases could lead a teacher to believe that the way he or she learned is the only effective way to teach. Based on their personal life experiences, socio-cultural and religious contexts, mathematics teachers may hold identical or different beliefs (Amirali & Halai, 2021). Notably, religious contexts and beliefs, no matter how strong or subtle they may be, impact teachers' beliefs about mathematics, and teachers apply them to their (mathematics) teaching (Chan & Wong, 2014). However, as religious beliefs are rich in content, teachers may apply only a portion of them to their teaching. As beliefs become more widely accepted and practised, they become part of the social context of teaching (Ernest, 1989). Subsequently, these beliefs become a widely shared belief system among the majority of the teachers, serving as the foundation for their thought process, behaviour, and interactions with each other and their students (Pajares, 1992). However, not all beliefs may be equally important for teaching and learning mathematics. "Beliefs vary along the central-peripheral dimension" (Rokeach, 1968, p. 3). Only the central beliefs are important because they have more connections with other beliefs in our belief system (Rokeach, 1968). These central beliefs are most difficult to change (Ertmer & Ottenbreit-Leftwich, 2010). Therefore, they need to be addressed and examined as they drive classroom actions and influence the teacher change process. (Richardson, 1996).

The beliefs become central if we learn them during childhood and involve a direct encounter with the object of belief (Rokeach, 1968). For example, a belief that mathematics textbooks are important for teaching does not only involve an object; it is further reinforced by the unanimous social consensus among all of one's reference persons and groups (Jamieson-Proctor & Carmen, 2008). However, if we consider beliefs to be a collection of attitudes, then beliefs that play a critical role within a person's belief system and help in determining his or her behaviour are important (Rokeach, 1968). We hold such beliefs within the innermost core of the belief system, and they act as central beliefs (Rokeach, 1968). Such biases and beliefs may inhibit a teacher's desire to change teaching practices related to the belief (Pajares, 1992) and are normally unaffected by new information (Ernest, 1989). Therefore, it is essential to determine the relative importance or centrality of various beliefs held by mathematics teachers.

Rokeach (1968) defines importance in terms of connectedness. They suggest that the more a given belief communicates or has functional connections with other beliefs, the more we can consider it a central belief (p. 5). Nonetheless, central beliefs are a small set of beliefs that has more implications and consequences for other beliefs in the belief system. Rokeach proposes criteria for functional connectedness or functional communication. They argue that beliefs that are directly concerned with our existence and identity and that we share with others are important. These beliefs have more functional connections and consequences than others. Whereas beliefs that are derived and are concerned with more or less arbitrary matters have fewer functional connections and consequences for other beliefs. Although, Rokeach's theory of central beliefs has been influential in psychology and sociology, it criticised due to its oversimplification. Rokeach's theory suggests that individuals have a relatively small set of central beliefs that guide their behaviour. However, some researchers (Festinger, 1962; Pajares, 1992) argue that beliefs are complex and multifaceted and that individuals may hold conflicting or contradictory beliefs. Under a particular social context of teaching and the teacher's level of thought processes and reflection, beliefs could change (Ernest, 1989). Influenced by the social context, teachers are likely to adopt the same teaching methods despite holding differing beliefs about mathematics (Pajares, 1992). For example, when teachers employ curriculum materials in their classrooms, they could potentially develop new mathematical and pedagogical beliefs and skills based on their design of lessons, conversations with students, use of technology, and so on (Lloyd, 2002). However, Yurekli et al. (2020) argue that several constraints (or factors) existing in the educational environment, such as assessment methods and students' understanding of mathematics, cause discrepancies between beliefs and practices.

In terms of measurement of central beliefs, Barton and Parsons (1977) assume that the degree to which an attitude is important or central to the individual is one of the most critical attributes requiring measurement. Barton and Parsons relate strength of belief to the degree to which an individual holds the attitude and the willingness to act on it. We can measure it by how much effort an individual is willing to expend to maintain an attitude or defend the belief. One can assess this through self-report surveys, attitude scales and experiments. For example, a survey might ask respondents to rate the importance of a certain belief on a scale of 1 to 5, with 5 being the highest score (Bautista et al., 2020). Other methods for measuring attitude strength include observing how quickly an individual responds to questions related to the attitude, how frequently the individual mentions it in conversation and how strongly the individual expresses agreement or disagreement with the attitude.

## Methodology

This paper presents quantitative results from an ongoing doctoral investigation that uses a mixed-methods approach. The main study utilized an online survey to acquire quantitative data. High school mathematics teachers from Pakistan volunteered to complete an online survey. The aim was to recruit teachers with varied number of years of teaching experience from both government and private schools in Pakistan's rural and urban areas. However, this study only presents the data and findings framed around the research question:

- What are the central beliefs of mathematics teachers and how are they connected with other beliefs?

We presented the seven (07) beliefs about the nature of mathematics teaching and learning as statements in the online survey, which are as follows: (1) Learning mathematics means exploring problems to discover patterns and make generalisations (B2_1); (2) Mathematics teaching is to teach students how to create and assign meanings to signs, symbols, and notations (B2_2); (3) Mathematics teaching is to share knowledge and ideas, and to discuss a variety of real-world contexts (B2_3); (4) Mathematics lessons should be followed by a critical discussion with students (B2_4); (5) Mathematics curriculum textbooks are the best medium of instruction and source of knowledge (B2_5); (6) Mathematical knowledge is retained more easily if it is acquired using multiple representations (B2_6); (7) While learning mathematics, it is important to memorize rules, facts and formulae (B2_7).

Based on Bautista et al. (2020), the high school mathematics teachers were asked to submit their agreement or disagreement on a Likert scale of $1\ to\ 5$ (1 strongly disagree and 5 strongly agree). More than 300 high school mathematics teachers in Pakistan responded to the survey. After initial screening and removing incomplete responses, the study selected 270 responses for analysis.

The study used descriptive statistics to measure the central position within the data and Spearman's Rho $(\rho)$ correlation test to understand how mathematics teachers' beliefs were correlated. Barton and Parsons (1977) consider correlation coefficients as a general method of measuring the structuredness of attitudes and beliefs. The test measured the strength and direction of association that may exist between teachers' beliefs. The two assumptions for Spearman's Rho were satisfied. First, we measured the scores using an ordinal scale—a measurement scale that uses labels to classify cases into ordered classes, such as the Likert scale (strongly agree to strongly disagree). Second, the relationships between items were monotonic, i.e., if the value of one item increases, so does the value of the other item.

## Findings

The study calculated descriptive statistics and correlations between the beliefs. Table 1 shows the results of descriptive statistics. The first column shows the code associated with each belief statement. The second column contained belief statement. The remaining columns shows the measure of central tendency i.e., Mean, Median, Mode and Standard deviation (SD).

**Table 1**

*Descriptive Statistics*

| Item No. | Item Statement | Mean | Median | Mode | SD |
|---|---|---|---|---|---|
| B2_1 | Learning mathematics means exploring problems to discover patterns and make generalisations. | 4.71 | 5.00 | 5 | 0.46 |
| B2_2 | Mathematics teaching is to teach students how to create and assign meanings to signs, symbols, and notations. | 4.24 | 4.00 | 5 | 0.62 |
| B2_3 | Mathematics teaching is to share knowledge and ideas, and to discuss a variety of real-world contexts. | 4.51 | 5.00 | 5 | 0.51 |
| B2_4 | Mathematics lessons should be followed by a critical discussion with students. | 4.50 | 5.00 | 5 | 0.61 |
| B2_5 | Mathematics curriculum textbooks are the best medium of instruction and source of knowledge. | 4.07 | 4.00 | 4 | 0.64 |
| B2_6 | Mathematical knowledge is retained more easily if it is acquired using multiple representations. | 4.59 | 5.00 | 5 | 0.63 |
| B2_7 | While learning mathematics, it is important to memorize rules, facts, and formulae. | 4.73 | 5.00 | 5 | 0.55 |

The descriptive results showed that apart from beliefs about signs, symbols, and notation (B2_2), and the importance of textbooks (B2_5), the rest of the mean values were 4.5 and above, which suggests teachers hold firm beliefs about mathematics teaching and learning. Teachers held robust beliefs about pedagogical practices, such as exploring problems to discover patterns and making generalizations, conveying mathematical knowledge through multiple representations, critical debate with students and sharing mathematical ideas. The study found the highest mean value (4.73) was for belief about memorizing rules, facts and formulae, indicating memorization as the most familiar pedagogical practice among teachers.

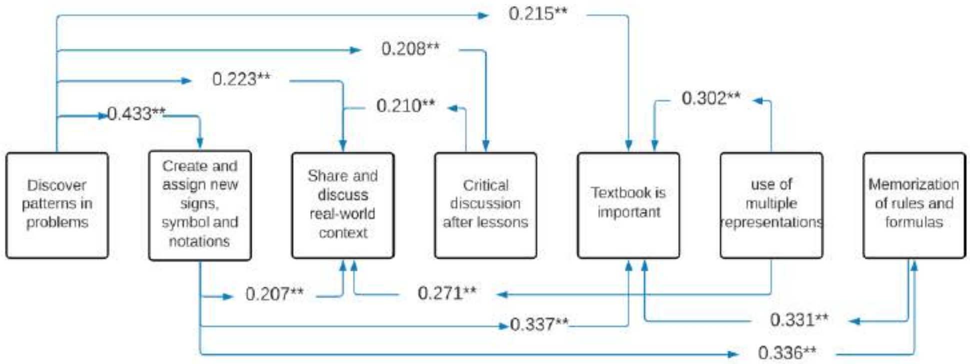



Figure 1. Correlation coefficient $\rho > 0.2$ ($p < 0.01$).

The results of the Spearman's correlation test are shown in Figure 1, showing correlation coefficient ($\rho$) marked with ($**$) representing the statistical significance (2-tailed) p-values at 0.01 for seven beliefs. To establish the importance of beliefs, the study only considered correlations coefficients with values greater than or equal to 0.2 ($\rho > 0.2, p < 0.01$).

The study found significant p-values for the Spearman Rho test. These values provide strong evidence that most beliefs are monotonically correlated. For example, the highest correlation ($\rho = 0.433, p < 0.01$) was found between learning mathematics through exploring patterns in mathematical problems and to generate new signs, symbols, and notations. Notably, the two beliefs 'learning mathematics means exploring problems to discover patterns and make generalizations' and 'how to create and assign meanings to new signs, symbols, and notations' reported significant correlations with all other beliefs. In particular, the correlation was relatively strong with teachers' beliefs about textbooks and memorization. Mathematics textbooks also revealed significant correlations with other beliefs about teaching and learning mathematics.

The findings showed that the teachers' beliefs about 'signs, symbols and notations', 'memorizing rules and formulae', 'textbooks' and 'exploring problems to discover patterns and make generalizations' revealed significant correlations with other beliefs about teaching and learning mathematics. The use of a textbook correlated positively with 'memorizing', 'multiple representations', 'critical discussion with students' and 'teaching real-world contexts', 'signs, symbols and notations', and 'exploring problems to discover patterns.' The correlations were relatively high with the 'memorization of rules and formulae; 'the use of multiple representations'; and 'signs, symbols, and notations.' The study also found that the belief about sharing 'knowledge and teaching mathematics using real-world contexts' was relatively weakly correlated with other beliefs about mathematics teaching and learning.

## Discussion

The study explored the central beliefs of mathematics teachers and how they influence other beliefs in the teacher's belief system. Due to constraints of space, we here primarily discuss the main findings of the study. The findings about memorizing are consistent with the body of literature on the beliefs of mathematics teachers. For example, the belief about memorizing facts and formulae received the highest mean score of 4.73. It is typical for teachers to believe that students should memorize, repeat, and imitate exact sequences of calculations and operations (Säljö, 2010). This notion is generally based on the idea that mathematics is a set of fixed rules, with little room for creativity or interpretation. With such an approach, teachers tend to focus on the correctness of students' answers, rather than the process of how they arrived at those answers (Schoenfeld, 2011). This suggests that beliefs about memorization may have more connections with other beliefs.

The findings suggest memorizing rules and formulae was relatively high in correlation with students' ability to create and assign meaning to new signs, symbols, and notations ($\rho = .336, p < 0.01$) and the teacher's use of mathematics textbooks ($\rho = 0.331, p < 0.01$). Considering that high school mathematics textbooks involve a wide variety of signs, symbols, and notations along with rules and formulae, their relationship with memorization may be significant. It suggests that the more teachers consider textbooks as an important source of information in the classroom, the more rules and formulae students are required to memorize. This is an intriguing finding in the context of problem-solving, implying that teachers believed students who memorize rules and formulas could create their own signs and symbols to assist them in solving mathematics problems. In education psychology, creative or innovative act of producing or interpreting signs or symbols is called an "inventive-semiotic act" (Goldin, 1998), an important teaching belief, which helps students understand mathematics better. Goldin (1998) argues that once students create and assign meaning to problems, they then move on to explore the logico-mathematical consequences of their inventions-in effect, learning to build their own (external and internal) mathematical representations. A teacher with such beliefs can help students learn the signs and symbols used in mathematics. Whereas Memorization is the process of committing information to memory through repetition or other techniques (Säljö, 2010). The only possible relationship between the two concepts is, both involve cognitive processes related to language and meaning. Memorization may be necessary to engage in inventive-semiotic acts, as the ability to recall and manipulate language and symbols is essential for creative expression. However, memorization alone does not guarantee the ability to engage in inventive-semiotic acts, as creativity involves the generation of new ideas and connections that go beyond rote memorization. We recommend further research into the two beliefs may help establish any possible connections.

An important factor that may play a role in teachers' adherence to memorization-based teaching strategies is their socio-religious affiliation. Amirali and Halai (2021) found that socio-religious experiences shape high school mathematics teachers' beliefs about mathematics in Pakistan. Most parents prefer their children to memorize the sacred book Qur'an even before starting school. A child (called "Hafiz") is expected to memorize exact written and exact vocal reproduction of verses. Typically, memorization is perceived as a tedious and repetitive form of learning. However, in Islam, scripture memorization is seen as a benefit from a moral, spiritual, and intellectual perspective (Kabir, 2021). Such a social, religious, and emotional influence may explain the teachers' deep connection with the practice of memorization as a core method of teaching in mathematics classrooms (Chan & Wong, 2014). It could further justify the weak relationship of "critical discussion after lessons" in the survey data with other teachers' beliefs. However, in students memorizing and imitating teachers' explanations of mathematics problems, the content learnt is limited to the textbook and its particular selection of information (Säljö, 2010).

Regarding the textbooks, only 39% of survey teachers strongly agreed, whilst 40% somewhat agreed that mathematics textbooks are the best source of information. A slightly surprising result considering previous literature on the use of textbooks. Historically, mathematics teachers have always placed a high value on and are closely linked to the textbooks to which they have access (Jamieson-Proctor & Carmen, 2008). It appears to be signalling a shift away from textbook-centered teaching to other contemporary educational resources, such as the use of digital resources. However, whether teachers make less use of the textbooks, might be or might not be the case considering that the data was collected during the lockdown when teachers were mostly using digital technologies (Jafri, 2022). The COVID-19 online teaching and learning may have influenced the results. Textbooks are widely available and have always played an important role in directing teaching approaches in Pakistan (Amirali & Halai, 2021). This could be the cause for the notable correlations between beliefs about the usage of textbooks and teachers' other beliefs about mathematics. The correlations were relatively high with the memorization of rules and formulas, the use of multiple representations, and the creation of new signs, symbols, and notations in mathematics teaching and learning.

Further, the belief about sharing knowledge and teaching mathematics using "real-world contexts" correlated with other beliefs about mathematics teaching and learning. However, its correlation with memorization was weakest ($\rho = 0.158, p < 0.01$). Teachers who are more likely to use real-world contexts in their teaching also tend to believe that mathematics is a creative subject (Karakoç & Alacacı, 2015). Such teachers emphasize problem-solving over memorization, and they encourage students to explain their mathematical understanding. Notably, over 63% of teachers think post-lesson critical discussion and the use of multiple representations in classrooms are important. This shows discrepancies, i.e., teachers' beliefs about post-lesson critical discussion may have fewer connections because of inconsistent beliefs. For instance, critical discussion requires teachers and learners to articulate their ideas, whereas memorization as previously discussed involves committing information to memory. It is unclear how students can critically discuss a lesson that they have just memorized or imitated, or how memorization and multiple representations can coexist. Inconsistencies in beliefs articulate teachers' desire to create an ideal or alternative teaching and learning environment (Pajares, 1992). Teachers may have flexible beliefs, which allows them to layer existing beliefs by accepting diverse sets of information and taking influences from different resources they use to teach mathematics (Schoenfeld, 2011). We can argue that changes in context and situations, such as COVID-19, may have induced changes in their practices and that their beliefs are changing (Jafri, 2022). However, the survey data cannot determine the extent to which these beliefs influence changes in teachers' pedagogical practices.

Overall, the above discussion shows that 'exploring problems to discover patterns and make generalizations' (B2_1) and the 'use of signs, symbols and notations' (B2_3) have more connections and most likely communicate with other beliefs about mathematics teaching and learning. These beliefs are relatively high in correlation with the beliefs about textbooks, which is relatively high in correlation with memorizing rules and formulae. These beliefs may potentially serve as the central beliefs in the teachers' belief system. They instil a sense of purpose and motivation in teachers as they strive to provide and teach mathematics in real-world contexts using an object of belief (textbook). Teaching students how to create and assign meanings to new signs, symbols, and notations may involve using logic, reasoning, and problem-solving skills to solve equations, understand concepts, and explore relationships (Goldin, 1998). Though memorization revealed fewer, albeit relatively higher correlations with other beliefs, it may have strong socio-religious belonging and acceptance. Rokeach (1968) argues beliefs based on faith or religion are more central than others. This socio-religious context could bring memorization as one of the central beliefs for mathematics teachers in Pakistan. The findings discussed in this article highlight the evidence-based relationship of teachers' central beliefs and their connections with other beliefs. As the field of mathematics education continues to evolve, it will be important to continue exploring and understanding the beliefs and how they influence practices of mathematics teachers.

## References

Amirali, M., & Halai, A. (2021). Teachers ' perceptions about mathematics in a socio-religious context : A case from Pakistan. *EURASIA Journal of Mathematics, Science and Technology Education*, *17*(12). https://doi.org/10.29333/ejmste/11434

Barton, A. H., & Parsons, R. W. (1977). Measuring belief system structure. *The Public Opinion Quarterly*, *41*(2), 159–180. https://www.jstor.org/stable/2748334

Bautista, R., Bilgen, I., & Bulgar-Medina, J. (2020). Measuring attitudes. In *SAGE research methods foundations* (p. 24). https://doi.org/10.4135/9781526421036

Beswick, K. (2007). Teachers' beliefs that matter in secondary mathematics classrooms. *Educational Studies in Mathematics*, *65*(1), 95–120. https://doi.org/10.1007/s10649-006-9035-3

Chan, Y.-C., & Wong, N.-Y. (2014). Worldviews , religions , and beliefs about teaching and learning : Perception of mathematics teachers with different religious backgrounds. *Educational Studies in Mathematics*, *87*(3), 251–277. https://doi.org/10.1007/sl0649-014-9555-l

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In *Mathematics teaching: The state of the art* (pp. 249–254). Falmer Press.

Ertmer, P. A., & Ottenbreit-Leftwich, A. T. (2010). Teacher technology change: How knowledge, confidence, beliefs, and culture intersect. *Journal of Research on Technology in Education*, *42*(3), 255–284. https://doi.org/10.1080/15391523.2010.10782551

Festinger, L. (1962). *A theory of cognitive dissonance*. Stanford University Press. https://books.google.co.nz/books?id=voeQ-8CASacC

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior*, *17*(2), 137–165. https://doi.org/10.1016/s0364-0213(99)80056-1

Jafri, M. (2022). Influence of the COVID-19 lockdown on high school mathematics teachers ' beliefs about using digital resources. *Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 314–321). Launceston: MERGA.

Jamieson-Proctor, R., & Carmen, B. (2008). Primary teachers' beliefs about the use of mathematics textbooks. *Navigating currents and charting directions. Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia* (pp. 295–302). MERGA.

Kabir, A. A. (2021). *Memorizing the sacred in the digital age: Exploring Qur'an memorization experiences using physical & digital formats*. University of Maryland.

Karakoç, G., & Alacacı, C. (2015). Real world connections in high school mathematics curriculum and teaching. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, *6*(1), 31. https://doi.org/10.16949/turcomat.76099

Lloyd, G. (2002). Mathematics teachers' beliefs and experiences with innovative curriculum materials. In G. Leder, E. Phekonen, & G. E. Torner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 149–159). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47958-3_9

Maasepp, B., & Bobis, J. (2014). Prospective primary school teachers' self-efficacy beliefs about mathematics. *Mathematics Teacher Education and Development*, *16*(2), 89–107.

Muijs, D., & Reynolds, D. (2015). Teachers' beliefs and behaviors: What really matters? *Journal of Classroom Interaction*, *50*(1), 25–40.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research*, *62*(3), 307–332. https://doi.org/10.3102/00346543062003307

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Information Age.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of research on teacher education* (2nd ed., pp. 102–119). Macmillan.

Rokeach, M. (1968). *Beliefs, attitudes and values: A theory of organization and change*. Jossey-Bass.

Säljö, R. (2010). Digital tools and challenges to institutional traditions of learning: Technologies, social memory and the performative nature of learning. *Journal of Computer Assisted Learning*, *26*(1), 53–64. https://doi.org/10.1111/j.1365-2729.2009.00341.x

Schoenfeld, A. H. (2011). Toward professional development for teachers grounded in a theory of decision making. *ZDM—International Journal on Mathematics Education*, *43*(4), 457–469. https://doi.org/10.1007/s11858-011-0307-8

Yurekli, B., Stein, M. K., Correnti, R., & Kisa, Z. (2020). Teaching mathematics for conceptual understanding: Teachers' beliefs and practices and the role of constraints. *Journal for Research in Mathematics Education*, *51*(2), 234–247. https://doi.org/10.5951/jresematheduc-2020-0021

# Making Mathematical Connections to the Order of Operations: Supportive and Problematic Conceptions

Fui Fong Jiew
*Queensland University of Technology*
fuifong.jiew@hdr.qut.edu.au

Lyn English
*Queensland University of Technology*
l.english@qut.edu.au

Mathematical connections play a critical role in developing a deeper understanding about mathematics, but misinterpretations may arise when connections are based on problematic conceptions. This paper reports on findings from a questionnaire and follow-up interviews with pre-service secondary mathematics teachers. Chin's (2013) Supportive and Problematic Conceptions framework is used to analyse the basis of pre-service teachers' conceptions when they establish mathematical connections pertaining to the order of operations. Findings reveal that the pre-service teachers connected the order conventions to the properties of operations and to the set concept. Findings also show that the conception of a quadratic equation as having two roots was problematic in that it impeded sense making of the order of operations.

The order of operations is generally seen as an arbitrary convention and taught by rote using mnemonics such as BODMAS or its variants (Zazkis & Rouleau, 2018). As a consequence, it causes misinterpretations such as division takes precedence over multiplication and raises concerns about the conceptual difficulties of the order of operations (Chin et al., 2022). To avoid these misinterpretations, there are suggestions that learning of the order of operations would benefit from a strong grasp of connections between the topic and other mathematical ideas (e.g., Coles & Sinclair, 2019; Toh & Choy, 2021). Viewing the order of operations as being connected with other mathematical ideas, such as the properties of operations could potentially develop "the total cognitive structure" (Tall & Vinner, 1981, p. 152) that is associated with the order precedence.

In a general sense, a mathematical connection is a relationship between mathematical ideas. In essence, mathematical connections or specifically known as *intra-mathematical connections* concern relationships between "representations, definitions, concepts, procedures and propositions within the context of mathematics" (Gamboa et al., 2021, p. 4). Some researchers propose *extra-mathematical connections* as referring to relationships between mathematics and contexts outside mathematics (e.g., Gamboa et al., 2021; NCTM, 2000). In this paper, the exploration of mathematical connections is in relation to intra-mathematical connections, that is, the mathematical ideas pre-service teachers connect when working with order of operations tasks.

Making mathematical connections is imperative because it plays a critical role in developing theoretical thinking and a deeper understanding about mathematics (Cai et al., 2014; Dorier & Sierpinska, 2001). The importance of making connections is also highlighted in some eminent teacher frameworks such as the Knowledge Quartet (KQ) and the Teaching for Robust Understanding (TRU) (Rowland et al., 2005; Schoenfeld, 2013). The TRU framework, for example, demands a classroom discussion to provide opportunities for making coherent connections between mathematical ideas (Schoenfeld, 2013). However, it is questionable what connections pre-service teachers make to the order of operations since it is commonly perceived as arbitrary.

Previous studies in relation to making connections in mathematics have focussed on conceptualisation of mathematical connections and the potential of using connections in support of student learning (e.g., Eli et al., 2011; García-García & Dolores-Flores, 2021; Gamboa et al., 2020, Rodríguez-Nieto et al., 2022). However, not all connections made are useful for learning. Although some mathematical connections are useful in helping students to better understand a mathematical concept, some might be problematic as the connections hinder students from making sense of the concept correctly. The current study differs from existing research in which this study analyses, not

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 291–297). Newcastle: MERGA.

only the connections that are supportive, but also the connections that are problematic and impede learning.

As part of a larger research project, this paper aims to understand the connections pre-service teachers make to the order of operations. The research questions addressed in this paper are:

- What mathematical connections do pre-service teachers make to the order of operations?
- How do the mathematical connections affect the pre-service teachers' evaluations of mathematical expressions?

This study is significant for several reasons. Other than extending the literature about mathematical connections by analysing those that can support or impede learning, this study also documents intra-mathematical connections that can be made to the order of operations.

## Theoretical Perspectives

Sfard (1991) described a *conception* as "the whole cluster of internal representations and associations evoked by the concept—the concept's counterpart in the internal, subjective "universe of human knowing" (p. 3). This implies that a conception is one of the building blocks that contributes to the divergence of performance in doing mathematics (Chin, 2013). In considering the conceptions in mathematics learning, Chin (2013) proposes supportive and problematic conceptions to describe how the personal conceptions affect the sense making of mathematical concepts. A supportive conception is a conception that supports generalisation whereas a problematic conception is a conception that impedes sense making.

Research examining conceptions of learning has dominated with studies about how previous knowledge impacts new learning (e.g., Bransford & Schwartz, 1999; Jiew & Chin, 2020; Wagner, 2010). Although it is agreed that much can be gained from analysing the influences of prior experience to what is learned later, such work should be complementary to research understanding how newly developed conceptions affects prior learning. In this field of research, Lima and Tall (2008) uses *met-after* to describe an experience encountered at a later learning stage that impacts the memories of old learning. When solving linear equations, some student participants in Lima and Tall's (2008) study misapplied the quadratic formula to solve the equations, and some made sense of the equals sign as multiplication. These were instances of *met-after*. As students were introduced to linear equations before quadratic equations, their experience working with the quadratic formula influenced what they remembered about linear equations. Lima and Tall (2008), however, refers to experience in general but not the conceptions developed within linear or quadratic equations.

Another construct proposed by Hohensee (2014) that aligns with *met-after* is *backward transfer*. It explains "the influence that constructing and subsequently generalizing new knowledge has on one's ways of reasoning about related mathematical concepts that one has encountered previously" (p. 136). Hohensee's (2014) construct, however, refers to ways of reasoning that students engage in doing mathematics. I suggest that the effects of new learning on old learning may also include the underlying mental structures such as conceptions of the mathematical concept itself.

In this paper, we use the notions of supportive and problematic conceptions to look closely at the basis of pre-service teachers' conceptions when they established mathematical connections to the order of operations. The framework of supportive and problematic conceptions may also provide insights into how the mathematical connections affect the evaluations of mathematical expressions. Knowing what mathematical connections are supportive is significant as it leads to meaningful learning. Knowing what conceptions are problematic is also essential as it avoids misinterpretations of a mathematical concept.

## Methodology

### *Setting and Participants*

This study is conducted as the first author's PhD research. There were 11 pre-service secondary mathematic teachers who were towards the end of their 4-year pre-service teacher education. All participants gave informed consent. Data collection involved a questionnaire and follow-up interviews. Some findings from the questionnaire and interview data are reported in this paper. Particularly, this paper discusses the participants' responses about two mathematical expressions involving multiplication and division as follows:

$$10 \div 5 \times 2$$

$$4 \times 6 \div 3$$

### *Data Collection*

A questionnaire about order of operations tasks was administered and follow-up interviews were conducted via the Zoom platform. The online platform was used because of prevailing COVID-19 restrictions at that time, which prevented in-person meetings from taking place. Follow-up questions were asked in the interviews to gain further insights into the participants' explanation about their responses provided in the questionnaire. The research session lasted between 50 and 60 minutes. Interviews were recorded and subsequently transcribed for analysis.

### *Data Analysis*

Data were analysed thematically following the steps suggested by Ary et al. (2013) and using the approaches proposed by Braun and Clarke (2021). The data were organised and annotated to identify responses that provided information about ways of evaluation of mathematical expressions and mathematical connections the participants made. Codes were generated to represent the most relevant data for the research questions. We used both "data-driven" and "theory-driven" approaches to avoid missing important data. After repeated iterations of coding, codes were then used to interpret themes. The generated themes were used to inform an overall understanding of the mathematical connections made by the participants.

## Results

The correct order to evaluate the expressions containing multiplication and division is from left to right. Of the 11 participants, six used the correct order, three performed division before multiplication, and two evaluated the expressions without a specific order. In this respect, evaluating an expression using no specific order means providing two different solutions to the expression. In other words, the participants evaluated the expressions without giving priority to any operation. The different orders of evaluation are discussed in turn.

### *From Left to Right*

Brendan's responses are reproduced in Figure 1 to illustrate the way that the six participants used the left-to-right order.

$$10 \div 5 \times 2$$
$$= 2 \times 2$$
$$= 4$$

$$= 4 \times 6 \times \frac{1}{3}$$
$$4 \times 6 \div 3 = 24 \div 3$$
$$= 8$$

Figure 1. Evaluate the expressions from left to right—Brendan's responses.

Brendan's explained that, "We can write division as multiplication. When they are all in multiplication, we can change the place, so any order doesn't matter. But when we maintain the question in multiplication and division, we have to follow the left-to-right order." Brendan connected the order of operations to properties of operations, particularly the multiplicative inverse and the associative property. Writing $\div 3 = \times \frac{1}{3}$ for the expression $4 \times 6 \div 3$ implies that he recognised division as the inverse of multiplication. Based on the associativity of multiplication, he realised that the numbers in an expression could be shifted if division was written as multiplicative. His conceptions about multiplicative inverse and associativity of multiplication support his explanation of using the left-to-right order. These conceptions are considered as supportive conceptions that allow Brendan to make generalisations.

On the other hand, Julie based her explanation on another mathematical concept, namely that of a set. She wrote $A \cup B \cap C$ and stated that, "For mixed operations, we must do from left to right. For example, A union B and intersection C. We need to do it from the left first, which is the union first." Julie linked the order of operations to set notation and set operations. Specifically, her conception about set notations and set operations is a supportive conception that allows her to successfully interpret the order of operations. This conception supports her generalisation in the context of simplifying numerical expressions.

### *Division Before Multiplication*

Audrey's responses are reproduced in Figure 2 to illustrate the way that the three participants used division before multiplication. For expression $4 \times 6 \div 3$, Audrey computed the right answer but used the wrong order.

$$10 \div 5 \times 2 = 2 \times 2$$
$$= 4$$

$$4 \times 6 \div 3 = 4 \times 2$$
$$= 8$$

Figure 2. Perform division before multiplication—Audrey's responses.

The participants who used division before multiplication explained their responses based on the acronym BODMAS. Audrey, for example, stated that, "Following BODMAS, D division has to be calculated before M multiplication. DM means division then Multiplication." Audrey connected the order of operations to another representation of the rules that is BODMAS. However, she misinterpreted that division precedes multiplication based on the order in which the letters were presented in the acronym. As the letter D comes before the letter M, Audrey perceived it as division takes precedence over multiplication. Even though the final answer is correct, she had a misinterpretation of the acronym.

*No Specific Order*

Howard's responses are reproduced in Figure 3 to illustrate the way that the two participants evaluated the expressions with no specific order.

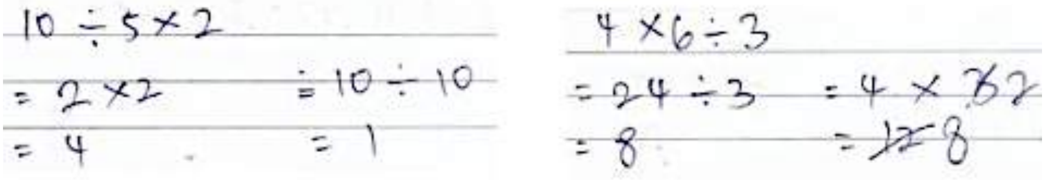


Figure 3. Evaluate the expressions with no specific order.

Howard explained that, "This kind of questions can have two answers. Just like an unknown, we can have two answers, for example $x$ equals to 1 and $-1$, sometimes we can have 4 answers." He explained his responses based on his conception of algebra. Obtaining two answers from an unknown implies that Howard made sense of the order of operations based on his understanding that a quadratic equation could have two roots. His conception of quadratic equation is correct in the context of algebra, but it is problematic in the context of simplifying numerical expressions. This conception is a problematic conception that impedes his sense making of the order of operations.

## Discussion and Concluding Remarks

The findings suggest that pre-service teachers made sense of the left-to-right order based on properties of operations. Their conceptions about associative and inverse properties are supportive conceptions that help them make sense of the order of operations. Building on these connections, they were able to perform calculations using the correct order. This finding contradicts Zazkis and Rouleau's (2018) argument, namely, the knowledge about the properties of operations was not used by pre-service teachers to interpret the order of operations. Another supportive conception revealed in this study is when Julie connected the order of operations to set notation and set operations. These connections work within the set context and continue to support the sense making of the order of operations.

The conception that a quadratic equation can have two roots is a problematic conception in the context of simplifying numerical expression. Although this conception is true and workable in the context of algebra, in particular with quadratic equation, this conception does not work when completing order of operations tasks in linear equations. It impedes the sense making of the order of operations.

The findings that making supportive connections to set and algebra align with Hohensee's (2014) notion of *backward transfer* and Lima and Tall's (2013) *met-after*. This shows that learning is not necessarily from simple to complex, it may be that new learning impacts on prior knowledge. Both the studies found that new learning of quadratic functions affected students' understanding of linear functions. The present study extended these prior studies in another mathematical area (i.e., the order of operations) and included conceptions rather than experience or ways of reasoning.

There was evidence that the pre-service teachers evaluated expressions based on the acronym BODMAS, which led to a misinterpretation. Although the use of an acronym in teaching the order conventions may lead to proficiency in recalling and performing the necessary calculations or procedures, the findings reveal the danger of using acronyms in the order of operations as reported in existing research (e.g., Dupree, 2016; Glidden, 2008; Zazkis & Rouleau, 2018).

Making mathematical connections is a key goal in the learning of mathematics (NCTM, 2009) and researchers encourage connection-making in learning mathematics with understanding (Bossé, 2003; Cai & Ding, 2017). In this study, further insights were gained from knowing the basis of conceptions when mathematical ideas were connected. Supportive conceptions are necessary to bring meaningful learning whereas problematic conceptions are essential to prevent misinterpretations of a mathematical concept or idea.

## Acknowledgments

The first author acknowledges the insights and support provided by Professor Lyn English and Dr Timothy Lehmann in their roles as the supervisory team during her PhD study. In addition, Dr Kin Eng Chin's insights into the conceptions used in making sense of the order of operations enabled this study to contribute to an underrepresented field. This study would not have been possible without financial support from QUT's Postgraduate Tuition Fee Sponsorship Waiver and QUT Postgraduate Research Award (QUTPRA)(International).

## References

Ary, D., Jacobs, L., Sorensen, C., & Walker, D. (2013). *Introduction to research in education* (9th ed.). Cengage Learning.

Bossé, M. J. (2003). The beauty of "and" and "or": Connections within mathematics for students with learning differences. *Mathematics Computer Education*, *37*, 105–114.

Bransford, J. D., & Schwartz, D. L. (1999). Rethinking transfer: A simple proposal with multiple implications. *Review of Research in Education*, *24*, 61–100. https://doi.org/10.2307/1167267

Braun, V., & Clarke, V. (2021). *Thematic analysis: A practical guide to understanding and coding*. SAGE.

Cai, J., & Ding, M. (2017). On mathematical understanding: Perspectives of experienced Chinese mathematics teachers. *Journal of Mathematics Teacher Education*, *20*, 5–29. https://doi.org/10.1007/s10857-015-9325-8

Cai, J., Ding, M., & Wang, T. (2014). How do exemplary Chinese and U.S. mathematics teachers view instructional coherence? *Educational Studies in Mathematics*, *85*, 265–280. https://doi.org/10.1007/s10649-013-9513-3

Chin, K. E. (2013). *Making sense of mathematics: Supportive and problematic conceptions with special reference to trigonometry* [Doctoral thesis, University of Warwick].

Chin, K. E., Jiew, F. F., & Tall, D. (2022). The Articulation Principle for making long-term sense of mathematical expressions by how they are spoken and heard: Two case studies. *The Mathematics Enthusiast*, *19*(2), 14. https://doi.org/10.54870/1551-3440.1567

Coles, A., & Sinclair, N. (2019). Re-thinking 'Concrete to Abstract' in mathematics education: Towards the use of symbolically structured environments. *Canadian Journal Science Mathematics and Technology Education*, *19*, 465–480. https://doi.org/10.1007/s42330-019-00068-4

Dorier, J.-L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 255–273). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47231-7_24

Dupree, K. M. (2016). Questioning the order of operations. *Mathematics Teaching in the Middle School*, *22*(3), 152–159. https://doi.org/10.5951/mathteacmiddscho.22.3.0152

Eli, J. A., Mohr-Schroeder, M. J., & Lee, C. W. (2011). Exploring mathematical connections of prospective middle-grades teachers through card-sorting tasks. *Mathematics Education Research Journal*, *23*, 297–319. https://doi.org/10.1007/s13394-011-0017-0

Gamboa, G. D., Badillo, E., Couso, D., & Márquez, C. (2021). Connecting mathematics and science in primary school STEM education: Modeling the population growth of species. *Mathematics*, *9*(19), 2496. https://doi.org/10.3390/math9192496

Gamboa, G. D., Badillo, E., Ribeiro, M., Montes, M., & Sánchez-Matamoros, G. (2020). The role of teachers' knowledge in the use of learning opportunities triggered by mathematical connections. In S. Zehetmaier, D. Potari, & M. Ribeiro (Eds.), *Professional development and knowledge of mathematics teachers* (pp. 24–43). Routledge.

García-García, J., & Dolores-Flores, C. (2021). Pre-university students' mathematical connections when sketching the graph of derivative and antiderivative functions. *Mathematics Education Research Journal*, *33*, 1–22. https://doi.org/10.1007/s13394-019-00286-x

Glidden, P. (2008). Prospective elementary teachers' understanding of order of operations. *School Science and Mathematics*, *108*(4), 130–136. https://doi.org/10.1111/j.1949-8594.2008.tb17819.x

Hohensee, C. (2014). Backward transfer: An investigation of the influence of quadratic functions instruction on students' prior ways of reasoning about linear functions. *Mathematical Thinking and Learning*, *16*(2), 135–174. https://doi.org/10.1080/10986065.2014.889503

Jiew, F. F., & Chin, K. E. (2020). Supportive and problematic conceptions in making sense of multiplication: A case study. *The Mathematics Enthusiast*, *17*(1). https://doi.org/10.54870/1551-3440.1483

Lima, R. N., & Tall, D. O. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics*, *67*(1), 3–18. https://doi.org/10.1007/s10649-007-9086-0

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. NCTM.

National Council of Teachers of Mathematics. (2009). *Focus in high school mathematics: Reasoning and sense making*. NCTM.

Rodríguez-Nieto, C. A., Rodríguez-Vásquez, F. M., & Moll, V. F. (2022). A new view about connections: The mathematical connections established by a teacher when teaching the derivative. *International Journal of Mathematical Education in Science and Technology*, *53*, 1231–1256. https://doi.org/10.1080/0020739X.2020.1799254

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*, 255–281. https://doi.org/10.1007/s10857-005-0853-5

Schoenfeld, A. H. (2013). Classroom observations in theory and practice. *ZDM—The International Journal on Mathematics Education*, *45*, 607–621. https://doi.org/10.1007/s11858-012-0483-1

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1–36. https://doi.org/10.1007/BF00302715

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169. https://doi.org/10.1007/BF00305619

Toh, T. L., & Choy, B. H. (2021). Mathematics-connection and beyond. *Yearbook 2020 Association of Mathematics Educators*. World Scientific. https://doi.org/10.1142/12279

Wagner, J. F. (2010). A transfer-in-pieces consideration of the perception of structure in the transfer of learning. *The Journal of the Learning Sciences*, *19*(4), 443–479. https://doi.org/10.108 0/10508406.2010.505138.

Zazkis, R., & Rouleau, A. (2018). Order of operations: On convention and met-before acronyms. *Educational Studies in Mathematics*, *97*(2), 143–162. http://dx.doi.org/10.1007/s10649-017-9789-9

# Remote Australian Primary School Parents' Attitudes Towards Their Children's Learning of Mathematics and the Role of Technology

Garry Jones
*Macquarie University*
garry.jones@mq.edu.au

Dung Tran
*Macquarie University*
dung.tran@mq.edu.au

Penny Van Bergen
*University of Wollongong*
pennyvb@uow.edu.au

Matt Bower
*Macquarie University*
matt.bower@mq.edu.au

This qualitative study investigates remote Australian primary school parents' attitudes towards their children's learning of mathematics and the role of technology. Using semi-structured interviews and thematic analysis, results revealed that parents aimed to provide the best mathematics learning opportunities for their children, yet they felt obsolete and isolated from their children's learning. Additionally, they valued the role of technology, but did not think school was providing adequate digital literacy skills. Discussion about what changes to the educational community are suggested to support parents in their children's learning.

Research has highlighted that in Australia, three groups of students regularly perform at a lower level than their peers—Indigenous students, students from lower socioeconomic backgrounds and those living in remote or rural communities (Sullivan et al., 2013). Understanding these disparities requires consideration of numerous variables. Some previous studies have focused on students' attitudes towards mathematics (Bragg et al., 2020; Perry, 2011) and attitudes towards technology in education (Hughes & Read, 2018). Given that parents are involved at the personal, behavioural and cognitive-intellectual level of their children (Hill & Tyson, 2009), parents' roles in the education of their child cannot be understated. However, very few studies have focused on parent's attitudes towards mathematics and technology, and a scarcity of research has focused on remote parents' attitudes. This study addresses this gap and answers the research question:

- What are remote Australian primary school parents' attitudes towards their children's learning of mathematics and the role of technology?

## Literature Review

### *Attitudes Towards Mathematics*

Attitudes are an integral part of student's learning of mathematics as there are connections between attitudes and learning outcomes (Grootenboer & Marshman, 2016). Attitudes can be conceptualised as "a predisposition with an emotional charge that directs and/or influences behaviours" (Palacios et al., 2013, p. 68). Previous studies have focused on students' attitudes (Perry, 2011) and how to make sustainable changes in students' attitudes (Long, 2016). Others have investigated those of preservice teachers (e.g., Tran & Javed, 2017), but rarely have any studies focused on this aspect with in-service teachers (see Ingram et al., 2020). Diverse methods have been used to study attitudes including questionnaires (e.g., Tran & Javed, 2017), video-diaries (e.g., Larkin & Jorgensen, 2016). Di Martino and Zan (2015) argue that there has been a gradual movement towards the interpretive paradigm in mathematics education to understand phenomena, instead of the attempt to explain behaviour through measurements, which is applicable in studying attitudes. Previous studies have underscored that attitudes are important, but no reviewed studies have focused on parents, who also play crucial roles in student learning.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 299–306). Newcastle: MERGA.

### *Parents' Attitudes Towards Technology and Mathematics Learning*

Parents, whether part of nuclear, blended, single, extended, step or grandparent families, are embedded in and influence communities and larger social systems (Hoff & Laursen, 2019). Parents and caregivers provide the bedrock for their child's academic, emotional and cognitive life (Hannah, 2020). Hoff and Laursen (2019) report that a parent's socioeconomic status (SES) influences their estimation of the development trajectory and milestones their child will attain at a given time, with higher SES parents estimating earlier attainment than lower SES parents' estimations. Additionally, Blevins-Knabe (2016) highlighted differences between high and low SES households. High SES parents were more engaged and interested in mathematics, with lower SES parents providing fewer mathematics opportunities. Some parents may experience higher confidence engaging with informal mathematics activities such as card and board games (Ramani & Siegler, 2021), in contrast to feeling disheartened and isolated trying to navigate an unfamiliar mathematics teaching and learning terrain. For some parents, the potential to integrate technology into mathematics and education may create opportunities for a shared dialogue. However, as Hughes and Read (2018) reported this integration may prove challenging, as more students in their research reported mathematics as the subject where they never used technology for learning.

Government, the wider community and schools may operate on the assumption that parents will be proficient in the domains of "twenty-first century literacies" (Downes et al., 2020), while concurrently being sufficiently digitally literate (Jin et al., 2020). However, this expectation may not always be achievable. For example, Downes et al. (2020) reported that parent beliefs about access to technology devices and usage was influenced by factors such as SES, culture and employment. Furthermore, Davies (2011) suggests that low SES parents experience a level of anxiety when their children are using technology.

Consequently, if parents are expected to possess the necessary capabilities and competencies to integrate mathematics, technology and education, there is a clear absence of research, particularly in remote Australia, which is such a disadvantaged location. Therefore, undertaking research which investigates and reports on the complexities involved, particularly in a remote location in Australia would be highly relevant area for educational research.

## Methodology

We adopted the semi-structured interview approach as it's flexible, versatile, able to adapt to the purpose of the study. Additionally, through the development of a structured interview guide, informative insights would be captured, as participants respond to, and elaborate on a series of open-ended, yet thoughtfully ordered and predetermined questions. The study utilised a culturally appropriate (Kariippanon & Senior, 2017) and socio-culturally sensitive design to ensure all participants experienced a safe and supportive interview environment. The interviews were conducted by a member of the research team who had been part of the remote community for a number of years. Therefore, the researcher was a sufficiently trusted member of the community, yet not so close as to potentially distort or introduce bias into the analysis. This placed the researcher as neither an insider nor an outside, but able to operate in the "space between" (Kerstetter, 2012).

### *Participants*

Participants for the study were drawn from an area in remote Australia, which is categorised as a *most disadvantaged* area of the nation (Australian Bureau of Statistics, 2016). Potential participants were contacted through Parents and Citizens' Associations and the local Aboriginal Education Consultative Group. Four participants agreed to participate in this project. Susan, is a female. She is married with two children, and one is attending Year 6. Bernadette was also female. She is currently living with her parents, and four of her children; two currently attend primary school. Tori, was female. She is currently married with four children. Two of the children are of pre-school age. The other two children currently attend primary school. Finally, Claire is also female. She is married with four children with one in Year 6.

### *Interview Protocol*

A comprehensive, questioning protocol was developed by the research team. The interview comprised of three interrelated and overlapping sections. The first section focused on establishing each parents' access to, and attitudes towards technology within the home, for themselves and their child. Parents were encouraged to discuss technology use, perceived level of competency and confidence, plus perceived capacity to support their eldest primary child in their learning and technology use. Next, insights were sought on their attitudes towards mathematics and the current use of technology in schools. Parents were encouraged to elaborate on their personal mathematics experiences, and whether these accord or not with their child's experience. Finally, attitudes towards informal and formal mathematics in the home was investigated, together with any recommendations about mathematics, technology or education. All participants selected an area open to the public, such as a café for the interview. To ensure appropriate cultural sensitivities, participants were invited to ask a trusted friend to join them for the interview. No participants brought a friend to the interview. The audio from the interviews were recorded by the researcher on two digital devices. Interviews lasted between 50 and 60 minutes.

### *Analysis*

Participant interviews were transcribed verbatim. These transcripts, with associated field notes were imported into NVivo for analysis. The researcher used the constant comparative approach (Glaser & Strauss, 1977), forensically analysing each interview transcript to establish the possibility of themes present in more than one transcript (Goulding & Lee, 2005). This approach enabled the codes to emerge from the data, with constant reference to all transcripts. Through an iterative approach, codes were renamed and revised taking into account the emerging understanding of participant responses. Following the emergent phase, axial coding aimed to identify and create "connections between a category and its subcategories" (Charmaz, 2000, p. 706). Through this process, preliminary connections were identified, with the data coalescing around four broad themes. These themes were evident by the breath of the data as they showed the highest frequency in NVivo.

## Results

### *Feelings of Being Obsolete and Isolated from Their Child's Mathematics Learning*

Parents generally felt obsolete and isolated from their child's mathematics learning expressing a sense of being undervalued and overlooked. This feeling was aligned with recognising an absence of a shared mathematics language or learning experience. For example, Bernadette, said,

> I find that in maths when I am trying to help the kids, and they will say to me, 'how do you do this?' And I do it one way, even like multiplication, and they go 'ooorrrr, we don't know how to do it that way!' Like I have learnt it one way and they are learning it another and I don't know how to meet in the middle and help them.

This feeling of being unable to "meet in the middle and help them" indicated a feeling of disconnection, resulting in them feeling old-school and unhelpful. This feeling was accompanied by an acute awareness that a boundary existed between the teaching approaches experienced by the parent and the child. As Bernadette further expressed, "You have to sort of stand back and trust that the way that Maria (teacher) taught him is the right way, and that he understands it, because I can't help him." Moreover, as Claire elaborated,

> I remember saying to Doris, 'you gotta carry the one'. And she goes, what do you mean carry the one? Do you mean lend or borrow? And I'm like, 'what are you talking?' She is like, 'you put the one over here, you put the one over here!'And I am like, 'umm I am so confused now.

Responses from parents expressed a practical inability to support their child with their mathematics, coupled with feelings of isolation, obsolescence and confusion around how to best support them. To counter these feelings, parents responded by questioning the current approach to teaching mathematics. For example, Claire, said,

> When one of her teachers actually said to me…'but we don't care if she gets the wrong answer or the right answer it's the strategy that she uses to get there'. And I thought, but she still needs to know how to get the right answer.

Claire was passionate at this point, as she was concerned about procedural aspects of mathematics, the importance of solutions and getting the "right answer", in contrast to her daughter Doris who was demonstrating strategic, conceptual mathematical understanding. This may indicate the presence of both a knowledge and understanding gap of the current mathematics pedagogy and may highlight that Claire has not been afforded with an opportunity to develop a deeper understanding of the current mathematics pedagogy.

### *Aiming to Provide the Best Mathematics Learning Environment*

Although aware of their knowledge and capacity gaps, parents expressed a determination to do their best to support their child's mathematics learning. They were actively creating and seeking opportunities to establish the best foundation, particularly around informal learning opportunities. For example, as Tori said, "from day one with the kids, when we have been using the microwave, we count down from ten or count up to ten. Yep, there are plenty of opportunities for counting". Claire reinforced this sentiment, saying,

> At the moment, Doris is counting down the days to Christmas. So, she needs to know that. She needs to know that today is the 5th and Christmas Day is the 25th. How many days is it to Christmas? She needs to know how to do that in her head.

Bernadette further added, "the children are heavily into UNO and monopoly and card games and even their grandfather teaches them to play crib. Well, that is counting." By consciously integrating informal mathematics into the everyday experiences of their children, the parents are striving to enrich their child's mathematical experience, and therefore provide the best environment possible.

This attitude was further reinforced when considering the role of technology and support networks for teaching and learning. For Claire, when it came to schoolwork, only the best would do, "My laptop is better than hers, so she uses my laptop for any school stuff." Attitudes expressing the desire to provide the best access to technology reached into the extended family, with Tori saying, "if he needs them (better technology) he has got things that he can access, and if he needs something that has a little bit more on it, we can always go to nan and pops, nan's got her computer." Parents are therefore, striving to source and secure the best resources possible. For Susan, in contrast to reaching out a family member, she sought to tackle the task herself, saying,

> if he was unsure of something I would probably have a go at doing it (solving the mathematics problem) without him seeing it. And see if I get it right, to then go, OK well this is, now I can now explain to you what you need to do.

Parents are clearly expressing, that to support their child, they will seek out whatever resources available. For Tori, a community network was instrumental, as she said,

> We had a thing called, and it was actually for the Aboriginal kids…called the Homework Centre. It wasn't necessarily for homework, but it was there to help and provide for these kids that don't necessarily have the help at home.

As highlighted by Tori, this merging of community, parent and student in a supportive learning environment, non-judgmentally accepted that, at times, parents may be willing, but due to circumstance, unable.

### *Technology Affords Individualised and Accelerated Mathematics Learning Opportunities*

Parents largely felt that technology was a powerful educational companion supporting learning in general and mathematics in particular. As explained by Susan,

> I think it's beneficial. I think like Prodigy can be used where the student just goes off and just uses it, or the teacher can set tasks in it and have like, so may set a particular period of time where kids are only exposed to a certain subject areas and can kind of guide it.

Here, Susan is highlighting the independence and autonomy that her child experiences when using technology for mathematics. The ability to "just go off and just uses it" captures a sense of trust and confidence that she feels when her son is using technology for mathematics. Importantly for parents, they expressed an opinion that the various technological tools available, could meet the child's needs depending on the specific learning activity. As Bernadette explained "I don't think he loves the laptop. He just doesn't seem to use it as much as the iPad. He'll jump on the iPad, he'll google things straight up, it's, he just takes it wherever he goes with him." Bernadette perceived that her son Mark valued both the portability and ease of accessibility of the iPad in comparison with a laptop. The capacity of technology to meet children's learning needs was also recognised by Susan. However, she expressed a salient educational caveat, saying, "I think it (technology) is good for revision, but, again I d on't think that it replaces the teaching that they get in the first place". Susan here expresses an attitude towards the timing, suitability and utility of technology use, which is conditional on its purpose. This point was further reiterated by Claire who said,

> If Doris were more, I suppose, more confident with maths I think that she would enjoy using some of those, the maths apps. But because she does not feel confident (with maths), she tends to stay away. And the more I try to encourage, the angrier she gets.

Consequently, parents expressly recognised technology's potential to support individualised and accelerated learning opportunities, but recognised the limitations.

The potential for technology to support accelerated learning was also highlighted in discussions about games and learning through playing. Tori was clear about her thoughts, saying, "My kids have always had them (maths apps) on their (iPad). Because if they are going to be playing a game, they may as well be playing something a little bit educational at least." Recognising the complementary opportunities for learning and enjoyment that co-exist in learning by playing was also important for Susan, who said, "I guess that Mathletics may be more game-based, but if the game-based app gets kids using it, they think that they are playing a game and the by-product is they are learning maths".

Rounding off the parent feelings that technology is ubiquitous, and that engaging their child in learning is paramount, Bernadette neatly captured the importance of using technology to unite teaching and learning, by saying,

> We are in a technological world, it's everywhere, and I think that drawing the kids in to having something whether it be an app, a computer programme, or something that ties in with their schoolwork, but it's a game. You've gotta catch them, you've gotta get their interests.

### *Current Technological Skills Taught at School are not Preparing Their Children for the Future*

Responses from parents broadly recognise technology's important role in the lives of their children, plus its value as an educative tool, particularly with its potential to support and accelerate learning. However, dissatisfaction was expressed regarding the level of preparation being provided in schools. For example, Susan expressed feelings of dissatisfaction, saying,

> I just think sometimes the focus becomes on doing elaborate things with technology like coding or using a variety of programs. But sometimes the basic skills are missed, and it's those basic skills that kids actually need to use to go in their day-to-day life using technology—like a kid might be used to texting on a phone, that is not going to help them on a keyboard.

Susan is expressing a level of frustration that basic competency skills are being overlooked, in favour of more "elaborate" skills, skills which she considers should be taught once the basics have been taught. Susan adds further,

> Like lots of kids do not know how to use a mouse and to be able to use a mouse to click on things is quite a different skill in terms of their development instead of just touching a screen.

When discussing this absence of foundational learning in technology, Susan became quite passionate and animated, clearly feeling that advanced technology, such as robotics or coding would not help her child succeed if they were deficient in basic computer skills.

> I think there is bit of an assumption that these kids are from a technological age and that they are all over that, but I think using a keyboard to type even a physical keyboard versus an on-screen keyboard and using one finger typing versus typing are quite different things.

Susan's level of disappointment was echoed by Tori. Tori felt that access to technology and the teaching of it needed to be more targeted, saying,

> You need to be able to research the stuff that you need off the internet, and you need to be able to type it up in Word, and you need to be able to use Excel, and you need to be able to use PowerPoint. I mean they probably need more (technology teaching) rather than less.

Parents clearly feel that schools are not preparing their children for a future they envisage awaits them, neither in basic operational skills nor software. Consequently, feelings of their children being left behind or abandoned by the system become relevant. For Bernadette, feelings of differential and unfair treatment are salient, as she said, "I think kids in the city, kids in more populated areas, they do have more access to things to learn in different ways and to learn more. Our kids are sort of shoved out here and forgotten about".

## Discussion

This qualitative study reports on the preliminary findings investigating remote Australian primary school parents' attitudes towards their children's learning of mathematics and the role of technology. Parents reported feeling isolated and disconnected from their child's mathematics learning journey, highlighting the absence of shared mathematical language, knowledge and understanding. This created a learning chasm between the parent and child. One suggestion to account for this gap could be the increased focus on the conceptual understanding of mathematics which followed the shift away from the *traditional curriculum* (Klein, 2007) of which many parents were unfamiliar. Additionally, as reported by McFeetors et al. (2020), parents were not seriously considered when reforms and changes to the mathematics curriculum were instituted. Consequently, barriers may have been created which prevent parents from engaging in their child's conceptual mathematics education. In disadvantaged, low SES areas, these barriers may be more pronounced. Therefore, as was identified in these preliminary results, parents tend to immerse themselves in informal mathematics activities. To close this gap, Biag and Castrechini (2016) reported on the contributions afforded by comprehensive community-based learning partnerships and environments. They reported that an extended learning program provided by the school bridged learning gaps. Additionally, Maier et al. (2017) reports on the benefits of community-orientated parent education programs and the improvement in student mathematics achievement. These community-based frameworks could be considered within the Australian remote locations to assist and upskill parents with their conceptual mathematical understanding, thereby providing additional support to their children.

Additionally, the potential role of technology in the both the mathematics education and teaching and learning was well understood by parents. Parents strived to provide their children with the best devices possible for learning, with the best chance to be successful. Parents too recognised that their children needed access to digital skills. As noted by Burns and Gottschalk (2019), being technologically aware and competent is required for full participation in the 21st century. However, while recognising the potential of robotics and coding, parents want their children have access to the basic skills first. This preliminary finding may support research by Harris et al. (2017) who reported a 'digital divide' between high and low SES areas, such that, low SES use home devices more for chat rooms and multimedia, whereas high SES use home devices to extend learning and promote academic skills. Consequently, children in high SES areas may be learning the basic computer skills in the home environment, therefore, making robotics and coding a natural extension. Whereas low SES students may not be developing the basic technology skills at home, so need these included in the curriculum. Therefore, schools may need to recognise that technology usage in the home of low SES families may not be providing students with the basic skills. Consequently, the teaching curriculum may require adjustments.

## References


Australian Bureau of Statistics. (2016). 2033.0.55.001—*Census of population and housing: Socio-Economic Indexes for Areas* (SEIFA), Australia, 2016 Australian Bureau of Statistics.

Biag, M., & Castrechini, S. (2016). Coordinated strategies to help the whole child: Examining the contributions of full-service community schools. *Journal of Education for Students Placed at Risk*, *21*(3), 157–173.

Blevins-Knabe, B. (2016). Early mathematical development: How the home environment matters. In B. Blevins-Knabe, & A. M. Berghout Austin (Eds.), *Early childhood mathematics skill development in the home environment* (pp. 7–28). Springer. https://doi.org/10.1007/978-3-319-43974-7_2

Bragg, L. A., Herbert, S., & Brown, J. P. (2020). Successful home-school collaboration: Let's talk about maths at home. *Australian Primary Mathematics Classroom*, *25*(3), 18–22.

Burns, T., & Gottschalk, F. (2019). *Educating 21st century children: Emotional well-being in the digital age*. OECD Publishing. https://doi.org/10.1787/20769679

Charmaz, K. (2000). Grounded theory: Objectivist and constructivist methods. In N. K. Denzin, & Y. S. Lincoln (Eds.), *The handbook of qualitative research* (2nd ed., pp. 509–535). Sage Publications.

Davies, C. (2011). Digitally strategic: How young people respond to parental views about the use of technology for learning in the home. *Journal of Computer Assisted Learning*, *27*(4), 324–335.

Di Martino, P., & Zan, R. (2015). The construct of attitude in mathematics education. In B. Pepin, & B. Roesken-Winter (Eds.), *From beliefs to dynamic affect systems in mathematics education: Exploring a mosaic of relationships and interactions* (pp. 51–72). Springer. https://doi.org/10.1007/978-3-319-06808-4

Downes, T., Di Cesare, D. M., Gallagher, T. L., & Rowsell, J. (2020). Parents' beliefs about and associations to their elementary children's home technology usage. *Education and Information Technologies*, *25*(5), 4557–4574. https://doi.org/10.1007/s10639-020-10188-2

Dwyer, S. C., & Buckle, J. L. (2009). The space between: On being an insider-outsider in qualitative research. *International Journal of Qualitative Methods*, *8*(1), 54–63.

Glaser, B. G., & Strauss, A. L. (1977). *The discovery of grounded theory: Strategies for qualitative research*. Routledge.

Goulding, C., & Lee, N. (2005). Grounded theory, ethnography and phenomenology. *European Journal of Marketing*, *39*(3/4), 294–308. https://doi.org/10.1108/03090560510581782

Grootenboer, P., & Marshman, M. (2016). *Mathematics, affect and learning: Middle school students' beliefs and attitudes about mathematics education*. Springer.

Hannah, U. (2020). Why parenting matters for children in the 21st century: An evidence-based framework for understanding parenting and its impact on child development. In *OECD education working papers, No. 222* (pp. 0–61). Paris: OECD Publishing.

Harris, C., Straker, L., & Pollock, C. (2017). A socioeconomic related 'digital divide' exists in how, not if, young people use computers. *PLoS One*, *12*(3)

Hill, N. E., & Tyson, D. F. (2009). Parental involvement in middle school: A meta-analytic assessment of the strategies that promote achievement. *Developmental Psychology*, *45*(3), 740–763.

Hoff, E., & Laursen, B. (2019). Socioeconomic status and parenting. In M. H. Bornstein (Ed.), *Handbook of parenting. Volume 2, Biology and ecology of parenting* (3rd ed., Vol. 2, pp. 421–447). Routledge.

Hughes, J. E., & Read, M. F. (2018). Student experiences of technology integration in school subjects: A comparison across four middle schools. *Middle Grades Review*, *4*(1).

Ingram, N., Hatisaru, V., Grootenboer, P., & Beswick, K. (2020). Researching the affective domain in mathematics education. In J. Way, C. Attard, J. Anderson, J. Bobis, H. McMaster, & K. Cartwright (Eds.), *Research in mathematics education in Australasia 2016–2019* (pp. 147–175). Springer.

Jin, K.-Y., Reichert, F., Cagasan, L. P., de la Torre, J., & Law, N. (2020). Measuring digital literacy across three age cohorts: Exploring test dimensionality and performance differences. *Computers and Education*, *157*, 103968. https://doi.org/10.1016/j.compedu.2020.103968

Kariippanon, K., & Senior, K. (2017). Engagement and qualitative interviewing: An ethnographic study of the use of social media and mobile phones among remote Indigenous youth. In *SAGE research methods cases part 1*. SAGE. https://doi.org/10.4135/9781473994843

Kerstetter, K. (2012). Insider, outsider, or somewhere in between: The impact of researchers' identities on the community-based research process. *Journal of Rural Social Sciences*, *27*(2), 99–117.

Klein, D. (2007). A quarter century of US 'math wars' and political partisanship. *Bulletin* (British Society for the History of Mathematics), *22*(1), 22–33. https://doi.org/10.1080/17498430601148762

Larkin, K., & Jorgensen, R. (2016). 'I hate maths: Why do we need to do maths?' Using iPad video diaries to investigate attitudes and emotions towards mathematics in year 3 and year 6 students. *International Journal of Science and Mathematics Education*, *14*(5), 925–944.

Long, B. (2016). Sustainable improvements in student mathematics learning and disposition as a result of Prepare 2 Learn intervention. *Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia*. Adelaide: MERGA.

Maier, A., Daniel, J., Oakes, J., & Lam, L. (2017). *Community schools as an effective school improvement strategy: A review of the evidence*. Learning Policy Institute.

McFeetors, P. J., McGarvey, L. M., & Palfy, K. (2020). Mathematics curriculum change: Identifying parental expectations. *Journal of Research in Science, Mathematics and Technology Education*, *3*(2), 51–72.

Merton, R. K. (1972). Insiders and outsiders: A chapter in the sociology of knowledge. *The American Journal of Sociology*, *78*(1), 9–47. https://doi.org/10.1086/225294

National Education Association. (2002). *Preparing 21st century students for a global society: An educator's guide to the "Four Cs"*. https://dl.icdst.org/pdfs/files3/0d3e72e9b873e0ef2ed780bf53a347b4.pdf

Palacios, A., Arias, V., & Arias, B. (2013). Attitudes towards mathematics: Construction and validation of a measurement instrument. *Journal of Psychodidactics, 19*(1), 67–91.

Perry, C. A. (2011). Motivation and attitude of preservice elementary teachers toward mathematics: Motivation and attitude. *School Science and Mathematics*, *111*(1), 2–10.

Ramani, G. B., & Siegler, R. S. (2021). How informal learning activities can promote children's numerical knowledge. In A. Dowker, & R. Cohen Kadosh (Eds.), *The Oxford handbook of numerical cognition* (pp. 95–125). Oxford University Press. https://doi.org/10.1007/978-3-030-64114-6_6

Sullivan, K., Perry, L. B., & McConney, A. (2013). How do school resources and academic performance differ across Australia's rural, regional and metropolitan communities? *Australian Educational Researcher*, *40*(3), 353. https://doi.org/10.1007/s13384-013-0100-5

Tran, D., & Javed, S. (2017). Examining non-traditional pathway preservice teachers' attitudes towards mathematics. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning. Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia*. Melbourne: MERGA.

# Mathematical Connections Evident in Secondary Students' Concept Maps on Transformations of the Parabola

Winfilda Kapofu
*University of Tasmania*
winfilda.kapofu@utas.edu.au

Helen Chick
*University of Tasmania*
helen.chick@utas.edu.au

Vesife Hatisaru
*Edith Cowan University*
v.hatisaru@.ecu.edu.au

Carol Murphy
*University of Tasmania*
carol.murphy@utas.edu.au

This study examined the types of mathematical connections established by four secondary school students while constructing concept maps on transformations of the parabola. The use of concepts maps revealed students' understandings—including some misconceptions—of transformations of the parabola and confirmed the usefulness of a model for categorising types of connections.

For many, mathematics is a coherent and connected discipline characterised by a network of ideas. Some students, however, view it as a collection of separate entities (García-García & Dolores-Flores, 2018). Establishing connections between seemingly distinct mathematical ideas is central to conceptual understanding and is broadly recommended in research literature (Choy & Toh, 2021) for its ability, among other reasons, to integrate intra- and extra-mathematical knowledge (Rodríguez-Nieto et al., 2020). Also, we "understand something if we see how it is related or connected to other things we know" (Charles & Carmel, 2005, p. 10). For students to understand a new mathematical concept or acquire a new skill, they must connect their pre-existing understandings to the new concept or skills (Anthony & Walshaw, 2009). To assess students' understanding of a mathematical concept, an examination of how students connect the concept with other concepts can be undertaken (Barmby et al., 2009).

Research evidence (e.g., Yanik, 2014) shows that students struggle with the idea of transformations—including translations, reflections, rotations, and dilations—and hold ill-formed conceptual understandings of transformations (Hollebrands, 2004). Existing work on transformations of the parabola provides a basis for students' understanding of transformations of hyperbolic, exponential, and trigonometrical functions. In transformations of the parabola, many concepts can be represented both geometrically and algebraically, so it is important that students have the capacity to traverse between the two representations. The need to explore students' conceptual understandings of transformations of the parabola, therefore, has become critical. The research reported in this paper sought to answer the question: *What types of mathematical connections are evident in concept maps drawn by secondary school students among concepts associated with transformations of the parabola and other mathematical concepts?*

## Mathematical Connections

There are two ways, among possible others, through which a deep understanding of a concept may be shown: (i) the connections a student makes between a concept and other mathematical ideas, and (ii) the student's various representations of the concept and the reasoning behind the connections made (Barmby et al., 2009). Borrowing from the definitions of mathematical connections available in the literature (Businskas, 2008; Eli et al., 2013; Garcia-Garcia & Dolores-Flores, 2018), in this paper, mathematical connections are considered to be the relationships a student constructs among mathematical ideas, representations, procedures, symbols, properties, definitions, and theorems. According to Businskas (2008), there are several ways of viewing a mathematical connection. These include: a relationship between ideas or processes; a process of making or recognizing links between

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 307–314). Newcastle: MERGA.

mathematical ideas; associations between two or more mathematical ideas; and, finally, a causal or logical interdependence between mathematical entities. There has been extensive research on mathematical connections established by practicing (Businskas, 2008; Eli, Mohr-Schroeder, & Lee, 2013; Hatisaru, 2022; Mhlolo, 2012) and pre-service teachers (Evitts, 2004). Less attention has been given to mathematical connections established by students.

In her study, Businskas (2008) gave definitions for five types of mathematical connections identified by secondary teachers: different representations, implications, part-whole relationships, procedures, and instruction-oriented connections. *Different representations* are either alternative or equivalent representations of a concept. *Part-whole relationships* are such that one concept is contained in or is a component of the other. *Implications* are connections that highlight that A implies B. *Procedures* associate a process, or method, for working with a concept. *Instruction-oriented connections* indicate that A is a concept or skill necessary for understanding B, and these are associated particularly with teaching.

García-García and Dolores-Flores (2018) explored the mathematical connections established by high school students while solving calculus problems, using task-based interviews for data collection. García-García and Dolores-Flores' (2018) model consisted of seven types of mathematical connections, of which four were consistent with the model of Businskas (2008). As they worked with students, the instruction-oriented connections were not evident in their study. They added *feature connections* (also found earlier by Eli et al. (2011) and later by Hatisaru (2022)), evident when properties of a mathematical concept are presented or described in terms of what associates or differentiates it from other mathematical concepts, and *reversibility* and *meaning*. A *reversibility* connection is evident when a person is able to establish a two-way relationship between concepts. The *meaning* connection is manifested when the properties of a concept are linked to its rules, formulae, or processes including definitions and contexts.

In this study, an amalgamation of the model by Businskas (2008) and by García-García and Dolores-Flores (2018) was conjectured to be applicable for analysing mathematical connections identified by secondary school students. This model consists of connections that incorporate meaning, different representations, part-whole relationships, procedures, features, and reversibility. Further elaborations of these type of connections are presented in Table 1, which provides an outline of the types of connections related to the specific topic of transformations of the parabola.

**Table 1**

*Types of Mathematical Connections Associated with Transformations of the Parabola*

| Type of connections | Examples |
|---|---|
| Meaning connections | Meanings or interpretations associated with concepts.<br>Analogies for properties: when a familiar concept is linked to an abstract intended domain such as:<br>(i) reflection over the x-axis as a “flip”;<br>(ii) an upright parabolic shape as a “smile” or a “U shape”; or<br>(iii) a translation of 5 units to the left as a “slide” or “shift” of 5 units |
| Different representations connections | Equivalent representations: (i) y-axis and $x = 0$ line; (ii) $y = ax^2 + bx + c$ and $y = a(x - h)^2 + k$<br>Alternate representations: e.g., A parabolic U shape in the Cartesian plane with key points labelled as an alternate representation of the function $y = x^2 - 3$ |
| Part-whole relationship connections | For instance:<br>(i) y- and x-intercepts and turning point are part of the parabola;<br>(ii) a parabola has an axis of symmetry hence the axis of symmetry becomes part of the parabola |

| Type of connections | Examples |
|---|---|
| Procedure connections | For instance, $y = a(x-h)^2 + k$<br>a: gives the shape and gradient of the parabola: (i) reflection of the parabola over the x-axis (when a < 0); (ii) if \|a\| < 1 the parabola is wider and less steep than when a = 1; (iii) if \|a\| >1 the parabola is narrower and steeper than when a = 1<br>h: horizontal translation: to the right (when h > 0) or to the left (when h < 0), $h \in \mathbb{R}$<br>k: vertical translation: upward (when k > 0) or downward (when k < 0), $k \in \mathbb{R}$ |
| Feature connections | Some or all the properties may be exhibited: The basic parabola has two roots, it is symmetrical about the y-axis and has a domain of $x \in \mathbb{R}$, $(-\infty, +\infty)$ and a range of $y \in \mathbb{R}$, $[0, +\infty)$ or $y \geq 0$ |
| Reversibility connections | Forward and backward relationship.<br>For instance: a parabola is given by $y = ax^2 + bx + c$ and the equation of the parabola can be obtained using information on the parabolic graph. |

## Methodology

This study is part of the first author's PhD investigation conducted in a year 10 'Mathematics General' class in an Australian secondary school. The mathematical content for this class enables above average students to continue on to a pre-tertiary mathematics course in year 11 and includes study of transformations of parabolas.

Concept maps are known to provide visual representations of dynamic structures of understanding within the human mind (Mls, 2004). A concept map is a visual technique reflecting the key perceptions of an individual regarding relationships between and among ideas (Wheeldon & Faubert, 2009). Concept maps were utilised as the data collection tool for this study in order to make visible each student's internally constructed connections associated with the transformations of the parabola.

### *Participants*

Twelve sixteen-year-old students were the informants of the study. For the purpose of examining the types of connections associated with transformations of the parabola, the concept maps of four participants were purposefully selected for closer examination in this paper because three of the maps had additional elaborations and one, without additional elaborations, included many arrows indicating an extensive set of connections. The four participants were given pseudonyms: Enoch, Irvine, Nathan, and Jimmy. They had all been at the study school since year 7. Nathan and Jimmy were attending extension classes during Mathematics extended sessions. They intended pursuing a mathematics-oriented course at tertiary level. Enoch and Irvine did not intend to take up pre-tertiary mathematics courses.

## Data Collection

Students received ten lessons of instruction on transformations of the parabola. Following this unit of work, participants were provided with concept cards for constructing their concept maps. The concept cards included concepts that were either related or unrelated to transformations of the parabola. There were also some blank cards. Participants were told to imagine they were writing a mathematics textbook on transformations of the parabola. They were given prompting questions to assist them with coming up with ideas they might wanted to include such as:

- How would you define the word "transformation" in mathematics?
- What are the different forms of transformation you know?
- Which topics are linked to transformations?
- What words, symbols, and representations are associated with transformations?

Participants could add their own concepts or information they felt had been omitted using the blank cards. Participants were encouraged to select and link their concepts and ideas with arrows and put labels on arrows to describe how the concepts were related. Some of the concept cards provided to the participants are seen in Figures 1 to 4.

Guided by the types of connections identified in Table 1, data were analysed to reveal the types of mathematical connections established by participants while they were constructing concept maps on transformations of the parabola.

## Findings

All four students made connections among concepts in their concept maps. Participants used arrows or lines to connect related concepts and ideas; however, some of the participants provided clustered related concepts without using connecting arrows. In some instances, participants provided elaborations on the arrows to describe how they thought the ideas are related. Figure 1 illustrates how Enoch used arrows and elaborations to describe the connection between the parabola and each of the algebraic equations. In Figure 2, Nathan's concept map provides a fine example of clustered concepts. These were concepts he identified as connected by putting them in close proximity of each other without using arrows to connect them.

### *Meaning Connections*

To some of the participants, "transformations of the parabola" meant *movement* of the parabola, changing its position on the Cartesian plane. This notion is captured, for instance, when Nathan puts "*position"* between "*flip"* and "*dilation"* and uses the phrases *"stretches along y or x axis"* to describe a dilation which is a form of transformation (see Figure 2). In the same concept map Nathan used certain phrases in his elaborations on horizontal shift: "$y = (x - h)^2$, *shifts h to the right"* and "$y = (x + h)^2$, *shifts h to the left"* (see Figure 2). Jimmy had a direct arrow from transformation to position then connected it with slide (see Figure 4), thus cementing the idea of transformation being perceived as movement. The meaning of translation as a form of transformation to all participants was either a vertical or a horizontal shift. This is evident in, for example, Irvine's concept map where he had arrows originating from translation to vertical shift and another to horizontal shift (see Figure 3). The meaning connection was also established as some participants clustered alternative words or phrases around a concept. For instance, *vertical shift* and *slide* were both linked to *translation* which is a form of transformation that can be described by these two terms (see Figure 2).

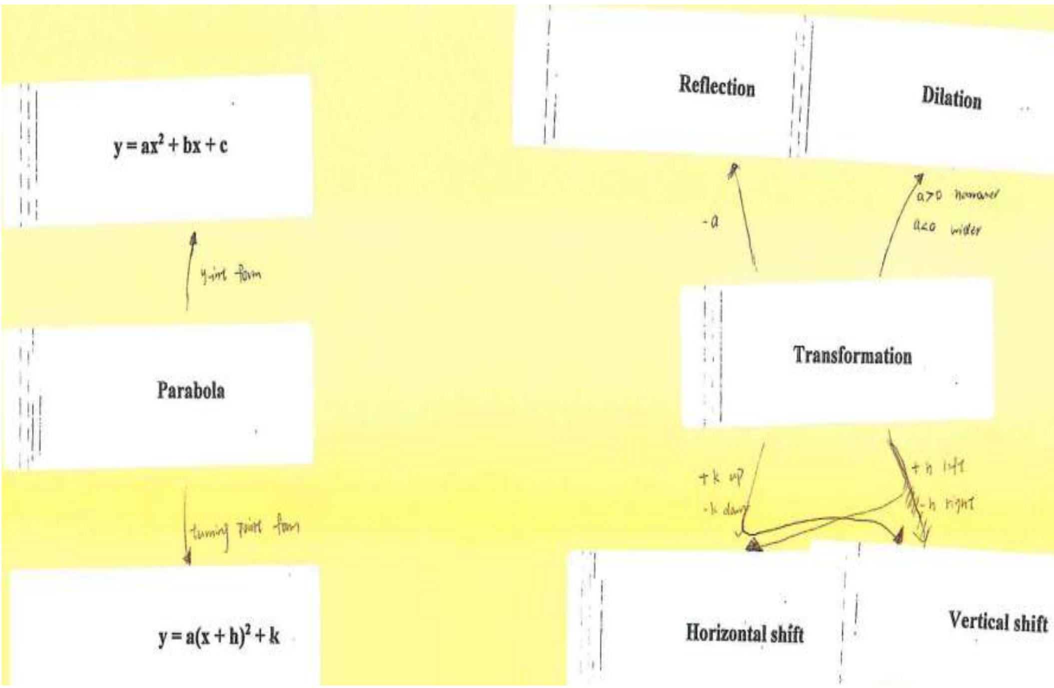



Figure 1. Enoch's concept map.

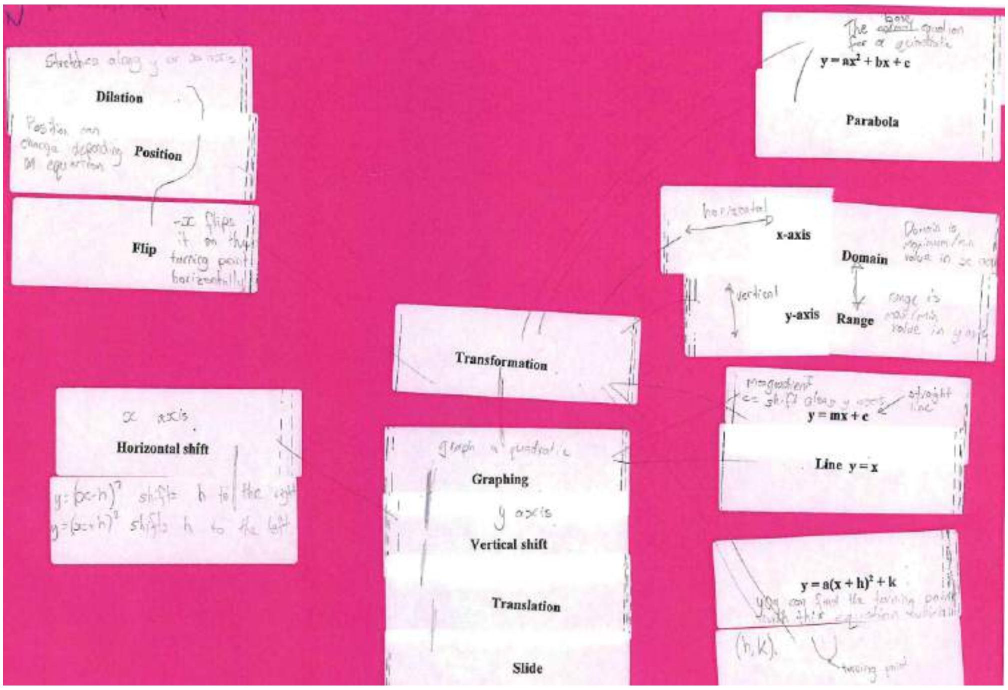



Figure 2. Nathan's concept map.

## *Different Representations Connections*

Some participants established different representations connections (Businskas, 2008). In Figure 3, Irvine used a graphical representation to illustrate a translation of the quadratic function $y = x^2$, which was presented in the form of a solid line parabola and the resulting images after translation as a dotted line parabola for the vertical and the horizontal shifts. Enoch connected the equations $y = ax^2 + bx + c$ and $y = a(x+h)^2 + k$ as alternate representations of a parabola. He also used the parameter *-a* as an equivalent of a reflection, and *+k up and -k down* to represent a vertical shift (Figure 1). Nathan identified "$y = (x-h)^2$," as an alternate representation for the horizontal graphical translation of the parabola (Figure 2). In Figure 4, Jimmy identified $y = x$ and $y = mx + c$ as alternate representations of a line, although he also associated $y = ax^2 + bx + c$ as a line as well.

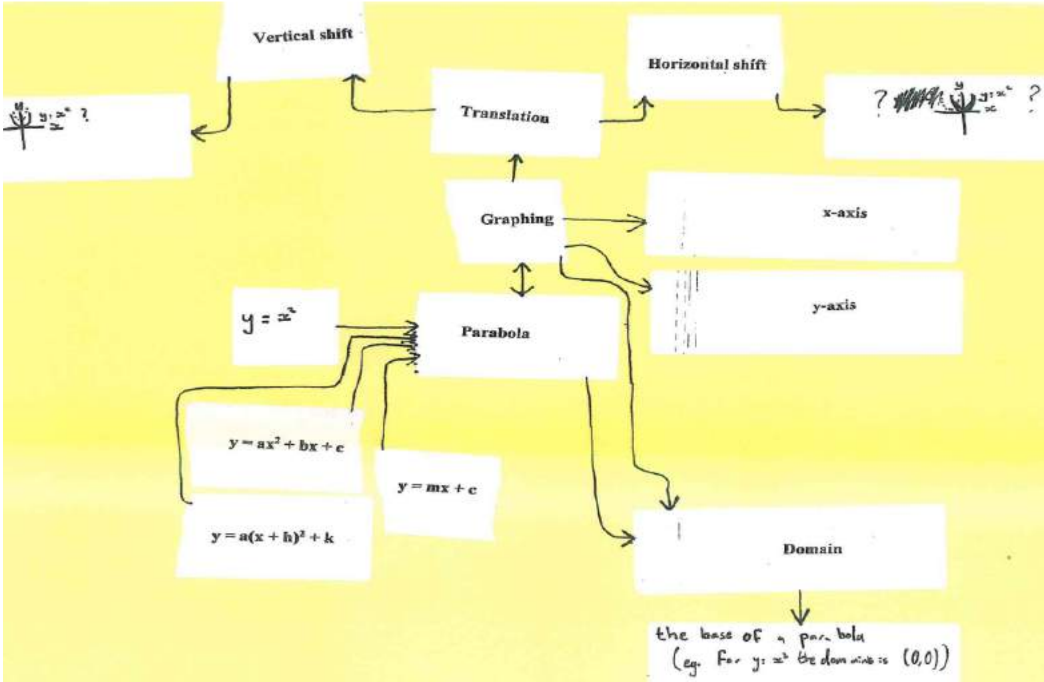



Figure 3. Irvine's concept map.

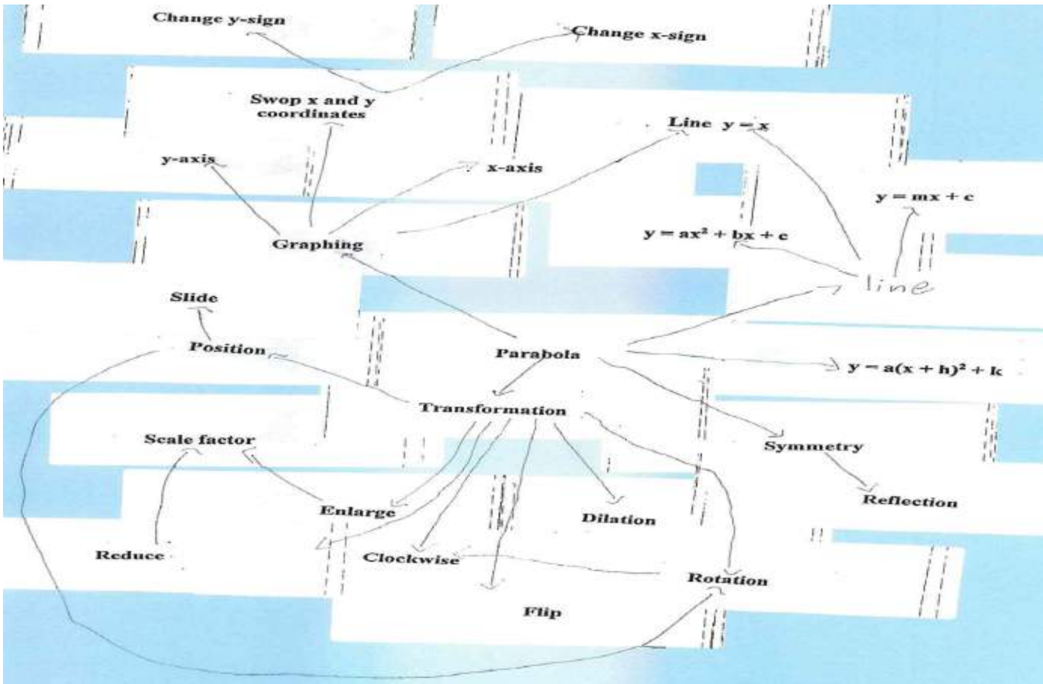



Figure 4. Jimmy's concept map.

## *Part-Whole Relationship Connections*

Another connection type that surfaced was the part-whole relationship. This connection type was also found in Hatisaru (2022). In Figure 3, Irvine linked *x-* and *y-axes* to *graphing* making them part of the graphing process. In the same concept map, "parabola" was linked to "domain" meaning that it is part of the parabola. Jimmy connected "scale factor" to "enlarge" and "reduce" meaning that scale factor is a part of each one of them (Figure 4). Jimmy also connected "symmetry" to the parabola since the parabola is a symmetrical shape.

### *Procedure Connections*

Procedural connections were evident in some participants' elaborations. For instance, Nathan gave the description of a flip as follows: *"-x flips it on the turning point horizontally"* (see Figure 2), suggesting that the shape (parabola) is reflected over the *x*-axis, implying that Nathan knew that the negative sign represents a reflection over the *x*-axis. Another emerging idea was that transformation influences the position of an object. This was evidenced by the connection made between *transformation* and *position* in Figure 2. This additional information: *"position can change depending on equation,"* showed that Nathan had developed an understanding that different parameters in an equation influence the outcome; thus, parameters in an equation determine processes required to be undertaken. Enoch connected transformation to dilation. He added the elaborations that "$a < 0$ *wider"* and *"*$a > 0$ *narrower"*, describing the process if one had to transform a shape by dilating it (see Figure 1, top right of concept map).

### *Feature Connections*

Some of the participants associated transformations of the parabola with algebraic equations of quadratic functions. In Figure 2, Nathan associated the equation $y = ax^2 + bx + c$ with "parabola" by having the concept cards adjacent to each other and linked—with an arrow—"transformation" with the equation $y = a(x + h)^2 + k$, where he highlighted the role of $h$ and $k$ in determining the turning point, and thus the translation of the basic parabola $y = x^2$. He also connected "transformation" to the $y = ax^2 + bx + c$ equation, although it was not clear how he thought transformation is associated with that equation. Irvine made connections between the parabola and its various forms of representation and the general form of a straight line (see Figure 3). These connections reveal that, besides being able to identify and describe the transformation, Irvine had the ability to navigate between the algebraic and geometric domains utilising the feature connection.

### *Reversibility Connections*

The reversibility connection from García-García and Dolores-Flores (2018) was only evident in one concept map and on one occasion. Bi-directional arrows were drawn to connect transformation to some of its forms. For example: *graphing* ↔ *parabola as* seen in Irvine's map in Figure 3 indicates that the relationship between graphing and parabola is reversible. This suggests that Irvine understood that one could use given information to graph a parabola. At the same time, a person could extract information from the graphical form of a parabola.

## Discussion and Conclusions

The types of connections established by participating students were consistent with those from the models of Businskas (2008) and García-García and Dolores-Flores (2018), with the exceptions of the instruction-oriented and implication connections. This may be attributed to the differences in the sample type as well as the data collection techniques. It was not expected to identify the instruction-oriented connection in the students' works. Implication connections and procedural connections were difficult to identify in the absence of detailed elaborations.

The students exhibited types of transformations using alternative wording and flexibility in the interpretation of algebraic transformation equations. Nevertheless, there were gaps in connecting concepts to their formal definitions. For example, the connection between "flip" and its formal name (reflection) was not evident in students' work. Further elaborations revealed some misconceptions held by the students. This was perhaps best evident in Nathan's response stating that "range is max/min value in the y axis" (Figure 2). It was also observed that participants made some ambiguous connections in their concept maps. For instance, in Figure 4, Jimmy did not provide descriptions of how the linked concepts were related. In other words, the concept map did not include enough information for some of the connections he might have established. Clearly much detailed information—such as from interviews—might be needed to gain better insights into the connections that students make.

These findings indicate that, as a data collecting tool, the concept map does not capture all the types of mathematical connections established by a student to reflect the full extent of their understanding of a learnt concept. Considering the emerging issues in this study, it is proposed that for most of the established connections to be captured, the concept maps could be accompanied by elaborations, and interviews, which would provide the participant with an opportunity to reveal some of the missing connections through their explanations and descriptions.

With the emerging issues mentioned above in mind, findings from this study can provide teachers with insight into what students regard as connections between and among concepts, and plausibly can assist in identifying misconceptions and gaps in students' conceptual understandings. They can also trigger teachers to reflect on their personal understandings of concepts creating opportunities for helping students develop well-formed conceptual understandings. Finally, education practitioners will be enlightened on how the formal concept definitions may be interpreted by students in personal ways.

## References

Businskas, A. M. (2008). *Conversations about connections: How secondary mathematics teachers conceptualize and contend with mathematical connections* [Master's thesis, Simon Fraser University].

Charles, R. I., & Carmel, C. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education, 7*(3), 9–24.

Eli, J. A., Mohr-Schroeder, M. J., & Lee, C. W. (2013). Mathematical connections and their relationship to mathematics knowledge for teaching geometry. *School Science and Mathematics, 113*(3), 120–134.

Evitts, T. A. (2004). *Investigating the mathematical connections that preservice teachers use and develop while solving problems from reform curricula* [Doctoral thesis, The Pennsylvania State University].

García-García, J., & Dolores-Flores, C. (2018). Intra-mathematical connections made by high school students in performing Calculus tasks. *International Journal of Mathematical Education in Science and Technology, 49*(2), 227–252.

Hatisaru, V. (2022). Mathematical connections established in the teaching of functions. *Teaching Mathematics and its Applications*: *An International Journal of the IMA*1-21. https://doi.org/10.1093/teamat/hrac013

Hollebrands, K. F. (2004). Connecting research to teaching: High school students' intuitive understandings of geometric transformations. *The Mathematics Teacher, 97*(3), 207–214.

Mhlolo, M. K. (2012). Mathematical connections of a higher cognitive level: A tool we may use to identify these in practice. *African Journal of Research in Mathematics, Science and Technology Education, 16*(2), 176–191.

Mls, K. (2004). From concept mapping to qualitative modeling in cognitive research. *In concept maps: Theory, methodology, technology: Proceedings of the first international conference on concept mapping* (pp. 271–274).

Rodríguez-Nieto, C. A., Rodríguez-Vásquez, F. M., & Moll, V. F. (2020). A new view about connections: The mathematical connections established by a teacher when teaching the derivative. *International Journal of Mathematical Education in Science and Technology*, 53, 1231–1256. https://doi.org/10.1080/0020739X.2020.1799254

Toh, T. L., & Choy, B. H. (Eds.) (2021). *Mathematics connection and beyond: Yearbook (2020)*. Association of Mathematics Educators. World Scientific.

Yanik, H. B. (2014). Middle-school students' concept images of geometric translations. *The Journal of Mathematical Behavior, 36,* 33–50.

# High-Stakes Examination Tasks as Impetus for Primary Mathematics Teachers' Reform in their Instructional Practice

Berinderjeet Kaur
*National Institute of Education*
Nanyang Technological University, Singapore
berinderjeet.kaur@nie.edu.sg

The study reported in this paper is on the professional development (PD) of primary school mathematics teachers. Teachers from two primary schools participated in the PD for two years. High-stakes mathematics examination tasks were used to kick start awareness and thinking about teaching for big ideas. Teachers did the tasks and discussed their solutions focusing on how their instruction could facilitate the acquisition of mathematical ideas as a body of connected knowledge. Data presented in this paper show that the tasks teachers worked with at the start of the PD did impact their understanding and instructional practice specific to big ideas in mathematics. Some challenges the teachers faced during the PD are also noted.

Charles (2005) defined a "Big Idea as a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p.10). The revised school mathematics syllabuses for primary schools in Singapore (Ministry of Education, 2019) reinforces that Big Ideas are central to mathematics as they connect ideas coherently from different strands and levels thereby facilitating a deeper and more robust understanding of individual topics in mathematics. The revised syllabuses list 6 big ideas (Notations, Diagrams, Proportionality, Models, Equivalence, and Measures) for primary schools.

There is a concerted push towards teaching for Big Ideas in Mathematics in Singapore schools. A research study, Big Ideas in School Mathematics (BISM) is presently underway in Singapore and a part of it is on professional development (PD) of primary school mathematics teachers related to the enactment of Big Ideas in their mathematics instruction. Research has documented that teachers' lack of relevant content knowledge of Big Ideas in mathematics translates into their lack of explicit attention to Big Ideas underpinning mathematics taught in schools and results in developing isolated compartments of mathematical knowledge in their students (Askew, 2013).

The study reported in this paper draws on part of the data from the BISM project. It involves the PD of primary mathematics teachers and attempts to answer the following research question:

- What are teachers' perceptions about high-stakes examination tasks (Primary School Leaving Examination (PSLE) mathematics tasks) as impetus for reform in their instructional practice?

In the context of this paper, the reform is specific to teaching for big ideas in their mathematics lessons.

## Review of Related Literature

The professional development of the mathematics teachers in the BISM project adopted the hybrid model (Kaur, 2011). This model integrates the "training model of PD" (Matos et al., 2009) with sustained support for teachers to integrate knowledge gained from the PD into their classroom practice. It is a form of CPD that exemplifies a shift of the centre of gravity for CPD from the "supply-side,", "offline" forms of knowledge transmission by professional development providers,

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 315–322). Newcastle: MERGA.

such as University academics, to “demand-side,” “online” in-situ forms of knowledge creation by teachers. The five critical features of the PD were:

- Content focus—it focused on what to teach and how to teach (Desimone, 2009; Stiff, 2002). The PD was not generic but specific to the pedagogy of mathematics related to primary school mathematics and teaching for big ideas.
- Coherence—it supported the instructional activities of teachers at school (Desimone, 2009; Stiff, 2002). They 2020 revised school mathematics syllabuses for primary schools places a heightened emphasis on teaching for big ideas and therefore the PD supported teachers in the adoption of the initiative ‘Big ideas in mathematics” (Ministry of Education, 2019).
- Duration—the PD was sustained (Borasi & Fonzi, 2002; Desimone, 2009; Elmore, 2002; Stiff, 2002). It was 2 years long comprising two cycles of teachers’ work with each cycle focussed on a big idea (Equivalence in the first year and Proportionality in the second year).
- Active learning—the PD was embedded in teacher’s work (Abdal-Haqq, 1995; Desimone, 2009; Wilson & Berne, 1999). Teachers learn best when observing, planning, enacting their plans and reviewing their work (Stiff, 2002). During the PD teachers work included ‘hands-on’ work—working through mathematical tasks, planning and enacting lessons, reviewing their lessons and mapping follow-up plans.
- Collective participation—the PD had collective participation. Collective participation by teachers allow for powerful form of teacher learning through prolonged interaction and discourse (Desimone, 2009; Stiff, 2002; Wilson & Berne, 1999). In addition, PD programmes that foster collaboration have been found to be effective (Borasi & Fonzi, 2002; Elmore, 2002; Hawley & Valli, 1999). In the PD teachers participated collectively as part of groups at two levels. The first involved all the mathematics teachers in a school and the second involved teachers teaching specific year levels in the school. Teachers also worked collaboratively during the ‘hands-on’ work.

## The Study

### *Participants*

Mathematics teachers from two primary schools, P1 and P2, participated in the BISM project. The profiles of both the schools were similar in that they were government funded schools and teachers were employees of the Ministry of Education in Singapore. In school P1 and school P2, there were 23 and 33 teachers respectively who participated in the project at the school level. At year 5 and 6 levels, in both schools, P1 and P2, 7 teachers participated.

### *Implementation of the Professional Development*

The PD was spread over two consecutive school years. In Singapore, a school year begins in January and ends in November. Each year the PD began with two knowledge-building workshops. Each workshop lasted 2 hours. These were attended by all the mathematics teachers in the school. During the first workshop, the mathematics educator presented the six big ideas in the primary school mathematics curriculum and facilitated whole group discussion about what these ideas are and their role in the learning of mathematics. During the second half of the first workshop and the second workshop teachers attempted some mathematics tasks (mainly taken from the Primary School Leaving Examination mathematics past papers). Following teachers work on every task, a member of the research team (who is a mathematics educator at the National Institute of Education) facilitated whole group discussion (review segment) the high point of which was “how the big idea of equivalence (in the first year) and big idea of proportionality (in the second year) facilitated the solution process”. Figures 1 and 2 show examples of the tasks and their respective solutions arrived at collaboratively by the teachers during the review segments of the workshops. The task in Figure

1, illuminates equivalence as a big idea, while task in Figure 2, illuminates proportionality as a big idea.

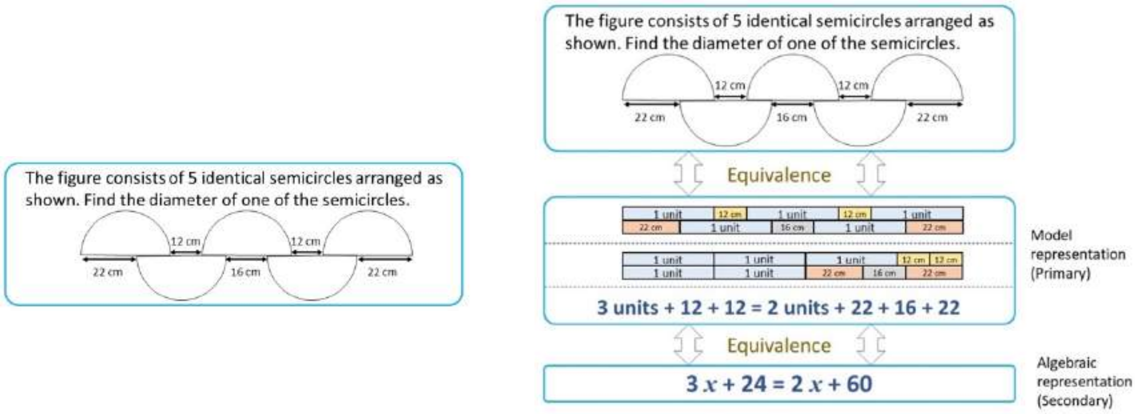
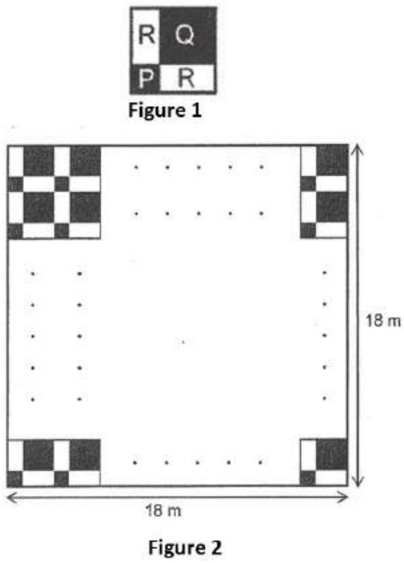



Figure 1. PSLE mathematics (2019-2021) question (SEAB, 2022).

Figure 1 shows a square tile made up of 2 black squares, P and Q, and 2 identical white rectangles R. The length of 1 side of square Q is twice the length of 1 side of square P.

a) What fraction of the square tile is made up of black squares?

b) Figure 2 shows a floor laid with the square tiles. The floor is 18m by 18m and is completely covered with the square tiles. Find the total area of the floor covered by black squares.




Solutions

a) In figure 1, there are 9 tiles altogether. 5 of them are shaded black. So the fraction of the sqaure tile that has black squares is $\frac{5}{9}$.

b) The total area of the floor is 18 x 18 $m^2$. The total area of the floor covered by black squares
$\frac{5}{9}$ x 18 x 18 $m^2$.

Figure 2. PSLE mathematics (2008-2013) question (SEAB, 2014).

During the next two meetings, we worked with teachers from the year 5 and year 6 levels. Each meeting lasted 2 hours. They deliberated on the mathematics they would be working on during the coming weeks in their lessons and how they may teach for the big idea (equivalence in the first year and proportionality in the second). In their groups they choose a sub-topic and planned a 40–50 minutes lesson. Members of the research team (a mathematics educator and another expert primary school mathematics teacher) were present during both the meetings and provided inputs where necessary. Following the two meetings, one of the teachers in the group enacted the lesson that was collaboratively planned. The BISM research team recorded the lesson.

During the fifth session the video record of the lesson enacted was reviewed. The review was guided by the following prompts.

- What were the lesson objectives? Were they as planned?
- Were the mathematical tasks sequenced and enacted as planned?
- How did the teacher support students in articulating their observations and ideas about the intended mathematical connections, aka big idea?
- Were they any actions other than the pre-planned ones the teacher undertook so as to accomplish his/her lesson goals?

The sixth session was devoted to focus group discussions (FGDs) involving teachers from year 5 and year 6 levels. The data reported in this paper is from the FGDs in the second year of the PD.

### *Data and Analysis*

The data presented in this paper is from the focus group discussions with the year 5 and year 6 levels teachers at the end of the second year of the PD. The following prompt steered the discussion.

- During the first two PD workshops last year and again this year, all of you worked on high stakes examination tasks (mainly taken from the PSLE (Primary School Leaving Examination) mathematics past papers). Please share with us your thoughts about any benefits that have arisen related to knowledge of big ideas and changes in your classroom instruction. You may also share with us any challenges that you encountered.

The discussions were transcribed. The transcripts were read several times and scanned for possible themes (Braun & Clarke, 2006). Four main themes emerged. Next, excerpts that were representative of the teachers' perceptions were collated for each theme and content analysis (Weber, 1990) of the excerpts carried out. In the next section, we present our findings.

## Findings

The following themes emerged from the thematic analysis of the discussion data. For each theme content analysis of the data is presented.

### *Big Ideas Pervade the Curriculum from Primary 1 to Primary 6*

Content analysis of the transcripts show that working on the high-stakes examination tasks allowed teachers to appreciate that big ideas pervade the mathematics curriculum from years 1 to 6. Teachers teaching the upper year levels felt that their peers in the lower year levels (1 to 4) too had a part to play in preparing students for the PSLE mathematics.

> P1T4: .. You were asking what is the benefit of using assessment questions to start of the learning. I thought, in a way, we are also telling the teachers this is what the students will have to achieve at the end of P6. It was a good way to tell our lower primary teachers that you need to help the students to see connections from young and not leave everything to P5 and P6. We can start doing all this from young. … all these big ideas come from P1 and 2. We can start connecting them from young. If you ask me, that (using assessment task as stimulus) was a good way as it helped the lower primary teachers to see that this is what the children will have to do and we don't have to leave all the teaching to the 5 and 6. Some of it can come from the P1 and P2. So I thought that was quite good to shape that we can start incorporating all these big ideas even in their lower primary topics.
>
> P2T2: I come to the realization that this Big ideas have always been in our syllabus. It is just that now we are made into awareness that this leads to the big ideas of equivalence, proportionality and so on. Some of the questions you got us to work on are past many years' PSLE questions. It's just that we were teaching it unknowingly. Now that we have greater awareness, we will link it to that concept. Then it makes us be able to see the connection between all the different kinds of questions to link it up to this common theme. Of course, the questions that came up as we discussed are good. So, we know where we are getting up in terms of exposing the students to big ideas as such. Some questions lend themselves to more than one big idea.

Figure 3. Excerpts from the FGDs about awareness of big ideas in the primary school mathematics curriculum

They also realized that 'big ideas' were not new as they had unknowingly been working with them. However, now with the realization they will be more deliberate in facilitating their students thinking when working through mathematical problems like the high-stakes assessment tasks used during the workshops. Figure 3 shows excerpts from the transcripts of the discussions.

### *Challenging Tasks + Active and Collaborative Learning*

Content analysis of the transcripts show that all the teachers found the tasks challenging and appreciated the struggle the tasks offered. This allowed them to experience the struggle their students also go through when doing such tasks. The content analysis also show that engaging in active and collaborative learning when doing the tasks, discussing the different approaches taken and reflecting on the mathematical ideas inherent in the tasks have led them to overcome their personal apprehensions. Figure 4 shows excerpts from the transcripts of the discussions.

> P1T1: Last year when I was doing the questions I was struggling, but after the discussions I saw the connections and could understand the solutions. This year I was more confident when I was doing the tasks. Now I am able to see the bigger picture and appreciate big ideas like equivalence that can make me think of how two figures may be related. Not just think about which formula to use and just calculate to get some answer.
>
> P2T4: I quite enjoyed the assessment tasks …challenging challenges you … yeah and it also gave me an opportunity to see things from my students perspective because sometimes I think as teachers we forget to do that or at least I forget to do that sometimes and I think these kinds of questions our students really struggle. Going through the assessment tasks as a group and talking gave me a better idea of different kinds of activities or questions that I can design to make my students think and make connections … not just this method or which formula to use

Figure 4. Excerpts from the FGDs about challenging tasks + active and collaborative learning.

### *What Appears to Have Changed in Classroom Instruction*

From the content analysis of the transcripts, it appears that teachers have attempted to make some changes to their pedagogy. In the past some of them when working with weak students tended to spoon feed them with methods and answers. But now they appear to allow their students to struggle when confronted with challenging tasks but support them by facilitating their discussions amongst peers and with them. By doing so, they have found that their students are able to self-correct their misconceptions and move along the solution path. They are placing greater emphasis on 'understanding the task' and examining possible relationships before 'carrying out' any calculations. They also appear to be mindful of their talk and deliberate about what they would like the students to focus on. Figure 5 shows excerpts from the transcripts of the discussions.

> P1T5: Very weak ones… in fact I kind of use a few examples to let them try out. It's really a struggle for them… and I realized what makes the change is … I am more cautious about the way I get the students to discuss. I think we always allow students … maybe because of the weakest ones, we tend to be very quick … we want to give them the solutions, we want them to answer quickly and get it over and done with… but for this lot, I find that what is interesting is when you allow … you throw it to them first and let them discuss first and then the idea actually bounced off … and from there, that is when I am able to find out what needs to be rectified … that means change to help them to … correct their misconception first, then move on to … like so call the big idea proportion and all these. But I think more importantly is what I noticed the students are able to do is … they don't see what they are learning in isolation, they are starting to relate it, like ratio, fractions and decimals… they are trying to relate it in their own ways …
>
> P1T1: So, in my lessons these days, I find myself … telling my students to also try and see the bigger picture … I always tell them, don't just keep zooming into all the nitty gritty details straight away. Take a step back and look at the entire thing first. Think about how … you can maybe chop up the shape into different parts, and I use this word a lot in my class now … relationship. So, I don't explicitly use the word proportion that much, but I do try to encourage them to see relationship between the different parts of the questions that they are trying to solve. So, in a way after … now that I have been exposed to this idea of proportionality, I do myself changing my classroom language or teaching language a little bit here and there. So, I am more conscious of what I am telling the students to look for in their work

Figure 5. Excerpts from the FGDs about some changes in instructional practice.

### *Planning to Teach for Big Ideas—Time a Concern*

From the content analysis of the transcripts, it appears that teachers found time taken to prepare lessons to teach for big ideas a concern. They valued the learning afforded when planning together for a lesson and appreciated the impact of the lesson as noted by teacher P2T5 that '*we can see the children go through so much thinking*'. Teachers in school P2 also through the planning of the lesson to teach pie-charts realised that proportional relationships are the basis for constructing and interpreting pie charts. This certainly has contributed towards their understanding of how the big idea of proportionality pervades topics in their curriculum. As teachers were grappling for the first time in trying to plan such lessons, it is expected that the time they need to plan would be much greater than the instructional time. But this aspect of time in the planning phase and the enactment phase was a concern to the teachers and as noted by P2T6, "*in reality if we can do this for 1 or two topics per year, I find this already a very big achievement*". Figure 6 shows excerpts from the transcripts of the discussions.

P2T5: Reflecting on the whole process, we did a very simple topic on pie charts. Most teachers will not even spend more than two weeks teaching it. Because of going through this process, made us really go down to the core of the concepts and we wanted so badly to fit in proportionality inside and that it made us really think through of where is the proportionality in it and I think that process benefited the teachers and also the children. After we have carried out our lesson we can see the children go through so much thinking. That is really good.

P2T6: We spend almost 6 to 7 hours together as a team. So many people, you know, to come up with a 1-hour lesson. The difficulty was on the questioning, how do we ask the questions. I agree that we are spending time (planning time), that is secondary. The primary aspect would be, in the class, if the topic is only given only 1 week and we want to use 2 weeks to teach, we must be very practical. … imagine if we want to extend for every topic. Realistically we have no time. In reality, if we can do this for one or two topics per year, I find this already a very big achievement.

P1T3: Seriously speaking, even though we had to spend hours, we discuss, we had headaches, it was fun. Because it was very good that we are discussing, we were trying ideas, we were questioning ourselves, it was very enriching. … So if you asked me, yes the number of hours, but I learnt a lot and I liked it.

Figure 6. Excerpts from the FGDs about time a concern when planning to teach for big ideas.

## Discussion and Conclusion

The research question that guided the study reported in this paper is "What are teachers' perceptions about high-stakes examination tasks (Primary School Leaving Examination (PSLE) mathematics tasks) as impetus for reform in their instructional practice?". The research team were deliberate in using past PSLE Mathematics tasks to kick start teachers work in the PD focussed on teaching for big ideas. Instead of providing teachers with lesson plans that are "offline" forms of knowledge transmission to teach for big ideas, the research team attempted to engage teachers with "online" in-situ forms of knowledge creation through high-stakes examination tasks to teach for big ideas.

As the nature of the high-stakes examination tasks were non-routine mathematical problems, there was little concern that teachers were being prepared to teach to the test (Phelps, 2011). From the data presented in this paper it is apparent that the high-stakes examination tasks that teachers worked with during the first two knowledge-building workshops every year of the PD programme have contributed to the PD of the teachers in some ways. The syllabus document of the primary school mathematics curriculum (Ministry of Education, 2019) outlines the six Big ideas and elaborates each of them. The elaboration of equivalence is as follows:

> Equivalence is a relationship that expresses the 'equality' of two mathematical objects that may be represented in two different forms. The conversion from one form to another equivalent form is the basis of many manipulations for analysing, comparing, and finding solutions. In every statement of equivalence, there

is a mathematical object (e.g. a number, an expression or an equation) and an equivalence criterion (e.g. value(s), or part-whole relationships) (Ministry of Education, 2019, p.15).

The elaborations of equivalence and proportionality were shared with the teachers at the beginning of the first knowledge-building workshops in the first and second years respectively. The high-stakes examination tasks engaged teachers in making sense of such elaborations and teachers realised that big ideas pervaded the school mathematics curriculum from years 1 to 6. As teachers from all year levels were working on the same tasks, it was beneficial for all of them to 'see' how big ideas manifest in the non-routine mathematical problems students will do in their PSLE Mathematics in due course. Working on the tasks the teachers also appreciated the challenge they confronted, and their students too confront or will confront when they are given such tasks to do.

The nature of the active work and discourse during the workshops, 'do the tasks—giving your best', 'discuss the solutions and seek inputs from all—engage with active whole class discussion', appear to have also provided the teachers with ideas of how to shift their classroom instruction when teaching for big ideas. It is noteworthy that teachers working with weaker students too attempted to hold back their inputs and facilitated student talk, directing them towards their resolutions of tasks at hand.

Teachers lamented about the investment of time warranted for designing lessons that facilitate students understanding of mathematical ideas as a connected web. This is understandable when a new push is added to any curriculum. Planning a few of such lessons yearly eventually will accumulate alongside deepening of teacher's knowledge for mathematics teaching. So, perhaps the concern to educators would be for teachers to undergo a mindset change and deepen their understanding of mathematics as a body of connected knowledge.

The limited data and findings presented in this paper do suggest that the high-stakes examination tasks the teachers worked with during the PD were an impetus for possible reform in their mathematics instructional practice. More research using such tasks for the PD of teachers is warranted to confirm the claim we are making in the conclusion of this paper.

## Acknowledgements

The study reported here has been funded by the Singapore Ministry of Education under the research project OER/31/19BK which is managed by the National Institute of Education, Singapore.

## References

Abdal-Haqq, I. (1995). *Making time for teacher professional development* (Digest 95–4). Washington, DC: ERIC Clearinghouse on Teaching and Teacher Education.

Askew, M. (2013). Big ideas in primary mathematics: Issues and directions. *Perspectives in Education*, *31*, 5–18.

Borasi, R., & Fonzi, J. (2002). *Professional development that supports school mathematics reform. Foundations series of monographs for professionals in science, mathematics and technology education*. Arlington, VA: National Science Foundation.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Charles, R. I. (2005). Big ideas and understandings as the foundations for elementary and middle school mathematics. *Journal of Mathematics Education Leadership*, *7*(3), 9–24.

Desimone, L. M. (2009). Improving impact studies on teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher*, *38*(3), 181–199.

Elmore, R. F. (2002). *Bridging the gap between standards and achievement: The imperative for professional development in education*. Washington, DC: Albert Shanker Institute.

Hawley, W. D., & Valli, L. (1999). The essentials of effective professional development: A new consensus. In L. Darling-Hammond, & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 127−150). San Francisco: Jossey-Bass.

Kaur, B. (2011). Enhancing the pedagogy of mathematics teachers (EPMT) project: A hybrid model of professional development. *ZDM—The International Journal on Mathematics Education*, *43*(7), 791–803.

Matos, J. F., Powell, A., & Sztajn, P. (2009). Mathematics teachers' professional development: Processes of learning in and from practice. In R. Even, & D. L. Ball (Eds.), *The professional education and development of teachers of mathematics* (pp. 167–183). New York: Springer.

Ministry of Education. (2019). *2021 Primary mathematics teaching and learning syllabus*. Singapore: Author.

Phelps, R. P. (2011). Teach to the test? *Wilson Quarterly*, *35*(4), 38–42.

Stiff, L. V. (2002, March). Study shows high-quality professional development helps teachers most. *NCTM News Bulletin*, *38*(7), 7.

Singapore Examinations and Assessment Board. (2014). *Primary school leaving examination (PSLE) mathematics questions (2008–2013)*. Singapore: Educational Publishing House Pte Ltd.

Singapore Examinations and Assessment Board. (2022). *Primary school leaving examination (PSLE) mathematics questions (2019–2021)*. Singapore: Educational Publishing House Pte Ltd.

Weber, R. P. (1990). *Basic content analysis*. Newbury Park, California: Sage Publications.

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. *Review of Research in Education*, *24*, 173–209.

# Teachers' Design of Instructional Materials: Locating Teachers' Appropriation of Usable Knowledge

Yew Hoong Leong
*National Institute of Education, Nanyang Technological University*
yewhoong.leong@nie.edu.sg

The phenomenon of teachers designing their own instructional materials is gaining more attention in research. Different aspects of this enterprise have been examined—its potential to reveal the complexity of teachers' instructional planning considerations, the design principles employed by teachers to realise instructional goals, among others. In the study reported in this paper, the focus was on its utility as a form of teacher professional development. In particular, evidence was sought for this claim: not only is teacher-designed instructional materials a useful tool for professional development, it can capture usable knowledge teachers appropriate from professional development.

## Introduction

While the study of teachers' interpretation and use of curriculum materials designed by others is an area of intense research for some time now, the focus on teachers themselves as designers of instructional materials for their own teaching is relatively scarce and only emerged very recently. By "instructional materials" I refer to materials that are classroom-ready and intended to be used in the classroom to engage students in learning. Defined this way, curriculum materials (CM) designed by others may indeed be used directly by teachers as instructional materials (IM)—a familiar case is one where teachers 'teach from the textbook'. However, in some jurisdictions, such as in Singapore, it is found that mathematics teachers do not usually use CM directly for classroom teaching; rather, they design their own IM—which may be adaptations of portions of CM—for use in their lessons (Cheng et al., 2021).

In studying teachers' design of their IM for mathematics classrooms, the emphasis has been on elucidating the design principles adopted by these teachers—as a way to understand the layers of complexity in their design processes. But more recently, another perspective has also emerged: the interaction between teacher professional development (PD)—itself an area that generates much interest—and teachers design of their IM (Kaur et al., 2022). The content in this paper is aligned to this new perspective. It is devoted to this particular question: Can teacher-designed IM document teachers' usable knowledge appropriated from PD?

## Professional Development and Usable Knowledge of Teachers

One major challenge of PD: how do we do PD in such a way that would result in positive changes in the classroom of the teachers who participated in the PD? This question is borne out of the reality that most PD—even in those where teachers who participated avowed that they have picked up useful ideas—have very little direct impact in changing how teachers conduct instruction in their classrooms (e.g., Hill, 2009; Wallace, 2009). Some have explained this phenomenon using the construct of "inert knowledge" (Renkl et al., 1996)—teachers may acquire some of these from PD but they are not activated during their instructional work. In contrast, "usable knowledge" is defined as "knowledge that teachers are able to access and use in a classroom situation" (Kersting et al., 2012).

This construct of usable knowledge helps us rethink and reframe PD. First, if teachers' usable knowledge is the goal, then how shall PD be done so that the content of PD is about matters that teachers are more likely to activate in their classroom teaching? Yet, this content of PD must remain substantial in the sense that it can effect change in instructional quality—which is the aim of PD. Second, if we can indeed make an argument for such a form of PD that targets usable knowledge,

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 323–330). Newcastle: MERGA.

how do we prove this claim—that the purported usable knowledge acquired during PD is indeed used by the teachers? At first glance, the answer to the second question seems obvious: "Well, observe the teachers' lessons!" Apart from the reality—as most education researchers would have experienced—that access to classrooms, especially to time the access to be immediately after PD, for all participants is resource-intensive; happenings in the classroom may not be easily traceable directly to PD. Teachers make a myriad of utterances and carry out many activities in the classroom; to make links between these words and actions to those conducted in PD can be likened to finding the proverbial needle in a haystack.

## Teacher-designed Instructional Materials in Relation to PD for Usable Knowledge

My claim is that teacher-designed IM can feature prominently in answers to the questions raised in the previous paragraph. The fact that teachers who designed their IM for use in the classroom do so instead of drawing directly from CM means that these teachers want to imbue their personal goals and characteristics into these IM as they are used in classroom teaching. In other words, these IM mirror very closely the chronology and content that actually occur during the in-class enactment of the lessons. That this is so has been reported in other studies (e.g., Chin et al., 2022, Leong et al. 2021). Thus, a careful examination of teacher-designed IM is also a careful examination of the resources the teacher intends to utilise in the classroom enactment that uses the IM. These IMs represent a space where the teacher would consider directly relevant to their in-class instructional work.

This renders IM a suitable object of focus for PD work if the aim of the PD is indeed to influence teachers' usable knowledge. That is, one efficacious way in which PD providers (PDP) can influence teaching quality in the classroom is via the IM that they design, since these teachers follow closely the IM that they bring into their classrooms. When PD revolves specifically around improvements to their IM-design, there is a higher likelihood that learning opportunities during these PD sessions be translated by teachers as usable knowledge since the content of these discourses is about stuff that matters to them in actual instructional work—as reflected in the IM. During these IM-focussed PD sessions, both PDPs and the teacher engage one another not in mere theoretical talk about what may be helpful for in-class teaching; rather, they are engaged in the joint work of realising theoretical ideals into the (re-)design of IM—in a way that incorporates the perspectives of both the teacher and the PDPs. This interaction between the teacher and PDPs that is centred on design work of IM is illustrated in the "PD context" box of Figure 1. During the PD setting, both PDPs and the teacher 'act on' (as shown by the one-directional arrow) the IM in the sense that their talk is directly about contents (and the underlying ideas behind them) in the IM, while they engage one another in the discourse (as shown by the bidirectional arrow between them).

Also, such a form of PD does not start with a 'clean slate' of IM-design; instead, the onus is on the teachers to present the draft of the IM (labelled as IM-A in Figure 1) that reflects their existing conceptions of teaching (a particular mathematics topic) prior to PD. This aspect is shown in Figure 1 as "Pre-PD design". Framed this way, the tone of the PD shifts away from "PDPs imposing their agenda" to that of "teachers retaining their agenda" (Leong et al., 2022). The teacher—having worked through a draft—now comes to the PD session with challenges they would have encountered and are thus more ready to look out for usable knowledge to fill the gaps. And since the PDP's focus during PD is on the IM—congruent to the teacher's agenda, suggestions and ideas will likely then be seen as directly usable for the improvement of IM and thus translatable to classroom instruction. While the PDPs' role is to 'value-add' to the quality of the IM, the prerogative to make changes (or not at all) to the IM rests on the teachers themselves. I think this heightening of the teachers' 'ownership' of the enterprise of IM-design is key to gearing the PD discourse towards usable knowledge acquisition.

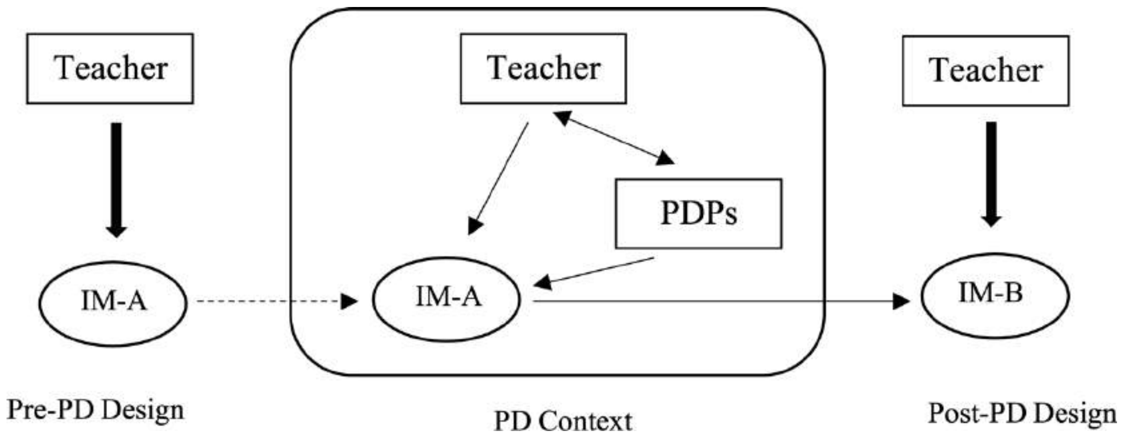



Figure 1. The place of teacher-designed instructional materials in relation to PD.

Following the PD session(s), the teachers work on amendments to the IM in response to the inputs they obtained from the PD setting. The final IM that is indeed classroom-ready is labelled as IM-B in Figure 1. Since there are no known inputs from other sources that are directly focussed on their IM-design between their initial draft and the final one, I claim that the differences between IM-B and IM-A come about exclusively from the PD encounter. In other words, the change in the IM—which is far easier to analyse than classroom enactments—is a reliable proxy to locate usable knowledge as appropriated from PD.

Conceived this way, teacher-designed IM is both a suitable PD resource (as IM-A) and a site to locate usable knowledge gained from PD (across IM-A and IM-B).

## Method

The purpose of this study is to explore if the theoretical construction as explicated in the preceding sections of this paper 'works'. That is, when I set up PD—in my capacity as PDP—with teachers as one that is centred on IM-design, and then comparing IM-A against IM-B, do the similarities and differences reveal usable knowledge for teachers that I would consider—as a mathematics educator—an improvement in instructional quality? [This last clause about what "I would consider …" is admittedly a non-rigorous way of judging instructional quality. This study can also be seen as an initial exploration towards establishing standards of instructional quality.]

The study reported here is part of a larger project on "Big Ideas in School Mathematics". The emphasis on teaching towards big ideas in mathematics is a rather recent one which is envisioned by the Singapore Ministry of Education (MOE, 2019). In brief, the push is towards teaching mathematics as connected instead of viewing contents as unrelated bits. As an example, "Equivalence" is highlighted as one such big idea to foreground to students. As students see equivalence as prominent in a number of school mathematics topics (e.g., congruence as a form of equivalence, equality as a special equivalence, equations can be rewritten into equivalent forms), they will then see them as connected by the undergirding big idea(s).

The context of this study is one where I provide PD to mathematics teachers to help them teach towards big ideas in mathematics. This cohered with the set up to 'test' the theory. Five mathematics teachers formed a team assigned by the research school to participate in the PD which focussed on the big idea of Equivalence. One of them, Teacher Benjamin (Pseudonym), was assigned the role to spearhead the design of a set of IM that is suitable for the teaching of the topic "Solving Quadratic Equations by Factorisation" for Year 8 students. Prior to the PD, Benjamin, in discussion with the rest of the teachers in the team, produced IM-A. During the PD sessions—two 1 hour sessions—we discussed many content and pedagogical issues related to the approach reflected in IM-A. In brief, I highlighted specific areas in IM-A where Equivalence can be made more prominent in a useful way for students. In particular, I pointed out that it is useful for students to see that "equivalence of statements" is the basis of typical working steps in the solution of quadratic equations; and it is also useful for students to spot places where such an equivalence is not maintained, resulting in erroneous steps and hence solutions. I offered specific suggestions as to how these can be represented and the locations within IM-A to flag them. After the PD sessions, Benjamin, with inputs from the other teachers in the team, re-designed IM-B. While Benjamin took on a more active role in this whole process as he led in the design and re-design of the IMs, the other four teachers participated in the PD sessions in terms of asking questions and supplying their inputs to changes. The understanding was that they shared in the ownership of the IM. In this sense, the changes in the IMs reflected not only usable knowledge adopted by Benjamin but also potential usable knowledge for the other teachers in the team. That "supporting teachers" can also benefit from such PD sessions with emphasis on just one teacher doing the enactment of the changes—such as in the case of *Lesson Study*—is shown in Leong et al. (2017). In the study reported here, we focus on the usable knowledge acquired by Benjamin.

The data collected were IM-A and IM-B, and the audio record of a post-lesson PD Session. This session was devoted to reflecting on the relevance of Equivalence in the teaching of this topic and the possibilities of extending its relevance to other future topics. There were three steps in the analysis process: The first was to compare IM-A and IM-B, section by section, for surface similarities and differences—inclusion/exclusion of texts, diagrams, or other types of scaffolds. The second step involved pulling these comparisons together to conjecture plausible overarching reasons—especially with respect to the goal of foregrounding Equivalence—for these moves. The last step was to go to relevant sections of the post-lesson PD session to either strengthen or refute the earlier conjectures.

## Findings

The surface similarities and differences between IM-A and IM-B are given in Table 1.

The conspicuous change in IM-B is the insertion of equation $x^2 - 3x = 0$ in the introductory section. Note that the students up to this point had no prior experience with solving quadratic equations. It seems that the teacher's intention for this insertion was for the students to experience for themselves the non-triviality of "maintaining equivalence" of each of the solution steps in this case—in contrast to the case of solving the linear equation $1 + 2x = x + 2$ which is but a recapitulation of content that was deemed familiar to students. This experience of 'being stuck' would then provide the motivation to know "Zero Product Principle" in order for the equivalence of statements in the working to proceed. This leads naturally to the next section of explicit teaching of the Zero Product Principle. Concretely, as illustrated in Figure 2, the teacher would have expected the students to be able to keep equivalence between Statement 1 and Statement 2, but knew that the students would not be able by themselves at this point to proceed further to reach the goal of finding the value(s) of $x$ that satisfies the equation.

**Table 1**

*Surface Similarities and Differences Between IM-A and IM-B*

| Section of IM | IM-A | IM-B |
|---|---|---|
| Equivalence | *Problem*: if ab = 0 what can you say about a or b? | *Problem*: Solve 1 + 2x = x + 2 (with about half a page of space thereafter) |
| | Watch video on Equivalence | [Removed] |
| | Recall process of solving linear equations as maintaining Equivalence: 2 examples | [Removed] |
| | Definition of Equivalence | [No change] |
| | Maintenance of Equivalence is to be continued for solving quadratic equations | [No change] |
| | | Solve $x^2 - 3x = 0$ (with about a third of a page of space thereafter) |
| Solving Quadratic Equations | Need Zero Product Principle to solve quadratic equations | [No change] |
| | Arithmetic example: 2 × 0 = 0, 0 × 8 = 0, -3 × 0 = 0, 0 × (-7) = 0, 0 × 0 = 0. | [Blanks are included in some of the Arithmetic example] |
| | Textual explanation: "If two numbers are non-zero, their product can never be 0 …" | [Blanks are inserted in the textual explanation] added this: "Just like in solving linear equation, we re-write equations into equivalent forms" |
| | Statement of Zero Product Principle: "If a and b are real numbers such that ab = 0, then a = 0 or b = 0. | Statement of Zero Product Principle: "If a and b are real numbers such that ab = 0 then we can also say its equivalent equations are a = 0 or/and b = 0. |
| | If P and Q are factors of an algebraic expression such that PQ = 0, then P = 0 or Q = 0" | If P and Q are factors of an algebraic expression such that PQ = 0, then P = 0 or Q = 0" |
| Exercises | Worked Example 1: Solve (a) $x(x-2)=0$; (b) $4x^2+6x=0$ | [No Change to all the Worked Examples, Discussion, and Practice Questions] |
| | *Practise Questions 1* | |
| | Discussion: Highlight error of dividing by "x" on both sides of the equation for Worked Example 1(b) | More spaces throughout for working for Practice Questions |
| | Worked Example 2: Solve (a) $(3x+7)(x-4)=0$; (b) $2y^2+7y-15=0$ | Given solutions of Worked Examples removed. |
| | *Practise Questions 2* | |
| | Worked Example 3: (a) Solve $25x^2-9=0$; (b) Explain why $25x^2+9=0$ has no real solutions | |
| | *Practise Questions 3* | |
| | Worked Example 4: Solve $(2y-1)(y-4)=9$ | |
| | *Practise Questions 4* | |

$$x^2 - 3x = 0 \text{ --- (Statement 1)}$$
$$\Leftrightarrow\ x(x-3) = 0 \text{ --- (Statement 2)}$$
$$\vdots$$
$$\Leftrightarrow \qquad x = \underline{\qquad}$$

Figure 2. Equivalent statements leading to the solution of $x^2 - 3x = 0$.

The emphasis of this approach is brought into sharper relief when placed against the introductory section of IM-A: the steps taken do not problematise the solution of quadratic equation by factorisation; it is a theoretical definition and explanation of what equivalence of statements mean in the case of solving linear equations and that one needs to maintain this same equivalence of statements in solving quadratic equations. The "usefulness" of this notion of equivalence was not meant to be experienced by the students. In contrast, IM-B was structured not only to explicitise the usefulness of maintaining equivalence as a working means to 'get to' the answer; it was for students to experience the usefulness for themselves—by getting stuck and then find out (through the equivalence of Zero Product Principle) how to be unstuck. In other words, the usable knowledge that Benjamin brought into the design of IM-B was that Equivalence as a big mathematical idea needs to be seen as useful for students in their work, and not merely as a theoretical idea to be 'covered' in teaching.

That this usable knowledge was derived from PD can be summarised by this exchange during the post-lesson PD Session:

3.50. PDP: [Directed at Benjamin]. Do you find [this emphasis on Equivalence] helpful for your lessons?

4.00. Benjamin: It is helpful. Sometimes we say it but writing it [referring to the symbol for Equivalence "⇔"] helps to draw students' attention to it. Sometimes I may not say it, but if I write it, they know they must maintain Equivalence. …

7.07. PDP: From the beginning [of the PD sessions], I say Equivalence must be seen as helpful. If it is not helpful, don't force it. Otherwise, students will just follow and do for the sake of doing which is not meaningful for them. …

9.17. Benjamin: For me, the parts where the equivalence breaks down in the students' method [referring to Discussion, Worked Example 3(a), Worked Example 4] are important in helping them see the usefulness of maintaining Equivalence.

Other evidences which suggest that Benjamin had the intention to lead students to realise for themselves the usefulness of maintaining Equivalence include the blanks and spaces inserted into IM-B—to provide room for students to grapple with underlying ideas related to equivalences instead of direct demonstration of steps by the teacher. This is alluded to in the above extract (at 9.17)—for example in Worked Example 3(a), he expected students' solution to include the step $x^2 = \frac{9}{25}$, then $x = \sqrt{\frac{9}{25}}$, which will give him the opportunity to again emphasise the relevance of maintaining equivalence.

I would say that the change in IM-B did improve instructional quality. By transforming the presentation of Equivalence from a theoretical "you need to know this term and so I have to tell you what it is" to an experiential "Equivalence is what makes the steps work; and without maintaining it, you will realise it does not work", Benjamin used the big idea of Equivalence as consistent language to help students reason through the steps in solving quadratic equations. This explicit foregrounding of Equivalence of equations/statements in this topic is unusual practice among

Singapore secondary mathematics teachers and it is aligned to the ideals of teaching towards Big Ideas in the recent curriculum revision.

## Discussion

The claim I made earlier was that teacher-designed IM is a suitable site to locate usable knowledge gained from PD. To subject this claim to a preliminary test, I set out, within the context of a broader project, to engage with teachers in my capacity as PDP using a set of IM designed by Benjamin, one of the teachers who participated in the PD. His first design, prior to PD, was based on how he interpreted the expectation of teaching Equivalence as a Big Idea. During the PD Sessions, the discussions were focussed on the IM. Among other suggestions and elaborations, one thing I emphasised was the need to help students see Equivalence as actually useful for them in the learning of the contents in the topic. There is evidence, based on comparing the first design and the final design of the IM, that this emphasis was accepted by Benjamin and explicitly incorporated into his instructional planning. This comparison of IMs allowed me to locate the usable knowledge he derived from the PD Sessions.

My argument actually goes beyond this claim—into the conditions under which teachers are likely to acquire usable knowledge from PD. Since the IM is designed by the teacher, he has high ownership of the contents in the IM and thus the usefulness of these contents in actual classroom instruction. This renders such IMs a suitable starting place to derive usable knowledge, and using the knowledge to tweak contents therein into forms that are even more usable for teachers—all along without losing ownership of the IMs. Although the data reported here were not crafted to address these other parts of the argument, the observation that Benjamin retained much of the practice items—including their sequencing and development—does not contradict the claim of continual ownership (and hence usefulness of changes made) of the IM throughout the whole process.

The points made in the above paragraphs mean that the setup of teachers' designing of IMs is a potentially fruitful endeavour in at least two ways: as a novel mode of PD that targets usable knowledge for teachers; as a research methodological tool to account for teachers' acquisition of usable knowledge. Each of these would have been considered significant contributions; the two-in-one offer renders it all the more tantalising for further exploration of its potentialities. As methodological tool, admittedly, much more needs to go into the rigorous formulation of an analytical frame to compare the IM development that goes beyond the "surface" comparisons attempted in this report. [The purpose of this report is to show that the IM-comparisons *can* show the location of usable knowledge employed by the teacher]. Clearly, further work must be done to develop IM developmental comparisons into a more robust method of analysis.

But as a school-based PD mode, I think some implications can be derived quite directly here. And so I end this paper with a description of the phases of such a PD mode that the reader may consider translating into actual practice as PDP: (1) Identify the intended additional instructional goal(s)—in the case of Benjamin and his colleagues, it is to foreground Equivalence as a Big Idea; (2) Choose a topic that lends itself easiest to the fulfilment of these goals—in this case, Solving Quadratic Equations; (3) First PD Session to provide motivations for the additional goals and ideas on how it is relevant to the topic at hand; (4) Teacher(s) design a first draft pf IM based on their understanding of these additional goals and in accordance with their own goals of teaching the topic; (5) PD Session(s) that focus on realising the additional goals in the topic using this first draft of IM as concrete materials for discussion. Questions by teachers pertaining to design of the IM are discussed/addressed; (6) Teachers' re-design of IM based on discussions of the first draft of IM; (7) Teachers carry out the lessons in class based on the re-designed IMs; (8) Post-lesson PD Session to clarify and summarise usable knowledge acquired by the teachers throughout the process.

The purpose of explicating the PD phases in greater detail is so that other PDPs may attempt this PD mode and hence open up a whole new domain of inquiry into "Teachers' design of instructional materials *as* professional development".

## Acknowledgements

The study reported in this paper is part of a larger research project known as "Big Ideas in School Mathematics Curriculum" (Grant number: OER 31/19 BK) funded by the Office of Educational Research, National Institute of Education, Nanyang Technological University, Singapore.

## References

Cheng, L. P, Leong, Y. H., & Toh, W. Y. K. (2021). Singapore secondary school mathematics teachers' selection and modification of instructional materials for classroom use. In B. Kaur, & Y. H. Leong (Eds.), *Mathematics instructional practices in Singapore secondary schools* (pp. 205–230). Singapore: Springer.

Chin, S. L., Choy, B. H., & Leong, Y. H. (2022). Overlaps and shifts of instructional goals in the design of a set of mathematics tasks. *Mathematics Education Research Journal, 34*(3), 523–549.

Hill, H. C. (2009). Fixing teacher professional development. *Phi Delta Kappan, 90*(7), 470–477.

Kaur, B., Leong, Y. H., & Attard, C. (2022). Teachers as designers of instructional tasks. *Mathematics Education Research Journal, 34*(3), 483–489.

Kersting, N. B., Givvin, K. B., Thompson, B. J., Santagata, R., & Stigler, J. W. (2012). Measuring usable knowledge: Teachers' analyses of mathematics classroom videos predict teaching quality and student learning. *American Educational Research Journal, 49*(3), 568–589.

Leong, Y. H., Cheng, L. P., Toh, W. Y., Kaur, B., & Toh, T. L. (2021). Teaching students to apply formula using instructional materials: A case of a Singapore teacher's practice. *Mathematics Education Research Journal, 33*(1), 89–111.

Leong, Y. H., Tay, E. G., Quek, K. S., Yap, S. F., Toh, W. Y. K. (2017). Uneven teacher learning in lesson study: Towards a theory of teacher learning. *The Mathematics Educator, 17*(1), 79–103.

Leong, Y. H., Yeo, B. W. J., & Choy, B. H. (2022). Instructional materials as a site to study teachers' planning and learning. *Mathematics Education Research Journal, 34*(3), 575–598.

Ministry of Education (2019). *Mathematics (express/normal (academic)) syllabuses*. Singapore: Author.

Renkl, A., Mandl, H., & Gruber, H. (1996). Inert knowledge: Analyses and remedies. *Educational Psychologist*, *31*(2), 115–121.

Wallace, M. R. (2009). Making sense of the links: Professional development, teacher practices, and students achievement. *Teachers College Record, 111*(2), 573–596.

# Planning and Anticipating Early Years Students' Mathematical Responses

Sharyn Livy
*Monash University*
sharyn.livy@monash.edu

Jane Hubbard
*Monash University*
jane.hubbard1@monash.edu

James Russo
*Monash University*
james.russo@monash.edu

This paper reports on early years teachers and how often they should devote planning time to anticipating student responses in advance of the lesson. Sixty-five Foundation to Year 2 teachers (students 5-8 years of age) completed questionnaires at the beginning and end of a year-long research-based professional development program. Participants were learning to teach with sequences of challenging tasks. Post-program data showed a shift in the frequency of time participants believed teachers should devote to anticipating student responses prior to teaching. Supporting teachers' mathematical knowledge for teaching with an emphasis on how they plan and anticipate student responses has implications for improving practice and student outcomes.

In preparation for teaching mathematics lessons, it is recommended that teachers anticipate student responses before the lesson. This practice is crucial as it allows teachers to consider what students might do and how they might respond during the lesson (Smith et al., 2020).

> Through planning, teachers can anticipate likely student contributions, prepare responses that they might make to them, and make decisions, about how to structure students' presentations to further their mathematical agenda for the lesson. (Smith & Stein, 2018. p. 9)

Smith and colleagues recommend that all mathematics teachers develop the practice of anticipating. Yet some teachers who teach Foundation to Year 2 students (5-8 years of age) may have a belief that they do not feel the need to anticipate student responses when planning. They may assume the "mathematics is easy" and they should be able to respond in the moment of teaching without detailed planning.

In our research project Mathematics Sequences of Learning (MSoL), we were interested in identifying if and why early years teachers when teaching sequences of challenging problems (tasks) devote planning time to anticipating student responses in advance of teaching. We sought to respond to the following research questions:

- How often should early years teachers devote planning time to anticipating student responses?
- How do early years teachers explain the practice of anticipating when planning sequences of challenging tasks?

## Literature

### *Theoretical Framework*

Mathematical Knowledge for Teaching (MKT) is informed by subject matter knowledge and pedagogical content knowledge (PCK) (Ball et al., 2008). PCK is a special kind of knowledge that is unique to teachers. When classifying pedagogical content knowledge (PCK), Ball et al. describe three domains, knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum. When demonstrating PCK teachers know what and how to teach a topic in combination with knowledge of instruction, activities, and evaluation tools (Brophy, 1991). Equally, MKT will influence what and how teachers plan for teaching.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 331–338). Newcastle: MERGA.

## *Planning for Mathematics Teaching*

Typically, primary school teachers plan with colleagues and a mathematics leader. These colleagues are often middle-school leaders who specialised in mathematics teaching and are responsible for leading team meetings (Davidson, 2019). Helping teachers consider what they can do prior to teaching has the potential to guide student thinking and discussion as well as mathematical instruction (Stein et al., 2008). Others agree that planning for mathematics teaching is complex (Davidson, 2019; Smith & Stein, 2018; Vale et al., 2019) and should be supported by sharing lesson ideas as a collaborative experience that teachers do together (Ebaeguin & Stephen, 2016). Depending on the size of the school approaches to planning for teaching may vary. In Australian schools, teachers are usually given one hour per week to plan mathematics lessons (Davidson, 2019). They might research curriculum materials and prepare detailed lesson plans (Ebaeguin & Stephen, 2016) filling in a planning proforma (e.g., Smith & Stein, 2018) that caters for different student learning needs.

Differentiating instruction includes teaching practices to support the needs of all students (Tomlinson, 2014). When planning, tasks can be chosen by teachers because of their differing potential (Bardy et al., 2021). Such as, sequences of challenging tasks have been reported to support differentiated learning experiences more effectively than other pedagogical approaches, although this is contingent on the teacher playing an active role to contextualise tasks, using open prompting questions, and facilitating sharing of student work (Russo & Hubbard, 2022). Such actions are supported by access to professional reading allowing teachers to extend their knowledge of theory and practice by identifying key mathematical concepts, ideas, skills, and language for each lesson (Davidson, 2019). Other recommendations when planning for differentiation includes the identification of common misconceptions, anticipating all possible solutions as well as likely student strategies (Smith & Stein, 2018).

## *Anticipating Student Solutions*

When reflecting on teaching problem-solving with secondary students, Wallace (2007) was concerned that rather than “problem-solving” her students were trying to solve the problem (task) the way she wanted. Not wanting to take over students’ thinking, when planning lessons, she began to anticipate both correct and incorrect responses to problems. By anticipating how students might respond to a problem Wallace believed this allowed her to think ahead during the lesson and assisted her questioning for guiding learning.

Teachers should be aware of the importance of planning with others to maximise opportunities for anticipating a range of possible solutions and strategies for solving a task before teaching. When planning with colleagues some teachers might be surprised at the possible number of solutions to a task. For example, a Year 3 teaching team identified 16 anticipated student solutions which were consequently produced by the students during the lesson (Vale et al., 2019). In addition, Stein et al., (2008) suggest that when students solve student-centred instructional tasks, they will solve them in more than one way. In summary, there is agreement that teachers first need to solve the problem themselves prior to teaching (Sullivan et al., 2015). Specifically, anticipating student solutions should include identifying a range of solutions/and or strategies (Smith & Stein, 2018). Others suggest the anticipated solutions can then be ordered as a trajectory of learning that can be considered by the teacher to scaffold student learning during the lesson (Vale et al., 2019).

Acknowledging that students will respond to tasks with a range of different answers, Smith & Stein (2018) designed a model to support teachers when considering ways to guide mathematical discussion during the lesson. They developed a model that included five practices: *Anticipating. Monitoring, Selecting, Sequencing* & *Connecting* for supporting lesson planning protocols. The teachers participating in the MSoL study were introduced to the five practices (Smith & Stein, 2018) and provided with research-informed sequences of lessons (Sullivan et al., 2023) and an instructional model (that included anticipating) for teaching challenging tasks (Bobis et al., 2021). By doing so we hypothesised that teachers would dedicate more planning time to anticipating student responses.

## Method

### *Context and Participants*

Participants included Foundation Year to Year 2 teachers and mathematics leaders (N=96) from 19 Catholic Primary Schools in Australia. They were participating in a year-long research-designed professional learning program MSoL related to teaching with sequences of challenging tasks (Sullivan et al., 2023). The aim of the program was to extend teachers' MKT when implementing research-designed resources 'Exploring Mathematical Sequences of Connected, Cumulative and Challenging Tasks' ($EMC^3$) (LP180100611). Fifteen sequences of lessons (provided in a resource book) and an instructional model (Bobis et al., 2021) were designed to support teachers to build new understandings of student-centred approaches to teaching mathematics in the early years.

The participants attended three professional learning days (April-July-October) with the research team (authors). Most participants were familiar with teaching challenging tasks, but not the resources provided as part of their professional learning. Three days of professional learning were designed to support teachers' PCK and provided strategies for planning and implementing the sequences of lessons.

Professional development can be effective when it focuses on situations in practice (e.g., Lipowsky & Rzejak, 2015). Therefore, Day 1 introduced teachers to the student-centred inquiry approach. Day 2 included a session on the '*5 Practices for Orchestrating Productive Mathematics Discussions*,' and planning approaches (Smith & Stein, 2018) and questioning strategies (Livy et al., 2021). Day 3 focused on assessment practices and rubrics (Hubbard et al., 2022). In addition, two different sequences were introduced to the teachers for each day of professional learning. This included anticipating possible solutions and strategies students might choose, including misconceptions or partial conceptions, when solving the task.

### *Data Collection*

Sixty-five participants responded to a 20-minute pre-program and post-program online Qualtrics questionnaire at the beginning of Days 1 and 3. In educational settings, online questionnaires are often used as a method of data collection because they have a better response rate and are more reliable when compared to pen-and-paper surveys (Seleh & Bista, 2017). The 16-item questionnaire included six demographic items and five Likert-style items each followed by an open question asking teachers to explain their responses to the previous Likert-style item. Two items (Q15 and Q16) are reported in this paper.

For the four-point Likert-style item Q15 pre- and post-questionnaire participants were asked, "When planning for teaching with challenging tasks should Foundation to Year 2 teachers devote planning time to anticipating student responses in advance of the lesson?" and selected *never*, *sometimes*, mostly, or *always.* Since we were particularly interested in participants' responses after the intervention, we have only reported on post-questionnaire responses to the question (Q16), "Explain why you think this?"

### *Data Analysis and Coding*

For Q15, data were entered into a statistical software suite (SPSS). A Wilcoxon signed-rank test was used to compare the data of the pre- and post-questionnaire responses to determine if the two samples showed a statistically significant change. The results and analysis included a report of the frequency of responses to four-point Likert-style responses of *never*, *sometimes*, mostly, and *always* (see Table 2 results). For Q16, participants' responses were entered into an excel spreadsheet for coding. The first two authors independently used open coding, they met to discuss, collate codes, and then agreed on six codes as shown in Table 1. Longer responses were coded using two or three codes. Codes were checked by the third author.

**Table 1**

*Open Coding Categories*

| | Code | Description of code | Example from text |
|---|---|---|---|
| 1 | Differentiation of student knowledge | Prepare prompts; plan for responses; cater for everyone | Create enabling and extending prompts |
| 2 | Misconceptions | Identifying misconceptions | Know what the misconception might be |
| 3 | Orchestrating discussion | Planning questions; when to discuss student responses; anticipating responses | It helps to have knowledge so that you can so you can talk about strategies |
| 4 | Scaffolding the lesson | Guiding learning; considering how to adapt the lesson; selecting student work samples to share | Being prepared for their responses can help us come up with ways to guide students |
| 5 | Time constraints | Reference to time as a barrier when planning | Limited planning time |
| 6 | Other | The response did not match previous codes | It's good to keep an open mind |

The frequency of responses to Q15 was tallied and converted to percentages (see, Table 2). For Q16 the number of responses for each code was tallied. Next the percentage of participants whose responses matched each code was calculated (see, Table 3).

## Results and Discussion

The results and discussion include a comparison of both pre-program and post-program Likert-style items (Q15) followed by the results and discussion of the open-response item (Q16) collected at the end of the program.

### *Frequency of Planning Anticipated Responses*

For Q15 the comparison of pre-program and post-program responses reveals differences between teachers' rankings ranging from 'Never' (1) to 'Always' (4) (Table 2). The results in Table 2 show a shift from 'Sometimes and Mostly' to 'Mostly and Always' when comparing the percentage of participants' responses pre-program and post-program. A total of 60% of participants shared the belief that teachers should either sometimes or mostly anticipate student responses before teaching. After participating in the project and extending their knowledge for planning and anticipating student solutions and strategies 86% of teachers agreed anticipating should occur mostly or always.

**Table 2**

*Q15 Pre- Post-frequency 14 (N=65)*

| Response | Pre-course frequency (%) | Post-course frequency (%) |
|---|---|---|
| 1. Never | 3% | 0% |
| 2. Sometimes (i.e., in a minority of lessons when teaching with challenging tasks) | 20% | 14% |
| 3. Mostly (i.e., most lessons when teaching with challenging tasks) | 40% | 34% |
| 4. Always (i.e., every lesson when teaching with challenging tasks) | 37% | 52% |

In addition, a statistical analysis of these results compared the mean ranking of the pre- and post-program questionnaire data using a Wilcoxon signed-rank test. The results showed that participating in the professional learning program elicited a statistically significant change when considering the importance of devoting planning time to anticipating student responses in advance of the lesson ($Z = 2.660$, $p = 0.008$). Indeed, the median rating shifted from 4.00 as something teachers should mostly do when teaching challenging tasks in the early years of primary school to 5.00 as something they should always do.

The next section reports and discusses the participants' explanations of their response to planning time and anticipating student responses.

*Reasons for Planning Anticipated Responses*

For Q16 the participants were asked to explain their responses to Q15 (Table 3).

**Table 3**

*Frequency of Codes Per Participant to Question 15 (Devoting Planning Time) Post Program*

| | Code | Frequency of responses (n=89) | Percentage of participants (n=65) |
|---|---|---|---|
| 1 | Differentiation of student knowledge | 31 | 48% |
| 2 | Misconceptions | 9 | 14% |
| 3 | Orchestrating discussion | 10 | 16% |
| 4 | Scaffolding the lesson | 26 | 40% |
| 5 | Time constraints | 9 | 14% |
| 6 | Other | 4 | 6% |

Table 3 includes the six codes used to code all participant responses and the frequency of responses (n=89) for all codes. The final column is the percentage of codes for the number of participants (n=65) and the total percentage is greater than 100 percent because sometimes more than one code occurred in one teacher's response. For example:

> When you plan for student responses, it allows you to scaffold their learning and plan for tasks based on the student's learning ability. (Codes 1 and 4)

The most common explanations related to Code 1 'differentiation of student knowledge' and Code 4 'scaffolding the lesson.'

### *Differentiation of Student Knowledge*

Nearly half (48%) of the participants' responses were coded as Code 1 highlighting teachers' acknowledgement of the importance of differentiation of student knowledge when anticipating. For example, one teacher considered the difficulty of the task, and another acknowledged the importance of considering the range of answers:

> To determine the level of difficulty and whether the students will be able to understand, [or] gain anything from the task. (Codes 1 and 4)

> To support their learning and understand their thinking. (Code 1)

Many of the other responses referred to supporting learning including the planning of *enabling* or *extending prompts* to use during the lesson. Enabling prompts are designed for students experiencing difficulties and extending prompts are designed for students who may complete the main task quickly (Sullivan et al., 2015). Participants were introduced to this practice of supporting student differentiation by using these prompts.

### *Scaffolding the Lesson*

More than one-third (40%) of participants' responses were coded as Code 4 and made connections with scaffolding the lesson. One teacher revealed, "It helps to have an idea of where the lesson might go and to be prepared for that." Other examples include, "to know what the next step is" as well as "to direct how the lesson will go."

These responses indicated the teachers' developing understanding of scaffolding learning by posing questions designed to extend student learning. However, it is important not to take over student thinking but for teachers to guide learning (Wallace, 2007).

Whereas another teacher mentioned how the school leader supported planning.

> I especially like doing the task together as a year level to guide us in anticipating student responses. (Codes 1 and 4)

This response confirms the importance of collaborative planning. Davidson, (2019) would agree that collaborative experiences will support teaching practices.

In summary, Codes 1 and 4 show evidence of PCK because these teachers were making connections with anticipating and differentiating or scaffolding the lesson, which could support both their KCS and KCT for teaching.

### *Code 5: Time Constraints*

Some responses (14%) focused on time constraints as a barrier. This should be addressed by school leaders and schools to ensure teachers have time to plan, especially when learning new approaches for teaching and planning. Examples of time constraints included:

> Time is restricting in planning, so it would be impossible to do it for each challenging task. (Code 5)

Another reason for not being able to plan for every lesson included the time needed for the detail expected when planning and anticipating student responses for each lesson:

> The planning process is very thorough and would be difficult to fit in with other curriculum requirements. (Code 5)

Whereas one teacher wrote about the importance of finding time to plan:

> It would be amazing to have the time to do it for all maths planning sessions however it is not realistic. It is so important to make the time if possible to prepare how you would respond. (Code 5)

Time constraints as barriers were because the teachers were not provided enough time to plan the detail they desired and needed prior to teaching a sequence of lessons. These findings were not surprising given that typically teachers are provided with one hour for planning mathematics lessons each week in primary schools (Davidson, 2019).

These results may have been influenced by different factors including participation across the three days of the intervention. Although not reported, perhaps teachers previously devoted time to sourcing tasks for teaching, limiting time for anticipating student responses. A benefit of using the EMC$^3$ resources and lesson approach they experienced during the intervention suggests that teachers were able to shift their thinking related to anticipating prior to teaching. Changing practices when planning could assist teachers to improve their MKT by developing an awareness of different ways to solve tasks. To support the practice of anticipating we recommended that teachers plan collaboratively to ensure they identify all possible solutions and think deeply about what students are likely to do and how they will approach the problem (Smith et al., 2020).

## Conclusion

As part of the MSoL project, we were interested in supporting teachers' PCK including when planning to teach with sequences of challenging tasks. Prior knowledge of teaching challenging tasks may have influenced these results. When participating in professional learning, prior knowledge can either assist teachers to extend their knowledge or hinder learning (Lipowsky & Rzujak, 2015). The findings of our study suggest that the program extended participants' PCK about planning and anticipating student responses because in the post-questionnaire three-quarters of Early Years teachers from 19 different schools reported that teachers should mostly or always anticipate student responses before teaching. When explaining the practices of anticipating they focused more on approaches for differentiation of student knowledge and scaffolding learning. Other approaches included a focus on the identification of possible misconceptions and orchestrating discussions. Each of these responses suggests enhancement of teachers' PCK including KCS and KCT.

A barrier for some teachers included time constraints. Not stated by the participants of this study but an implication worth noting for teachers is that they would have more time for anticipating student responses to the task because they were provided with fourteen sequences of learning experiences (challenging tasks) (Sullivan et al., 2023). As a result, the project teachers had more time during their planning to extend their knowledge for teaching by anticipating student responses. Such practices suggest these teachers can be better prepared for teaching than in the past. It would be expected as teachers become more familiar with the tasks, anticipating student responses and planning teacher actions will be less demanding the following year if teachers consolidate their teaching with the same year level. However, teachers may begin to plan their own sequences as they become familiar with the EMC$^3$ approach, which would take time away from anticipating student responses. In summary supporting teachers' MKT with an emphasis on how they plan and anticipate student responses has implications for supporting student learning and approaches for improving teachers' KCS and KCT. This study was limited by its focus on self-reported teacher data. Further research including longitudinal data, observing teachers' planning and follow-up interviews are needed to extend the current study's findings.

## References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5)*,* 389–407. https://doi.org/10.1177/0022487108324554

Bardy, T., Holzäpfel, L., & Leuders, T. (2021). Adaptive tasks as a differentiation strategy in the mathematics classroom: Features from research and teachers' views. *Mathematics Teacher Education and Development, 23*(3), 26–53.

Bobis, J., Russo, J., Downton, A., Feng, M., Livy, S., McCormick, M., & Sullivan, P. (2021). Instructional moves that Increase chances of engaging all students in learning mathematics. *Mathematics*, *9*(6), 582. https://doi.org/https://doi.org/10.3390/math9060582

Brophy, J. E. (1991). Conclusion to advances in research on teaching: Teachers' knowledge of subject matter as it relates to their teaching practice. In J. E. Brophy (Ed.), *Advances in research on reaching researchers' subject matter knowledge and classroom instruction* (Vol. 2, pp. 347–362). JAL Press.

Davidson, A. (2019). Ingredients for planning student-centred learning in mathematics. *Australian Primary Mathematics Classroom*, *24*(3), 8–14.

Ebaeguin, M., & Stephens, M. (2016). Key elements of a good mathematics lesson as seen by Japanese junior high school teachers. In B. White, M. Chinnappan, and S. Trenholm (Eds.), *Opening up mathematics education research. Proceeding of the 39th annual conference of the Mathematics Education Research Group of Australasia (pp. 206–213).* MERGA

Hubbard, J., Russo, J., & Livy, S. (2022). Assessing mathematical competence through challenging tasks. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 282–289). Launceston: MERGA.

Lipowsky, F., & Rzejak, D. (2015). Key features of effective professional development programmes for teachers. *Ricercazione, 7*(2), 27–51.

Livy, S., Bobis, J., Downton, A., McCormick, M., Russo, J., & Sullivan, P. (2021). Teacher actions for consolidating learning in the early years. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in mathematics education: Foundations and pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 273–280). Singapore: MERGA.

Russo, J, & Hubbard, J. (2022). Changing teacher practices: Differentiating mathematics instruction through sequences of challenging tasks in the early primary years. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 16–19). MERGA.

Smith, M., Bill, V., & Sherin, M. G. (2020). *The 5 practices in practice: Successfully orchestrating mathematics discussions in your elementary classroom*. Corwin Press.

Smith, M., & Stein, M. (2018). *5 practices for orchestrating productive mathematical discussion* (2nd ed.). National Council of Teachers of Mathematics.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340. https://doi.org/10.1080/10986060802229675

Sullivan, P., Bobis, J., Downton, A., Livy, S., McCormick, M., & Russo, J. (2023). *Maths sequences for the early years: Challenging children to reason mathematically.* Oxford University Press.

Sullivan, P., Walker, N., Borek, C., & Rennie, M. (2015). Exploring a structure for mathematics lessons that foster problem solving and reasoning. In V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th annual conference of Mathematics Education Research Group of Australasia* (pp. 41–56). MERGA.

Saleh, A., & Bista, K. (2017). Examining factors impacting online survey response rates in educational research: Perceptions of graduate students. *Journal of MultiDisciplinary Evaluation, 13*, 63–74. https://eric.ed.gov/?id=ED596616

Tomlinson, C. A. (2014). *The differentiated classroom: Responding to the needs of all learners*. ASCD.

Vale, C., Widjaja, W., Doig, B., & Groves, S. (2019). Anticipating students' reasoning and planning prompts in structured problem-solving lessons. *Mathematics Education Research Journal*, *31*(1), 1–25. https://doi.org/10.1007/s13394-018-0239-5

Wallace, A. H. (2007). Anticipating student responses to improve problem solving. *Mathematics Teaching in the Middle School*, *12*(9), 504–511. http://www.jstor.org/stable/41182474

# Using a Triple Number Line to Represent Multiple Constructs of Fractions: A Task Design Process and Product

Tarryn Lovemore
*Rhodes University,*
*Nelson Mandela University*
tarryn.lovemore@mandela.ac.za

Sally-Ann Robertson
*Rhodes University*
s-a.robertson@ru.ac.za

Mellony Graven
*Rhodes University*
m.graven@ru.ac.za

This paper reports on a key representation, a triple number line, designed as part of the first author's doctoral study. The study sought ways to represent multiple constructs of fractions in the context of merging music and mathematics to support learners' understanding of fractions. A problem scenario was designed guided by Realistic Mathematics Education principles. Findings shared in this paper are based on the process of designing and implementing the tasks around the triple number line. Data for this qualitative, participatory dual-design experiment in task design were collected via formal and informal interviews in two micro-Communities of Practice. We conclude that the key representation of the triple number line can be a powerful tool for supporting learners in their fraction understanding.

The focus of this paper is the use of a triple number line (three number lines in parallel) as a key representation designed to support grade 4 and 5 learners (9 to 11 years old) in moving flexibly between different constructs of fractions (in particular, the part-whole, ratio, and measure constructs). We also share some of the process we went through in arriving at this key representation. The broader design-research study sought ways for integrating music and mathematics to develop and support fraction understanding. The research question we intend to answer in this paper is:

- How might one connect problem scenarios and music-mathematics integrated representations to deepen understanding across multiple constructs of fractions?

Guided by Realistic Mathematics Education (RME) (Freudenthal, 1991), an imaginary problem scenario was designed followed by a sequence of eight lessons in which music and mathematics representations were used. This paper hones in on the final two lessons of the sequence to show some of the ways in which the key triple number line representation was used to support learners in solving problems relating to fraction understanding. Data for the study were collected via two micro-Communities of Practice (micro-CoPs), adapting Lave and Wenger's notion of CoP (1991). A designer/researchers' micro-CoP and a researcher/teachers' micro-CoP were initiated. Analysis of these data provides insight into the design process that guided us towards implementing the triple number line. Although the broader study was designed and trialled in a South African context, we see it as having potential value for wider contexts given the universal challenges so often encountered in the teaching and learning of fractions (Cortina et al., 2015; Getenet & Callingham, 2021; Siemon, 2003; Streefland, 1991).

## Literature Review

### *Multiple Constructs of Fractions*

Fractions are a vital part of learning mathematics. Independently of their usefulness in everyday life situations, they contribute to developing proportional reasoning and algebraic thinking (Barbieri et al., 2020; Siemon, 2003). A challenge in teaching and learning fractions is that learners often have the misconception that whole number properties can be applied to fractions, for example, adding numerators and denominators as separate numbers, rather than considering a fraction as a number

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 339–346). Newcastle: MERGA.

on its own (Cortina et al., 2015; Getenet & Callingham, 2021; Siemon et al., 2015; Streefland, 1991). Working with the multiple constructs of fractions (namely, fraction as measure, fraction as quotient, fraction as ratio, fraction as operator, and the part-whole fraction model) poses a further challenge in the teaching and learning of fractions (Behr et al., 1983; Getenet & Callingham, 2021; Siemon et al., 2015). These constructs (or meanings) of fractions, despite being referred to separately, are not discrete categories, but rather allow for multiple, interrelated ways of understanding and sense-making of the same situation (Siemon et al., 2015). Lamon (1999, p. 41) explains, the "meaning of fractions derives from the contexts in which they are used".

Often the sole focus for fractions at the primary school level is on the part-whole construct (dividing a pizza into equal parts, for example). This, however, is pedagogically insufficient (Barbieri et al., 2020; Getenet & Callingham, 2021). Much more advantageous is providing learners opportunities to work with the multiple constructs of fractions and to recognise the connections across them (Charalambous & Pitta-Pantazi, 2007; Shahbari & Peled, 2014; Siemon et al., 2015). For this reason we strove to find a problem scenario and representation that allowed for movement across the multiple constructs of fractions, creating opportunities to view a situation, and solve problems around it, using multiple meanings of fractions. As noted, we focused specifically on three constructs we see as connecting well with our intention to integrate music note values and rhythm into the teaching and learning of fractions: the part-whole construct—a set of "discrete objects or a continuous amount that can be divided into parts of equal size" (Shahbari & Peled, 2014, p. 373); the fraction as ratio construct—a comparison between two quantities (Charalambous & Pitta-Pantazi, 2007) conveying their relative magnitude (Shahbari & Peled, 2014); and the fraction as measure construct—a representation of the size of measured lengths as measured from a point on a number line (Cortina et al., 2015).

### *The Number Line as a Supportive Representation in Fraction Understanding*

Number lines are well-recognised as having the potential to be a key representation for developing fraction understanding (Barbieri et al., 2020; Saxe et al., 2013; Soni & Okmoto, 2020). Barbieri et al. (2020) identify number lines as mathematically accurate ways for visually representing fractions. Monson et al. (2020) too, note that a number line is a useful model for demonstrating that a fraction is a single point on a number line, and therefore, a measure from 0 (thus emphasising the fraction as measure construct). Despite their usefulness for representing fractions, however, as Barbieri et al. (2020) note, number lines are seldom used in teaching and learning fractions due to various challenges. Barbieri et al. (2020) and Charalambous and Pitta-Pantazi (2007), for instance, caution that learners often count the partition markings of the number line instead of the spaces between them (for example reading quarters on a number line as fifths or thirds). Siemon and Luneta (2018) observe that learners might misconstrue the full number line as the unit that needs to be divided into equal parts rather than the distance 0 to 1 iterated a number of times. Recognising that learners may find it confusing to see equivalence on a single point on a number line, Siemon and Luneta (2018) recommend the use of fractions strips to introduce fractions on a number line.

Possible misconceptions notwithstanding, a single number line is a powerful means of supporting learning when working with equivalent fractions and fractions greater than one whole (Monson et al., 2020; Siemon et al., 2015). A double number line, where two lines run parallel and where there is a relationship between the scales of each line (Orrill & Brown, 2012), is a supportive visual representation when working with proportional reasoning and ratios. Double number lines are often used, for example, to solve problems relating to speed involving a scale of distance and time (Nabb, 2023). The value of a triple number line is highlighted in Cher's (2022) design of an electronic interactive triple number line to support learners in visualising solutions to equations.

In considering the literature on the challenges of teaching and learning fractions, the benefits of experiencing the multiple constructs of fractions, and the potential of single, double and triple number lines as key representations, we resolved to include them in our task design for integrating fractions and music. We believe our trialling of and reflecting on, in particular, the use of a triple number line to solve problems involving multiple, interrelated constructs of fractions can contribute both to the extant literature body and to practice.

### *Realistic Mathematics Education as a Starting Point*

With the theoretical framing of RME (Freudenthal, 1991) we designed an experientially real starting point or problem scenario for the 8-lesson sequence. We found Cobb et al.'s (2008) three tenets of RME useful in the design of the integrated music-mathematics tasks: namely, that a meaningful mathematics task should (i) have an experientially real starting point, (ii) allow for informal reasoning and representing, which subsequently leads to (iii) formal representation and vertical mathematisation. The starting point need not be real-world. It should, however, be a *real experience* with which learners can engage. Even an imaginary fairy tale or folktale could be a meaningful starting point from which a need to use mathematics authentically emerges (van den Heuvel-Panhuizen, 2003).

For the start of the lesson sequence we contrived a folktale-type story (inspired by southern African wildlife and culture) to serve as our problem scenario: different animals crossing a river in different ways during a seasonal migration (see Lovemore, 2023). In this initial task, teachers guided learners in playing a game, imagining that they were different animals that had to make jumps on stones across a river, represented by a marked constant distance. A zebra, for example, would take four equal-sized jumps to cross the constant river-crossing; an ostrich would take two equal-sized jumps per river-crossing. This constant river-crossing unit became the unit that guided our follow-up tasks. With each jump the learners took to cross the imaginary river, the rest of the class would clap, thus resulting in rhythmical beats as the different animals crossed the river. Thus, the movement and clapping were a real experience for the learners from which to base follow-up tasks. Learners were then asked to informally represent their jumps per river-crossing unit. This was extended to continuing the jumps beyond one river-crossing unit, enabling representations and problem-solving with fractions greater than one whole. We recognised in our designer/researchers' micro-CoP the potential this problem scenario had as the starting point in our task design journey towards supporting fraction understanding.

## Methodological Decisions

The broader study was a participatory dual-design experiment in task design, after Gravemeijer and van Eerde's "dual-design experiment" (2009, p. 259) whereby, through an intervention, both learners and teachers are afforded opportunities to learn something new. In the case of the present study, the dual-design involved the researchers' and teachers' learning. Researchers learnt through the design process, including feedback received from the participating teachers. In turn, the teachers were learning through their implementation of, and careful reflection on, the designed tasks. The designer/researchers and teachers operated within their respective micro-CoPs as co-researchers (Makar, 2021). The first author met with the second and third authors in the designer/researchers' micro-CoP for initial task design. (All three authors agreed to have their first names used in publications from the study). The first author then shared the task resources with the participating teachers in the researcher/teachers' micro-CoP. The two teachers discussed in the present paper were a Grade 4 teacher, Ms Savuka, and a Grade 5 teacher, Ms Clegg (pseudonyms). They both taught at the same independent school in the Eastern Cape Province of South Africa. Their participation in the study was wholly voluntary. Both were given the assurance at the outset that they were at liberty to withdraw from it at any stage. They were also given the assurance that their identities and that of their school would remain confidential. Their contribution throughout the study was to interrogate and reflect on the intended design of the tasks and to suggest possible adjustments ahead of their actual implementation and to then trial these tasks with their learners. Their post-implementation reflections were taken back to the designer/researchers' micro-CoP to allow for ongoing refinement of the task design. Data were collected mainly through unstructured discussions and interviews within the two micro-CoPs. The thematic analysis process for this paper was then achieved using NVivo software to deductively code for RME principles and for multiple constructs of fractions.

## Findings and Implications

Discussion in this section is divided into two phases: Phase 1, dealing with the designer/researchers' process of designing the two-lesson task and the triple number line representation; Phase 2, dealing with the teachers' implementation of, and reflection on, the task.

### *Phase 1: The Task Design Process and Product*

In our task design, learners could relate the animal jumps and musical claps to the fraction as ratio construct (for example, four zebra jumps/claps per river-crossing unit); the fraction as measure construct (distance and time of jumps); and the part-whole construct (equally dividing a whole river-crossing unit into smaller jumps). Having recognised the potential of the problem scenario for learners to experience using multiple constructs of fractions we then sought ways for learners to informally represent these constructs in the form of animal river-crossing jumps. Figure 1, below, shows a learner's informal representation of Zebra, Ostrich, Kudu (a South African antelope), and Monkey jumps, as told in the made-up folktale.

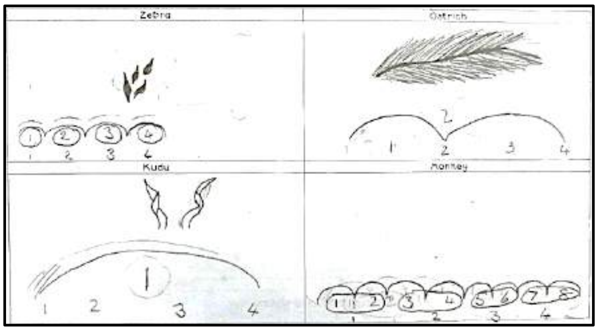


Figure 1. A Grade 5 learner's representation of animal river-crossing jumps.

The informal representations also held the opportunity to guide learners towards more formal representations on a number line, just as is suggested in RME theory (Cobb et al., 2008). In our designer/researcher micro-CoP we initially encountered an obstacle as to how we could represent the fraction as ratio and measure constructs on the same linear model. The following excerpt from the third author's reflection summarises our 'AHA-moment' when we realised that it was not necessary to merge the two constructs. Rather, we could use the problem scenario as a way to *align* the various fraction constructs as well as their musical and mathematical representations. We recognised that we could work simultaneously with the jumps per river-crossing (fraction as ratio) and the iterations of the jumps (fraction as measure of distance and time).

> Mellony: I think this is a big AHA, on exactly why these concepts are getting confused. Rate and fraction [as measure] are of course interrelated, but they're conceptually so different, and yet so similar, so one conflates… What you need to help teachers see is that we're working simultaneously with two fraction concepts, fraction as rate (jumps per river crossing). … but with this animal crossing thing, we can also link this to fraction as measure. [Designer/researcher micro-CoP, 2022-01-25].

We therefore built on the learners' informal representations of animal jumps per river-crossing by getting the learners to draw their animal jumps onto a river-crossing unit number line. To link the animal jumps and claps to a linear musical representation, we designed note value cards printed on transparency film that could be placed to fit exactly into the musical bars aligning with the river-crossing unit distance. (See more in Lovemore et al., 2022; Lovemore, 2023). This option mirrors Siemon and Luneta's (2018) recommendation of using fractions strips to introduce fractions on a number line. Our transparent music note value cards served as a form of fraction strip. Figure 2 below shows the alignment between the river-crossing unit animal jumps, the linear musical representation, and a subsequent matching task requiring learners to match animal fraction strips to both the musical note value strips and the animal river-crossing jumps.

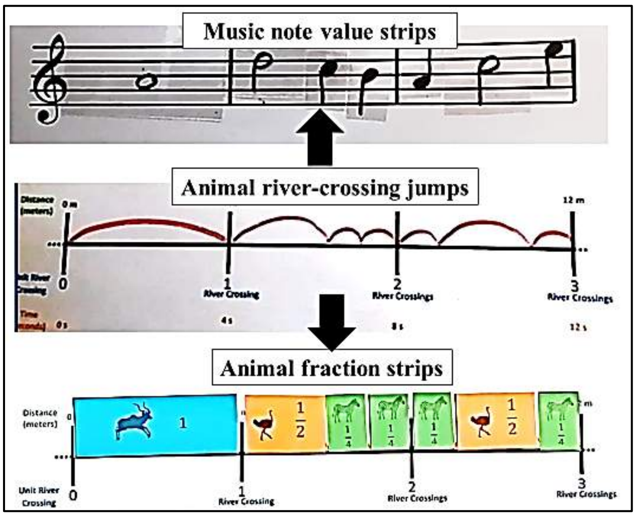



Figure 2. Linear musical and mathematical representations aligned.

The insights shared in Figure 2 led us to the realisation that in fact our problem scenario allowed for the creation of a triple number line (Figure 3). We saw the three parallel number lines as further facilitating the making of conceptual links between and across the fraction as ratio and fraction as measure constructs.

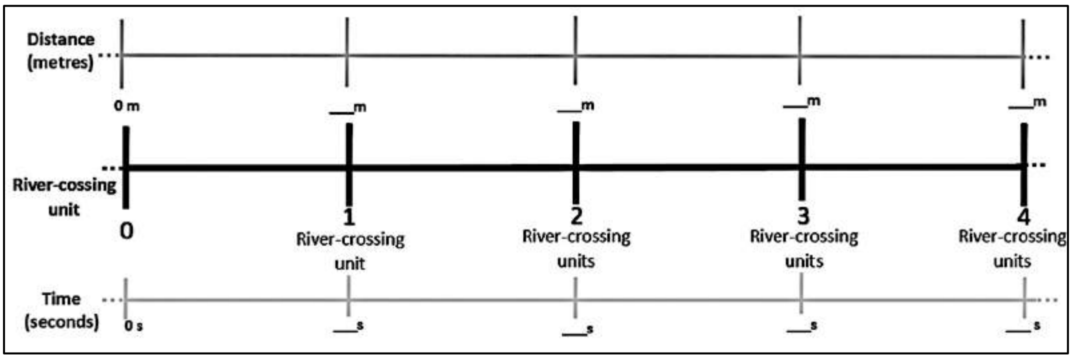



Figure 3. Triple number line.

In our triple number line representation, the river-crossing unit stays constant. The number line above the river-crossing unit indicates measurement of distance and the number line below it represents a timeline. The units of distance and time could indicate different variables parallel with the river-crossing unit, thus creating meaningful opportunities for problem-solving. Figure 4 below illustrates how the triple number line might be used to solve a problem exploring the speed at which the animals jump.

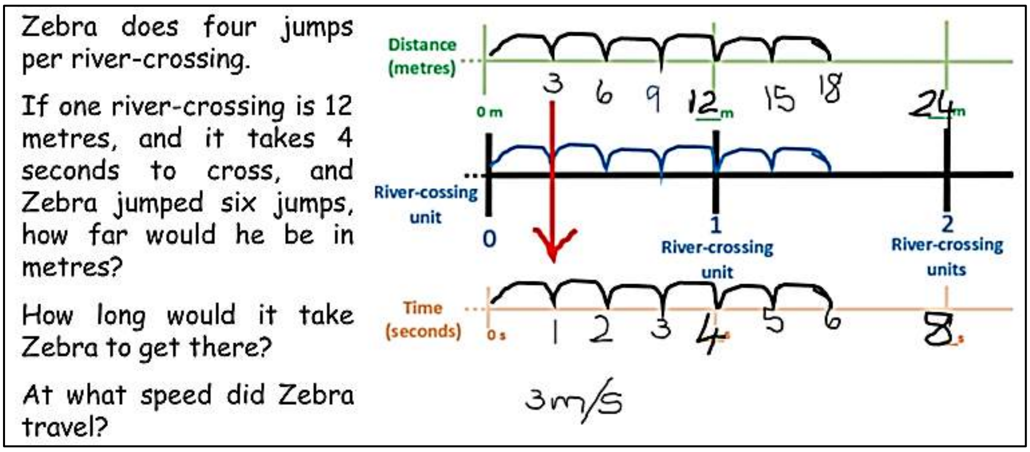



Figure 4. Designer/researchers' demonstration of using the triple number line to solve problems.

We anticipated that this triple number line representation had the potential to support learners in solving complex fraction problems. Below is an excerpt from our designer/researcher micro-CoP discussion on the value of using the constant river-crossing unit while varying the distance and time lines so as to allow flexibility in working with fractional understanding and proportion.

> Mellony: When we get to problem-solving, that's when we're really getting to the powerful fraction stuff, because we can change that. We will say, if the river is 10m wide... now what? So the river-crossing is the main unit, completely in the foreground in our head, and then the other two variables change.
>
> Tarryn: And we're getting to developing the deep conceptual understanding.
>
> Mellony: So the teachers have to know that the kids are not expected to answer any of those questions without the image of the triple number line... So speed is coming alive, ratio and proportion is coming alive. [Designer/researcher-CoP, 2022-02-01].

### *Phase 2: Teachers' Reflections on the Task Design Product*

Within the second micro-CoP, the two participating teachers reflected on their implementation of the tasks, including the use of the triple number line. For the purpose of the current study, learners' work was not analysed, but rather teachers' feedback on their experiences of implementing the task. Both teachers reported on how their learners relied on the visual representation of the number line to support them in their understanding and in then solving the problems requiring fractional understanding. The Grade 4 teacher, Ms Savuka, noted that her learners relied heavily on the triple number line as a 'crutch' to solve the problems. The Grade 5 teacher, Ms Clegg, explained, by contrast, that there had been more variation in her class: some learners used the triple number line representations to solve the problems, others appeared not to need to do so. Shared below are some of Ms Savuka's and Ms Clegg's reflections on the value of the triple number line as a key representation.

> Ms Savuka: It's something that's new to them. So showing that not only [whole] numbers on the number-line but fractions too.
>
> Ms Clegg: I don't think they would have managed this without [the number line].
>
> Ms Savuka: They definitely needed it… I was a bit apprehensive. I thought, now introducing distance and time, how are they going to be able to answer problem-solving questions? And I was blown away… especially in Grade 4, problem-solving and word sums is something that so many of them struggle with. And I just saw the benefit of having a visual representation. That really helped them a lot.
>
> Ms Clegg: Maybe that's something that we can look at just in our general maths. Because I agree, where they go from very concrete, and then we should be semi-abstract, and suddenly we jump into, now they must read an entire paragraph, know what all the fancy maths words mean, and there's no pictures.
>
> Ms Savuka: Nothing they can use to answer the questions.

> Ms Clegg: Yes, whereas, having a basis like a number line, or a picture that forms a number line. Or something like that, that they can actually draw on. [Researcher/teacher micro-CoP, 2022-08-15].

These reflections highlight the teachers' recognition of the importance of visual representations for supporting learners' fraction understanding and their application in problem solving. Their responses align with the literature on the use of number lines in supporting fraction teaching and learning (Barbieri et al., 2020; Saxe et al., 2013; Siemen & Luneta, 2018; Soni & Okmoto, 2020), including using double number lines to aid proportional reasoning (Nabb, 2023; Orrill & Brown, 2012). In line with the RME principles guiding task design (Cobb et al., 2008; van den Heuvel-Panhuizen, 2003), Ms Savuka and Ms Clegg both indicated that they felt their learners were successfully guided from the starting point of the problem scenario, through to informal representations (animal river-crossing jumps and claps), and then on to the formal abstract representation of fractions on a number line, whereafter they were able to use a triple number line to solve complex problems requiring flexible movement between and across multiple constructs of fractions.

## Conclusion

As noted, this paper shares a part of the first author's doctoral study which strove to integrate music into mathematics through RME-guided task design. The authors here report on how the problem scenario and designed triple number line allowed for the design of tasks requiring that learners move flexibly between the interrelated constructs of fraction as measure, fraction as ratio and part-whole. From the data, the designer/researchers' realisation that the multiple constructs could be aligned, yet not conflated, is an example of the process of a task design journey. The participatory nature of the journey included teachers' reflections on the triple number line as a key resource to support young learners in solving complex problems. Literature and the findings from this study show the value of using a number line, and particularly a triple number line, to support teaching and learning of complex fraction constructs. While this was a small-scale, qualitative study, future opportunities for research exist in, for example, the form of a quantitative study comparing pre- and post-test results to evaluate the effectiveness of using a triple number line as a key representation in solving complex fraction problems.

## References

Barbieri, C. A., Rodrigues, J., Dyson, N., & Jordan, N. C. (2020). Improving fraction understanding in sixth graders with mathematics difficulties: Effects of a number line approach combined with cognitive learning strategies. *Journal of Educational Psychology, 112*(3), 628–648. https://doi.org/10.1037/edu0000384

Behr, M., Lesh, R., Post, T., & Silver E. (1983). Rational number concepts. In R. Lesh, & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–125). Academic Press.

Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics, 64*(3), 293–316. https://doi.org/10.1007/s10649-006-9036-2

Cher, D. (2022, May 26). *A triple number line model for visualizing solutions to equations*. Albegra, Math Software. http://www.sineofthetimes.org/a-triple-number-line-model-for-visualizing-solutions-to-equations/

Cobb, P., Zhao, Q., & Visnovska, J. (2008). Learning from adapting the theory of realistic mathematics education. *Education & Didactique*, *2*(10), 154 –124. https://doi.org/10.4000/educationdidactique.276

Cortina, J. L., Visnovska, J., & Zúñiga. C. (2015). An alternative starting point for fraction instruction. *International Journal for Mathematics Teaching and Learning,* 1–28. http://www.cimt.org.uk/ijmtl/index.php/IJMTL/issue/archive

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Kluwer Academic.

Getenet, S., & Callingham, R. (2021). Teaching interrelated concepts of fraction for understanding and teacher's pedagogical content knowledge. *Mathematics Education Research Journal, 22*, 201–221. https://doi.org/10.1007/s13394-019-00275-0

Gravemeijer, K., & van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *The Elementary School Journal, 109*(5), 510–524. https://doi.org/10.1086/596999

Lamon, J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Erlbaum.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University.

Lovemore, T. (2023). *Integrating music and mathematics for connecting across multiple constructs of fractional understanding: An RME task design journey* [Doctoral thesis, Rhodes University].

Lovemore, T., Robertson, S-A., & Graven, M. (2022). Aligning mathematical and musical linear representations to support fractional reasoning. In N. Fitzallen, C. Murphy, & V. Hatisaru (Eds.), *Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 354–361). Launceston: MERGA.

Makar, K. (2021). Engaging in longitudinal research with classroom teachers: Theoretical and practical considerations. In J. Novotná, & H. Moraová (Eds*.), International symposium elementary maths teaching SEMT '21 proceedings* (pp. 439– 440). Faculty of Education. ISBN 978-80-7603-260-6.

Monson, D., Cramer, K., & Ahrendt, S. (2020). Using models to build fraction understanding. *The Mathematics Teacher, 113*(2), 117–123. https://www.jstor.org/stable/10.5951/mtlt.2019.0105

Nabb, K. (2023). Flexible thinking with double number lines. *PRIMUS, 33*(1), 42–50. https://doi.org/10.1080/10511970.2021.2019150

Orrill, C. H., Brown, R. E. (2012). Making sense of double number lines in professional development: Exploring teachers' understandings of proportional relationships. *Journal of Mathematics Teacher Education, 15*, 381–403. https://0-doi.org.wam.seals.ac.za/10.1007/s10857-012-9218-z

Saxe, G. B., Diakow, R., & Gearhart, M. (2013). Towards curricular coherence in integers and fractions: A study of the efficacy of a lesson sequence that uses the number line as the principal representational context. *ZDM, 45*(3), 343–364. https://doi.org/10.1007/s11858-012-0466-2

Shahbari, J. A., & Peled, I. (2014). Modelling in primary school: Constructing conceptual models and making sense of fractions. *International Journal and Science and Mathematics Education, 15*, 391–391. https://doi.org/10.1007/s10763-015-9702-x

Siemon, D. (2003). Partitioning: The missing link in building fraction knowledge and confidence. *Australian Mathematics Teacher, 59*(3), 22–24. https://doi.org/10.3316/aeipt.130543

Siemon, D., & Luneta, K. (2018). *Teaching mathematics: Foundation to senior phase* (2nd ed.). Oxford University.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R., & Warren, E. (2015). *Teaching mathematics: Foundation to middle Years* (2nd ed.). Oxford.

Soni, M., & Okamoto, Y. (2020). Improving children's fraction understanding through the use of number lines. *Mathematical Thinking and Learning, 22*(3), 233–243. https://doi.org/10.1080/10986065.2020.1709254

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Kluwer Academic.

van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*(1), 9–35. https://doi.org/10.1023/B:EDUC.0000005212.03219.dc

# Developing Student Teacher Knowledge of Instructional Strategies for Teaching Proportions: The Important Role of Practicum

Xiaowen Ma
*Victoria University of Wellington*
xiaowen.ma@vuw.ac.nz

This paper reports on the development of six student teachers' knowledge of instructional strategies (KOIS) for teaching proportions during a 2-month practicum in China. Development of four subcomponents was explored through Content Representation (CoRe) questionnaires and follow-up interviews. Data was analysed deductively and levels of each subcomponent determined based on a scoring rubric. Implications include that practicum is capable of developing KOIS effectively as final scores were at the maximum level with some participants showing greater initiative within their KOIS than others. Implications include that a further level in the rubric could be considered to reflect when creativity is shown in PCK.

Knowledge of instructional strategies (KOIS) is widely accepted as an important component of pedagogical content knowledge (PCK), which is crucial professional knowledge for mathematics teachers for effective teaching and student learning. Research has emphasised that teachers' mathematical PCK is significantly associated with student gains in mathematical understanding (Baumert et al., 2010; Hill et al., 2005), leading to the imperative of cultivating teachers' PCK in all stages of their teaching career. However, evidence shows that student teachers' PCK is not usually at a desired level at the end of initial teacher education and still needs support both in China (e.g., Bao, 2016; Li, 2016), as well as internationally (e.g., Callingham et al., 2012; Şahin et al., 2016). As the final chance for Chinese student teachers to enrich their PCK, practicum provides student teachers opportunities for their PCK to be explored and expanded upon in authentic classroom contexts with support from mentor teachers (Hume & Berry, 2013). Based on Ball et al.'s (2018) Mathematical Knowledge for Teaching (MKT) framework and Hanuscin et al.'s (2018) conceptualization of PCK components, this study reports on six student teachers' development of one PCK component (i.e., KOIS) during their final practicum. KOIS was chosen for this study as this aspect of PCK had been developed more than other PCK components (i.e., knowledge of curriculum, knowledge of instructional strategies) in a large study investigating the development of student teachers' PCK in practicum. The research question is:

- To what extent is KOIS developed during practicum?

## Background Literature

### *Knowledge of Instructional Strategies*

Since Shulman (1986) introduced the notion of PCK (i.e., teachers' knowledge of transferring subject matter into comprehensible forms for all learners) and its two central domains of instructional strategies and understanding of students, many researchers have drawn inspiration to conceptualise their own PCK models. Among the various components identified as constituting PCK, KOIS and knowledge of students (KOS) have consistently been identified as the most important by researchers (e.g., Park et al., 2011; Sæleset & Friedrichsen, 2021). As a component of PCK, KOIS has been interpreted in different forms for teaching different subject areas. For teaching science, Park and Oliver (2008) speculated two components of KOIS, namely topic-specific activities and representations, and subject-specific strategies. This was almost consistent with the conceptualization of Hanuscin et al.'s (2018) work, which added a subdomain of "strategies for

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 347–354). Newcastle: MERGA.

adapting instruction for diverse learners" for science education and identified subcomponents of KOIS as including:

> Knowledge of topic-specific and science-specific strategies for teaching. Teachers should have knowledge of the conceptual power of a particular activity and/or representation, the extent to which it facilitates student learning, and how they might adapt that instruction to better facilitate the learning of diverse students. (p. 668)

Many mathematics education scholars have contributed to the development of ideas relating to PCK, including KOIS. For instance, Ball et al.'s (2008) MKT model, Krauss et al.'s (2008) three dimensions of PCK, and Chick et al.'s (2006) Clearly PCK framework. The MKT model has been used as a theoretical framework by researchers investigating teachers' or student teachers' mathematics PCK (e.g., Jacob & McConney, 2013; Livy & Downton, 2018) and has become a foundational tool in mathematics education (Melhuish et al., 2021). It divides mathematics knowledge for teaching into two aspects: subject matter knowledge and PCK. KOIS in the MKT model is referred to as knowledge of content and teaching, and combines with two other domains—knowledge of content and students and knowledge of content and curriculum—to constitute the entire MKT framework. According to Ball et al. (2008), teachers must know a series of strategies about how to design instruction, including being able to arrange specific instructional content, choose appropriate representations and activities, be aware of the advantages and disadvantages of representations and activities in particular contexts, and identify instructional affordances of different methods and procedures. "Instruction" is another appellation of KOIS in the three dimensions of PCK in Krauss et al.'s (2008) model (i.e., tasks, student, and instruction), representing knowledge of multiple representations and explanations of mathematical problems. In the Clearly PCK framework, KOIS represents the strategies and representations appropriate for teaching specific mathematics concepts (Chick et al., 2006).

Overall, literature suggests the following aspects should be included in KOIS for mathematics teaching: strategies for teaching mathematics, activities and representations for specific mathematics topics, and strategies for adapting the activities, representations or other instructional strategies. Development of the four subcomponents of KOIS- subject-specific strategies, topic-specific activities, topic-specific representations, and strategies for adapting instruction for diverse learners—have therefore been investigated in this research.

### *Developing KOIS in Practicum*

Research has focused on the effect of various factors on developing student teachers' KOIS in practicum. For example, investigating the contribution of one task (e.g., classroom observation) in the practicum (Livy & Downton, 2018), and the impact of the relationship with mentor teachers (Msimango et al., 2020), or interventions (Sæleset & Friedrichsen, 2021). Livy and Downton (2018) reported a study of developing student teachers' PCK in which they observed student teachers teach a single geometric reasoning lesson during practicum. Data from the observations and fieldnotes contributed to researchers' understanding of what student teachers noticed and learnt about teaching geometric reasoning. However, their research investigated student teachers' PCK development in the observation of one lesson and did not seek to explore their development over time. Similarly, Sæleset and Friedrichsen (2021) explored the integration of knowledge of strategies and knowledge of learners in the intervention of stimulated recall, where student teachers watching video recordings of their lessons and reflected on their teaching. Their findings indicated that student teachers' instructional strategies and knowledge of learners were frequently integrated in the process, with topic-specific strategies being developed most. Although the two studies mentioned above were conducted in practicum, they focused on just one task within the practicum. In contrast, Msimango et al. (2020) reported on how mentoring relationships might impact student teachers' PCK development during practicum. Data from separate interviews of student teachers and their mentor teachers showed inconsistencies in communication had acted as barriers for developing PCK, while harmonious communication with mentor teachers was helpful for PCK development, particularly for instructional strategies. These researchers investigated the influence of mentoring on PCK development through interview data only, again without systematically measuring changes in student teachers' PCK.

## Methods

In this study, the researcher explored the development of the four aspects of student teachers' KOIS (i.e., subject-specific strategies, topic-specific activities, topic-specific representations, and strategies for adapting instruction for diverse learners), which together comprise one component of their PCK, over the entire practicum. Triangulation of multiple data sources from interviews and questionnaires were used. By analysing all the data sources in relation to the different levels specified in the scoring rubric developed by Hanuscin et al. (2018), strong evidence of the development of KOIS during practicum was revealed.

### *Participants*

Participants were six student teachers who were studying a four-year undergraduate program in a university in China. The program comprises four semesters of foundational learning including university-based learning, plus a two-month practicum in a local primary school as the final component of the qualification. Student teachers spend this practicum in schools solely under the supervision of school mentor teachers, experiencing a range of school-based educational activities (e.g., planning lessons, observing classroom teaching, and teaching lessons to the whole class). The six participants were recruited due to the abundant learning opportunities relating to teaching proportions within their practicum, and their degree theses are on a topic outside of mathematics education, making a high level of prior knowledge of mathematics unlikely to contribute to PCK development in practicum. There were five student teachers working in year four of primary school, experiencing the instruction of decimal-related topics, and one student teacher working in year five, experiencing the instruction of fraction-related topics. They are referred to using pseudonyms.

### *Instruments*

This was a qualitative multiple case study (Yin, 2016) with multiple data sources from questionnaires and interviews. The CoRe questionnaire (Loughran et al., 2012) is a matrix that includes a series of topics about a particular content area and a set of eight pedagogical questions corresponding to the components of PCK, which has been widely utilized by PCK researchers (e.g., Hume & Berry, 2013; Nilsson & Karlsson, 2019). The present study also used the CoRe questionnaire, which was adapted by adding prompts and two further questions to make it as suitable as possible for participants to understand the questions explicitly and to maximise the usefulness of the collected data. The adaption was informed by the trial of data collection tools when the author piloted the tool with six primary mathematics teachers at other schools. In this study, all student teachers completed the questionnaire on three topics. There was a unified CoRe topic over the three topics, "the meaning of decimals (or fractions)", because in China it is the first topic that primary school children need to be taught when they start learning about decimals (or fractions) and it can be a challenging topic for teaching and learning within primary education.

The CoRe questionnaire was followed by a supplementary interview to explore the reasons for participants' questionnaire responses and to add to the content of the questionnaire, thus ensuring rich and detailed data. All student teachers' changes on their PCK level were explored using two CoRe questionnaires and follow-up interviews, once at the beginning and the other at the end of the practicum.

### *Data Analysis*

Data was systematically analysed in a deductive way, where all participants' CoRe and interview responses corresponding to the identified KOIS subcomponents were identified and evaluated with a four-level scoring rubric (Hanuscin et al., 2018). With this rubric, each participant's responses in relation to the KOIS subcomponents were scored from Level 1 (limited knowledge) to Level 4 (robust knowledge) considering the quality of their strategies, the reasonableness of their choices on particular strategies, and their understanding of the strengths and weaknesses of these strategies in specific contexts. Through this process, the development of student teachers' KOIS in practicum was able to be made explicit.

## Findings

Findings from the CoRe questionnaires and interviews indicated that practicum had improved student teachers' understanding of instructional strategies for teaching proportions, as all student teachers' level of each subcomponent of KOIS had been developed to the highest level of Level 4 when most initial scores were Level 1 with the rest Level 2. Student teachers' initial understanding of different topics was varied, as there were differences in their initial scores among topics within one subcomponent domain (e.g., topic-specific activities). It seemed that due to their varying understandings of the individual topics, there was thematic variability in the level of KOIS subcomponents reflected in their initial responses. However, this variability was not evident in their final data, according to the rubric. Although most student teachers had developed their KOIS to the highest level of the CoRe by the end of practicum, it seemed that for most, their understanding of teaching strategies was based on imitating the strategies from their mentor teachers and key resources as the strategies identified by them were aligned with their mentor teachers' lessons and the textbook. Only Xue showed creativity on the instructional strategies, rather than imitating others.

### *Initial KOIS*

In the initial data, student teachers showed a limited understanding of instructional strategies, especially strategies that are specific to particular topics and strategies for adjusting their teaching. They provided general instructional strategies and seemed to lack confidence about answering the CoRe and interview questions. When answering questions on the activities or representations that they would like to apply while teaching specific topics, they simply mentioned the category of activities or representations, but did not seem to be able to explain further detail about the timing and purpose of the strategies. For example, Shang noted that she would use a ruler to help children understand decimal counting units, but she did not seem able to explain specifically how or when she would use the ruler. Moreover, student teachers did not show appropriate awareness on adjusting their instructional strategies. They seemed to be inclined to teach as they had planned, because they had some trepidation about the failure of varying their teaching from the plan. As Wangrn said, "The instructions should be the same to all children in one lesson, for the purpose of achieving the instructional goals quickly and smoothly".

### *Final KOIS*

In student teachers' final data, their understanding of instructional strategies had noticeably improved. They were able to identify appropriate strategies that are specific to mathematics learning in general or specific to particular topics, and the strategies of adapting activities or representations, as well as the importance of applying these strategies and potential barriers in particular contexts.

*Subject-specific strategies.* Several subject-specific strategies featured in student teachers' responses, including group inquiry activities and reasoning, in-class games, and math note-booking (i.e., organising mathematics concepts in a particular form, such as a mind map). Although their explanations for using these strategies were specific to mathematics, they were not specific to particular topics. Group inquiry activities had been cited by all student teachers for all of their three topics. The group inquiry activities were usually combined with reasoning, where children were guided to generalize conclusions from specific contexts presented by teachers in the process of group cooperation and discussion. According to Xue, group inquiry activities contribute to not only enhance children's understanding of new concepts as they are working out the principles for themselves, but also exercise and improve children's abilities of independent learning and cooperating in the process, rather than simply to learn certain content knowledge. The limitations of group inquiry were indicated from the perspectives of teacher and children respectively. From the perspective of teachers, according to Chen, the in-class time might be tight for novice teachers, as group activities are time consuming. From the perspective of children, according to Song and Xue, some children may not pay attention to the discussion or to the tasks given to them by the teacher, but just chat or play with others:

> Some individual students may not be focused in class and not discuss with peers carefully. The teacher may ignore them. (Song)

> It may be a bad design for children who need to be supervised deliberately by teachers, or children with poor learning autonomy and concentration. Because he/she usually can not concentrate on the class independently. (Xue)

*Topic-specific activities*. The topic-specific activities that were identified by student teachers included hands-on activities, topic-specific scenarios (e.g., a scenario of "ranking four children's long jump scores" that two student teachers who included the "size comparison of decimals" topic noted), and "reactivating" activities (i.e., review the key prior knowledge and understanding of it at the beginning of the new lesson, before introducing the new concept). All five student teachers who worked in year four noted a hands-on measurement activity for the topic of "meaning of decimals", which is an example activity in the textbook (PEP, 2016, p. 32). The activity encourages children to use hands-on tools (e.g., ruler) to measure the length of objects in the classroom (e.g., desk and blackboard) at the beginning of teaching the meaning of decimals, aiming at introducing the creation of decimals (i.e., to represent numbers that are not integers). According to Chen, measurement is an interesting activity that enables children to operate with their hands, which motivates children's enthusiasm for mathematics learning and engages them in the classroom, as well as being a good introduction to the new concept. In addition, children could also adopt the measurement results as general knowledge, as was indicated by Wangrn. However, the barriers of the practical measurement activity identified by student teachers included that it may distract children from the learning that follows, and it is time-consuming. According to Chen, it is difficult for the teacher to pull children who are engrossed in the measuring activity back into the normal classroom, which is not conducive to the smooth progress of the whole lesson. This was similar with Wangxy's statement:

> This group measurement activity actually takes longer and is less likely to ensure effective teaching. Firstly, there is some variability in the learning ability of each group, and secondly, some students may not be very motivated to learn, which may further affect the teaching progress. (Wangxy)

*Topic-specific representations.* Multiple topic-specific representations were pointed out by student teachers, including physical objects (i.e., meter stick, 1dm$^2$ squared paper, and 1dm$^3$ cubes), analogy (i.e., life-related analogy and prior knowledge related analogy), and online interactives and simulations (i.e., animations and graphic representations). Student teachers discussed representations for facilitating children's learning consistent with textbooks and literature, as well as the strengths and weaknesses of the representations for particular children. For example, physical objects were referred to as the best pedagogical tools to introduce the decimal-related concepts. All five student teachers who worked in year four identified the physical object of a meter stick for the "meaning of decimals" topic and some of them further included 1dm$^3$ cubes (1cm$^3$ units, 1mm$^3$ units). The three student teachers' whose CoRe involved the "properties of decimals" topic identified meter stick and 1dm$^2$ squared paper (1cm$^2$ units, 1mm$^2$ units). According to them, the meter stick was used as a visual display of length, enabling children to understand abstract concepts of the meaning of decimals and decimals' counting units through observing the conversion of different length units. The meter stick was applied first because it is a carrier of length units and is intuitive and visual, as was indicated by student teachers, and units of length are more familiar and easier than units of area or weight to primary school children. The visuals of squared paper and cubes were applied following the meter stick in the topic of "meaning of decimals" and "properties of decimals" respectively, as they were verification of the conclusions drawn from the units of length, from the perspective of area and volume respectively:

> The last session was about the law of the properties of decimals derived from length units, and children might wonder, is it possible that the law derived from this particular context could applied to other context? So we take this conjecture from the particular to the general, by the area of the square paper. (Wangxy)

Information that fits the analogy can be extracted from four student teachers' responses in their CoRe questionnaire. Features of the analogy that was presented by student teachers were that it was life-related or existing knowledge-related. For example, for the "meaning of fraction" topic, Xue indicated that he would like to use common objects that are normal in children's real-life to represent the concepts of "unit 1" and "fractional unit" (e.g., consider a class of 30 students as unit 1, and the fraction unit is $\frac{1}{30}$). The application of analogies allows children to understand abstract proportion-related concepts in a simple and familiar way, according to the student teachers, as well as deepening children's enthusiasm for learning mathematics. As Xue said, "children will find mathematics existing everywhere in their lives".

*Strategies for adjusting teaching for diverse learners.* Student teachers identified several strategies for adapting instruction for diverse learners, which were represented by several strategies for children of different knowledge levels and for different classes. They were able to explain the ways of adapting strategies, as well as the strengths and limitations of the adjustment. The adjustment strategies shared by all student teachers included a strategy of adjusting group inquiry activities and a "little teacher" pattern. The former was increasing or decreasing the amount of inquiry, based on the overall level of children in one classroom. For example, for the class of lower level, according to student teachers, the teaching design should be simpler and the group inquiry should be reduced as they felt that children with poor comprehension skills are better suited to a teacher-led or teacher-explained approach, whereas the collaborative inquiry approach would not be conducive to their learning or stimulate their divergent thinking, and would be time-consuming. The "little teacher" pattern refers to children acting as teachers to instruct their classmates in concepts. According to student teachers, children think in a more similar way to each other than to the teacher. Therefore, it may be more effective for children to understand a concept when a peer instructs it. It can exercise children's logical thinking skills at the same time, as the "little teachers" have to find a clear and logical way to organize and represent their ideas:

> First of all, it can test whether students have understood the concept and can express it correctly. Secondly, it can exercise the students' expression ability, where teachers are able to see whether they can express it clearly in a logical way. (Chen)

## Discussion and Conclusion

Findings of this study include that practicum is effective in cultivating student teachers' KOIS, as they were developed from initial scores at lower levels to all final scores being at the maximum level. Due to the characteristics of topic and context specificity of PCK (Park & Chen, 2011), it was reasonable that student teachers did not perform well on their initial score of topics that they did not know and in unfamiliar contexts, even though they had completed almost 4-year university-based coursework before practicum. Therefore, this study raises a question for further investigation about how student teachers had made such huge progress in only 2-months of practicum. The reasons for the efficiency may be related to student teachers in China having strong understanding of the concepts of proportions and only needing to learn how to teach these concepts to primary children. Their mathematics content knowledge entry to initial teacher education programme is high (equal to around the year 13 level), or it may be related to assistance from mentor teachers or utilisation of other sources as was indicated by Xue (e.g., expert teachers' classroom videos). Further research could help probe what aspects of practicum have most effect on KOIS development.

An important implication is that incorporating a higher level in the scoring rubric could be useful. Although student teachers' KOIS had been developed to the maximum level identified by the rubric, their understanding of strategies was seemed consistent with the textbooks and their mentor teachers' lessons but lacked independent innovation, especially the representations and activities specific to particular topics. Although all student teachers' KOIS were developed to the same highest level, the example of one student teacher (i.e., Xue) who is more confident to be creative with strategies rather than only adapting ideas from the textbook and mentor teachers indicated that there could be a higher level (i.e., Level 5) included in the CoRe rubric to measure and report on further levels of competence, such as more innovation and creativity. It would also be useful for future research to investigate ways of cultivating student teachers' creativity in designing effective teaching strategies, rather than imitating others.

Limitations of this study include that student teachers might have felt anxious or not confident to share things that they were not certain of in the first interview because they felt their PCK would be judged, but they felt more confidence or trusted the researcher more as the study progressed. For future research investigating student teachers' PCK, it would be useful for the researcher to have established relationships with participant student teachers before the practicum. Despite the study limitations, the findings add to understanding of PCK development in that they show that KOIS—especially the development of knowledge of representations and activities that are specific to particular topics—can be effectively developed during practicum.

## References


Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389–407. http://dx.doi.org/10.1177/0022487108324554

Bao, Y. (2016). Analysis of MPCK of primary mathematics teachers with different background in Guangdong province. *Journal of Shanghai Educational Research,* (08), 14–20. https://kns.cnki.net/kcms/detail/detail.aspx?FileName=SJYY201608005&DbName=CJFQ2016

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., . . . Tsai, Y. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133–180. http://dx.doi.org/10.3102/0002831209345157

Callingham, R., Chick, H. L., & Thornton, S. (2012). Editorial: Evidence-based approaches to developing content knowledge and pedagogical content knowledge in pre-service mathematics teacher education. In R. Callingham, H. L. Chick, & S. Thornton (Eds.), *Mathematical teacher education and development* (Vol. 14, pp. 1–3): MERGA.

Chick, H. L., Baker, M., Pham, T., & Cheng, H. (2006, July). Aspects of teachers' pedagogical content knowledge for decimals. *In Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297–304).

Hanuscin, D. L., Cisterna, D., & Lipsitz, K. (2018). Elementary teachers' pedagogical content knowledge for teaching structure and properties of matter. *Journal of Science Teacher Education, 29*(8), 665–692. http://dx.doi.org/10.1080/1046560X.2018.1488486

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371–406. http://dx.doi.org/10.3102/00028312042002371

Hume, A., & Berry, A. (2013). Enhancing the practicum experience for pre-service chemistry teachers through collaborative CoRe design with mentor teachers. *Research in Science Education, 43*(5), 2107–2136. http://dx.doi.org/10.1007/s11165-012-9346-6

Jacob, L., & McConney, A. (2013). The Fitzroy Valley numeracy project: Assessment of early changes in teachers' self-reported pedagogic content knowledge and classroom practice. *Australian Journal of Teacher Education, 38*(9), 94–115. http://dx.doi.org/10.14221/ajte.2013v38n9.1

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM, 40*(5), 873–892. http://dx.doi.org/10.1007/s11858-008-0141-9

Li, Y. (2016). The analysis and reflection on the defects of pre-service mathematics teachers' PCK. *Journal of Chifeng University (Natural Science Edition), 21*, 218–219. https://kns.cnki.net/kcms/detail/detail.aspx?FileName=CFXB201621097&DbName=CJFQ2016

Livy, S., & Downton, A. (2018). Exploring experiences for assisting primary pre-service teachers to extend their knowledge of student strategies and reasoning. *The Journal of Mathematical Behavior, 51*, 150–160. http://dx.doi.org/10.1016/j.jmathb.2017.11.004

Loughran, J., Berry, A., & Mulhall, P. (2012). *Understanding and developing Science teachers' pedagogical content knowledge.* Springer Science & Business Media. http://dx.doi.org/10.1007/978-94-6091-821-6

Melhuish, K., White, A., Sorto, M. A., & Thanheiser, E. (2021). Two replication studies of the relationships between mathematical knowledge for teaching, mathematical quality of instruction, and student achievement. *Implementation and Replication Studies in Mathematics Education, 1*(2), 155–189. http://dx.doi.org/10.1163/26670127-01010007

Msimango, N., Fonseca, K., & Petersen, N. (2020). Mentoring and preparing primary school mathematics teachers. *Perspectives in Education, 38*(2), 272–284. http://dx.doi.org/10.18820/2519593X/pie.v38.i2.18

Nilsson, P., & Karlsson, G. (2019). Capturing student teachers' pedagogical content knowledge (PCK) using CoRes and digital technology. *International Journal of Science Education, 41*(4), 419–447.

Park, S., Jang, J., Chen, Y., & Jung, J. (2011). Is pedagogical content knowledge (PCK) necessary for reformed science teaching?: Evidence from an empirical study. *Research in Science Education, 41*(2), 245–260. http://dx.doi.org/10.1007/s11165-009-9163-8

Park, S., & Oliver, J. S. (2008). Revisiting the conceptualisation of pedagogical content knowledge (PCK): PCK as a conceptual tool to understand teachers as professionals. *Research in Science Education, 38*(3), 261–284. http://dx.doi.org/10.1007/s11165-007-9049-6

People's Education Press. (2016). *Primary mathematics.* Beijing: People's Education Press.

Şahin, Ö., Gökkurt, B., & Soylu, Y. (2016). Examining prospective mathematics teachers' pedagogical content knowledge on fractions in terms of students' mistakes. *International Journal of Mathematical Education in Science and Technology, 47*(4), 531–551. http://dx.doi.org/10.1080/0020739X.2015.1092178

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14. http://dx.doi.org/10.3102/0013189X015002004

Sæleset, J., & Friedrichsen, P. (2021). Pre-service science teachers' pedagogical content knowledge integration of students' understanding in science and instructional strategies. *Eurasia, 17*(5). https://doi.org/10.29333/ejmste/10859

# Big Ideas in Mathematics: Exploring the Dimensionality of Big Ideas in School Mathematics

Jahangeer Mohamed Jahabar
*National Institute of Education*
jahangeer.jahabar@nie.edu.sg

Toh Tin Lam
*National Institute of Education*
tinlam.toh@nie.edu.sg

Tay Eng Guan
*National Institute of Education*
engguan.tay@nie.edu.sg

Tong Cherng Luen
*National Institute of Education*
cherngluen.tong@nie.edu.sg

Big Ideas in school mathematics can be seen as overarching concepts that occur in various mathematical topics in a syllabus. For teachers, this knowledge can be used to help students develop a better understanding of mathematics by making visible the central ideas, and connection across topics and across levels. For students, this knowledge can further be helpful affectively by engendering an appreciation of mathematics as a subject that is coherent and comprehensible. Although there has been much interest recently in the understanding of Big Ideas, there is little research done in the assessment of Big Ideas thinking. In this paper, we discuss our development of an instrument to measure the Big Ideas of equivalence and proportionality. Our analysis of some pilot items suggests that Big Idea thinking is a multidimensional construct within most school environments.

We concur with Charles' (2005) definition of a Big Idea in mathematics as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p.10). Each Big Idea connects various concepts and understanding across topics, strands and levels. The notion of Big Ideas became prominent in the teaching and learning of mathematics when it was highlighted by the National Council of Teachers in Mathematics in 2000, where it stated: "Teachers need to understand the Big Ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise. Their decisions and their actions in the classroom—all of which affect how well their students learn mathematics—should be based on this knowledge." (NCTM, 2000, p17)

While there has been an interest in the understanding and teaching with the understanding of Big Ideas, to date, there has been little research done on how the knowledge and understanding of Big Ideas can be assessed. One reason can be attributed to the different classifications and definitions of the Big Ideas by various educational bodies and researchers. Charles (2005) listed twenty-one Big Ideas in mathematics. Niemi (2006) identified a list of twenty Big Ideas for school algebra alone and wrote about the development of an assessment instrument to measure the Big Ideas based on his list. Hurst (2019) had further categorized his list of Big Ideas into three categories: sequential Big Ideas, umbrella Big Ideas and process Big ideas. The Singapore Ministry of Education (MOE) syllabus identified six Big Ideas for the Primary grade levels in Singapore (Grades 1 to 6), and an additional two for the Secondary grade levels (Grades 7 to 10) (MOE, 2018a, 2018b).

Another possible reason for the scarcity of Big Ideas assessment instruments could be the lack of clarity on the intent of the assessment in the first place. Educational institutions may not rate it important and pressing enough to know how well their students are able to think along the Big Ideas to incorporate specific items into their class and grade assessments. With a proper instrument available at the national level to measure Big Ideas thinking, professional development can be provided to educational institutions on how to assess Big Ideas thinking and enactment in their environments. Teachers could also use this information for curriculum development and design, resource design and formative assessment. It would seem then that any instrument designed to assess Big Ideas should be an addition to the current suite of assessments already put in place and, in particular, to be done outside of high stakes mathematics assessments. Assessment instruments

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 355–362). Newcastle: MERGA.

developed to assess Big Ideas thinking will have to factor in the testing load on the students as well as the amount of time taken away from curriculum to administer the instrument.

A third reason could be that it is difficult to create items that assess Big Ideas thinking because Big Ideas link numerous mathematical understandings and cuts across topics and grade levels. Thus, a major consideration for items created to assess Big Idea thinking is that they must be able to test specifically such cross-topic thinking and not be confounded by specific topical familiarity and expertise. Another aspect to consider, akin to Carroll's three-stratum theory of cognitive abilities (e.g., Carroll, 1993; 1997; Warne & Burningham, 2019) is when to see Big Ideas as a unidimensional construct and when as a multi-dimensional construct.

This paper reports the initial stage of developing an instrument, to measure Equivalence and Proportionality, two out of the eight Big Ideas in the mathematics curriculum document (MOE, 2018a, p5). The pilot instrument was administered to students in two primary schools and two secondary schools in Singapore. We took into consideration the need to create an item with questions that test across different topics. The items were analysed to explore if Big Ideas thinking in schools is a unidimensional construct or if it is 2-dimensional along the two Big Ideas to be assessed.

## Equivalence and Proportionality

There are no generally accepted formal definitions yet on any of the Big Ideas. For example with regard to Equivalence, Warren & Cooper (2009), in their EATP project, highlighted 5 key aspects: equations as equivalence; the balance principle; sign systems for unknowns; identity and inverses; and finding solutions and generalisations. On the other hand, Fyfe et al. (2018), restricted equivalence to the idea in which both sides of an equation are equal and interchangeable. While understanding the broad nature of the construct, their study was focused on the symbolic understanding of equivalence: the understanding of the equal sign.

For this study, we will specify our understanding of two Big Ideas, Equivalence and Proportionality, and use them in developing our instrument. We look at two Big Ideas common to the list proposed by Charles (2005) and the list of Big Ideas developed by MOE: Equivalence and Proportionality.

Charles (2005) described Equivalence as when "any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value". MOE described equivalence as: "A relationship that expresses the equality of two mathematical objects that may be represented in two different forms. In every statement about equivalence, there is a mathematical object (e.g., a number, an expression or an equation) and an equivalence criterion (e.g., value(s), part-whole relationships)." (MOE, 2018b, p.15) The former defines equivalence focusing on representations having the same value while the latter goes beyond the equivalence of values. Consider the mathematical problem as follows:

Example 1: Ali had \$220 and Colin had \$310. After each of them bought an identical T-shirt, Colin had thrice as much money as Ali had left. How much does each T-shirt cost?

In the mathematical problem stated, the difference in money Ali and Colin had at the beginning is the same as the difference in money at the end. The two equivalent entities are the difference in money before and after having bought identical T-shirts.

When defining the Big Idea of proportionality, Charles (2005) explained that "when two quantities vary proportionally, that relationship can be represented as a linear function". MOE describes proportionality as a "relationship between two quantities that allows one quantity to be computed from the other based on multiplicative reasoning" (MOE, 2018b, p.16). We operationalise proportionality by making explicit the two entities that vary in direct proportion. This can be seen in the following example:

Example 2: 60 workers can make 180 chairs in one day. All workers work at the same rate. How many chairs can 36 workers make in one day?

It can be clearly seen from this example that the number of chairs made in one day varies proportionally to the number of workers.

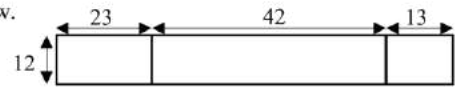
## Methodology

Our design of the items for the instruments is guided by the three characteristics of Big Ideas as detailed by Hsu et al. (2007). The items that we develop serve to (a) connect different parts of the curriculum under the umbrella Big Idea; (b) be a basis for understanding other topics; and (c) make choices about curriculum. Objectives (a) and (b) are meant for the instruments for both teachers and students, and (c) is for the teachers. In this paper, we shall only consider the instrument for the students. We shall call this instrument Big Ideas in School Mathematics, BISM for short.

**Part 1:** The diagram below shows three rectangles joined together to form a bigger rectangle.

You may use the diagram to fill in the blanks below.

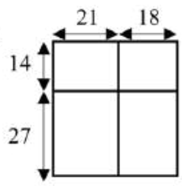
23 x 12 + 42 x 12 + 13 x 12 = ? x 12




**Part 2:** The diagram below shows four rectangles joined together to form a bigger rectangle.

Fill in the missing numbers below.

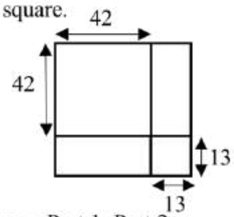
14 x 21 + Δ x 18 + 27 x 21 + 27 x 18 = ♥ x 39




**Part 3:** The diagram below shows 2 squares and 2 rectangles joined together to form a bigger square.

Fill in the missing numbers below.

55 x 55 = 42 x 42 + 13 x 13 + 13 x Δ x ♥




**Part 4:** Which of these following statements best describes the common mathematical idea across Part 1, Part 2 and Part 3?

a) I used diagrams for the parts

b) I used Equivalence for the parts

c) I used Guess and Check for the parts

d) I used Proportionality for the parts

e) Others (Please elaborate)

**Part 5:** The shaded area of the figure below can be used to show a mathematical statement. Which of the following statements matches the shaded part of the figure?

a) (1 + 6) + (2 + 5) + (3 + 4) … + (6 + 1) = 6 x 7

b) 1 + 2 + 3 + … + 6 = (6 x 7 ) ÷ 2

c) 1 + 2 + 3 + … + 7 = (7 x 8) ÷ 2

d) 1 + 2 + 3 + … + 8 = (8 x 9) ÷ 2

e) 1 + 2 + 3 + … + 7 = (7 x 8)

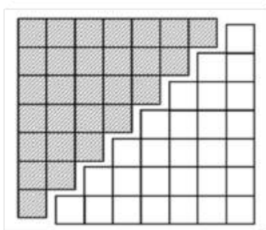
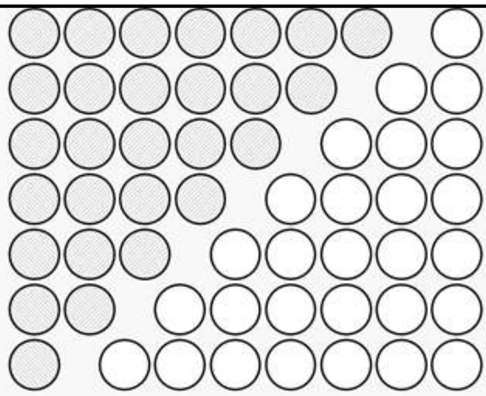


Figure 1. An item on the *Big Idea of Equivalence* consisting of five parts.

Each item in BISM tests only one Big Idea of either Equivalence or Proportionality and has five parts. Part 1 to Part 3 each consists of a selected response question focusing on the same Big Idea and are from the same topic, for example ‘Area of rectangles’. Ability to correctly answer these parts could be due to topical and procedural knowledge and/or knowledge of the Big Idea under focus. To facilitate thinking beyond the topical and procedural knowledge, Part 4 seeks to assist participants to look for the link connecting the three parts. Part 4 also seeks to trigger students’ Big Idea concepts, if any. Participants then attempt Part 5, a question that focuses on the same Big Idea but based on a different topic. Figure 1 shows one of the items under the Big Idea of Equivalence.

Each of the first three parts can be solved by seeing the equivalence between the area of the large rectangle and the sum of the areas of its component parts. Part 5 needs also the Big Idea of Equivalence but not exactly in terms of areas of rectangles. The focus is on the equivalence of two mathematical forms, a diagram and its equivalent mathematical equation. It can be subsumed within the strand of Whole Numbers. Part 5 was refined later to dissociate it from the idea of Areas with changes in the figure as well as the phrasing as shown in Figure 2.

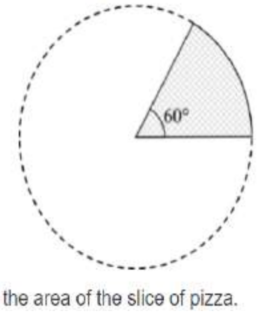


The figure above can be used to show a true mathematical statement.
Which of the following is that mathematical statement?

Figure 2. Refined part 5.

Figure 3 shows Part 5 of an item on the Big Idea of proportionality. Students at this level have not learnt about areas of circles but are expected to have knowledge of areas of rectangles as well as that the sum of angles at a point is 360°. The problem can be solved by seeing that the area of the “slice” varies proportionally with the angle at the centre.

The diagram shows a slice of pizza cut off from a full round pizza of 420 cm².




Find the area of the slice of pizza.

Figure 3. Part 5 of an item on the *Big Idea of Proportionality*.

## Analysis of Instrument and Item Performance

The items have been validated by three content specialists at the National Institute of Education, with a doctorate degree in Mathematics, Educational Assessment and Mathematics Education, respectively. To validate the items further, the instrument was analyzed using Rasch analysis via the Winsteps software.

Firstly, the instrument was analysed to check if the items collectively are unidimensional, i.e., that it measures only one latent trait, Big Ideas thinking.

The items were administered to 360 students from two participating schools at grade levels five and six. Each student is given two complete sets of items, one each from the different Big Ideas of Equivalence and Proportionality. The items given to various students overlapped and interlaced to allow for item equating to be done. Each student was given 45 minutes to complete the items.

Figure 4 shows the analysis report from Winsteps on the standardized residual variance based on the students' responses.

TI_Std_Pri_Dimen_NoPart4 - Notepad
File Edit Format View Help

TABLE 23.0 All_data_Testpoint 1 trial.xlsx ZOU827WS.TXT Mar 1 2022 9: 3
INPUT: 360 PERSON 8 ITEM REPORTED: 360 PERSON 8 ITEM 2 CATS WINSTEPS 5.2.0.0

Table of STANDARDIZED RESIDUAL variance in Eigenvalue units = ITEM information units

| | | Eigenvalue | Observed | | Expected |
|---|---|---|---|---|---|
| Total raw variance in observations | = | 10.7998 | 100.0% | | 100.0% |
| Raw variance explained by measures | = | 2.7998 | 25.9% | | 26.4% |
| Raw variance explained by persons | = | 1.4784 | 13.7% | | 13.9% |
| Raw Variance explained by items | = | 1.3214 | 12.2% | | 12.4% |
| Raw unexplained variance (total) | = | 8.0000 | 74.1% | 100.0% | 73.6% |
| Unexplned variance in 1st contrast | = | 1.4077 | 13.0% | 17.6% | |
| Unexplned variance in 2nd contrast | = | 1.3068 | 12.1% | 16.3% | |
| Unexplned variance in 3rd contrast | = | 1.1508 | 10.7% | 14.4% | |
| Unexplned variance in 4th contrast | = | 1.1369 | 10.5% | 14.2% | |
| Unexplned variance in 5th contrast | = | 1.0840 | 10.0% | 13.6% | |

Figure 4. Standardized residual variance report.

If the items are unidimensional, the variance explained by the items is expected to be considerably larger than the unexplained variance in the first contrast. As can be seen from Figure 4, the variance explained by the items are about equal in magnitude with all the unexplained variances in the other contrasts, signaling that the items may not be unidimensional. A closer analysis is needed to understand how the items function. We shall proceed to analyse the first two contrasts. Figure 5 shows the Principal Component Analysis report for the first contrast, Contrast 1.

CONTRAST 1 FROM PRINCIPAL COMPONENT ANALYSIS
STANDARDIZED RESIDUAL LOADINGS FOR ITEM (SORTED BY LOADING)

| CON-TRAST | LOADING | MEASURE | INFIT MNSQ | OUTFIT MNSQ | ENTRY NUMBER | ITEM | LOADING | MEASURE | INFIT MNSQ | OUTFIT MNSQ | ENTRY NUMBER | ITEM |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .60 | -.46 | .93 | .82 | 1 | 1 E-P-013 | -.53 | -.66 | .93 | .80 | 6 | 6 P-P-032 |
| 1 | .45 | 1.06 | 1.10 | 1.09 | 4 | 4 E-P-016 | -.41 | .48 | 1.20 | 1.23 | 5 | 5 P-P-031 |
| 1 | .35 | .54 | 1.02 | 1.04 | 7 | 7 P-P-034 | -.38 | -.38 | .88 | .78 | 3 | 3 E-P-015 |
| 1 | .16 | -.44 | 1.07 | 1.19 | 8 | 8 P-P-035 | -.32 | -.14 | .88 | .84 | 2 | 2 E-P-014 |

Figure 5. Principal component analysis for contrast 1.

From the report, the contrasting questions are identified based on their polarizing loading. The two most positively loaded questions are those belonging to items testing for Equivalence, recognizable from the 'E-P' prefix (the second 'P' refers to the Primary track) on the question identifications while the most negatively loaded questions are those belonging to items testing for the Big Idea of proportionality, recognizable from the 'P-P' prefix on the question identifications. Figure 6 shows the Principal Component Analysis report for the second contrast, Contrast 2.

CONTRAST 2 FROM PRINCIPAL COMPONENT ANALYSIS
STANDARDIZED RESIDUAL LOADINGS FOR ITEM (SORTED BY LOADING)

| CON-TRAST | LOADING | MEASURE | INFIT MNSQ | OUTFIT MNSQ | ENTRY NUMBER | ITEM |
|---|---|---|---|---|---|---|
| 2 | .60 | .48 | 1.20 | 1.23 | 5 | 5 P-P-031 |
| 2 | .46 | -.44 | 1.07 | 1.19 | 8 | 8 P-P-035 |
| 2 | .38 | .54 | 1.02 | 1.04 | 7 | 7 P-P-034 |

| LOADING | MEASURE | INFIT MNSQ | OUTFIT MNSQ | ENTRY NUMBER | ITEM |
|---|---|---|---|---|---|
| -.51 | -.14 | .88 | .84 | 2 | 2 E-P-014 |
| -.38 | 1.06 | 1.10 | 1.09 | 4 | 4 E-P-016 |
| -.28 | -.46 | .93 | .82 | 1 | 1 E-P-013 |
| -.24 | -.38 | .88 | .78 | 3 | 3 E-P-015 |
| -.23 | -.66 | .93 | .80 | 6 | 6 P-P-032 |

Figure 6. Principal component analysis for contrast 2.

Similar to the first contrast, the two most positively and negatively loaded questions are from the items testing the Big Ideas of Proportionality and Equivalence respectively. From these two PCA analyses, it can be preliminarily concluded that the two Big Ideas of Equivalence and Proportionality may require very different skillsets and that Big Ideas thinking may be multidimensional in nature.

LARGEST STANDARDIZED RESIDUAL CORRELATIONS
USED TO IDENTIFY DEPENDENT ITEM

| CORREL-ATION | ENTRY NUMBER ITEM | ENTRY NUMBER ITEM |
|---|---|---|
| -.24 | 4 E-P-016 | 5 P-P-031 |
| -.24 | 1 E-P-013 | 5 P-P-031 |
| -.24 | 1 E-P-013 | 6 P-P-032 |
| -.22 | 2 E-P-014 | 7 P-P-034 |
| -.19 | 3 E-P-015 | 8 P-P-035 |
| -.18 | 4 E-P-016 | 7 P-P-034 |
| -.18 | 2 E-P-014 | 4 E-P-016 |
| -.17 | 4 E-P-016 | 8 P-P-035 |
| -.17 | 3 E-P-015 | 4 E-P-016 |
| -.17 | 6 P-P-032 | 7 P-P-034 |
| -.16 | 2 E-P-014 | 5 P-P-031 |
| -.16 | 5 P-P-031 | 7 P-P-034 |
| -.16 | 2 E-P-014 | 8 P-P-035 |
| -.14 | 3 E-P-015 | 7 P-P-034 |
| -.14 | 6 P-P-032 | 8 P-P-035 |
| -.13 | 4 E-P-016 | 6 P-P-032 |
| -.13 | 1 E-P-013 | 3 E-P-015 |
| -.13 | 7 P-P-034 | 8 P-P-035 |
| -.12 | 5 P-P-031 | 6 P-P-032 |
| -.11 | 5 P-P-031 | 8 P-P-035 |

Figure 7. Item correlation report.

The items are further analysed by examining how the questions are correlated to each other. A positive correlation of the questions will reinforce the unidimensionality of the latent trait. Figure 7 shows the item correlation report.

From the report, the largest negatively correlated items are between Proportionality and Equivalence items. We do see items within the same Big Ideas also being negatively correlated, as seen from the 7$^{th}$ entry in the table as well as some more pairs of items down the table. A closer study of the items showed that these items are from different topics. Both item E-P-014 and E-P-016 are shown as Part 2 and Part 5 respectively in the earlier Figure 1. Part 2 is from the measurement topic of area while Part 5 is a counting question under the strand of Whole Numbers. Item P-P-031 under the topic of whole numbers is shown earlier as Example 2 while P-P-034, under the measurement strand is shown as Part 5 in Figure 3. In both these pairs of questions, students may have solved them using topical knowledge instead of knowledge of equivalence or proportionality. Hence, they may be able to do an item from a topic but not another item from another topic with the same Big Idea.

## Findings and Discussion

At this point in the research, we have tested some items at both the primary and secondary levels. Only two complete sets of items were tested for each Big Idea of Equivalence and Proportionality at the primary level. The findings and discussions are based on this small number of items that was tested at the primary level. From the standardized residuals and principal component analysis, it can be concluded preliminarily that it may not be feasible to construct just one instrument that measures the latent trait of knowledge of Big Ideas. Big ideas in mathematics may likely be multidimensional. At this juncture, it may seem more appropriate that two separate instruments be created, one for each Big Idea. Thus, the final BISM instrument may consist of a battery of subtests. At the next data collection stage, a detailed analysis will be made for the separate instruments to check once again for the unidimensionality of the items for each instrument first before making further analysis on students' performance.

For some of the items we have created, especially those in the first three parts of an item, participants may have solved them using topical content knowledge and not because they have the knowledge of Big Ideas. These earlier parts tend to be easier to solve and thus could be solved well by students equipped with sound content knowledge. Part 5 of each item was administered after the participants were asked to reflect and find similarities for each of the first three parts, guiding them to think beyond the topical content and more towards Big Ideas thinking. We have decided to analyse Big Ideas thinking through the participants' performance on Part 5 of the items only.

Finally, this study is part of a larger project that includes the professional learning development (PLD) of teachers in their knowledge of Big Ideas. The instrument that we are developing will also be used in the formative assessment of the teachers in the PLD and subsequently for the students in their classes.

## Acknowledgements

The study reported here has been funded by the Singapore Ministry of Education under the research project OER/31/19BK which is managed by the National Institute of Education, Singapore.

## References


Carroll, J. B. (1993). *Human cognitive abilities: A survey of factor-analytic studies*. NY: Cambridge University Press.

Carroll, J. B. (1997). The three-stratum theory of cognitive abilities. In D. P. Flanagan, J. L. Genshaft et al., *Contemporary intellectual assessment: Theories, tests, and issues* (pp. 122–130). NY: Guilford Press.

Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education Leadership, 7*(3), 9–21.

Fyfe, Matthews, P. G., Amsel, E., McEldoon, K. L., & McNeil, N. M. (2018). Assessing formal knowledge of math equivalence among algebra and pre-algebra students. *Journal of Educational Psychology, 110*(1), 87–101.

Hsu, E., Kysh, J., Ramage, K., & Resek, D. (2007). Seeking Big Ideas in algebra: The evolution of a task. *Journal of Mathematics Teacher Education, 10*, 325–332.

Hurst, C. (2019). Big Ideas of Primary Mathematics: It's all about connections, in Toh, T. and Yeo, J. (Eds.), *Big ideas in mathematics: Yearbook 2019, Association of Mathematics Educators* (pp. 71–93). Singapore: World Scientific.

Ministry of Education. (2018a). *2020 Secondary mathematics syllabuses* (draft). Singapore: Author.

Ministry of Education (2018a). *2021 Primary mathematics syllabuses* (draft). Singapore: Author.

National Council of Teachers of Mathematics (NCTM). (2000). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

Niemi, D., Vallone, J., & Vendlinski, T. (2006). *The power of big ideas in mathematics education: Development and pilot testing of POWERSOURCE assessments*. CSE Report 697, National Center for Research on Evaluation, Standards, and Student Testing (CRESST).

Warren, E., & Cooper, T. J. (2009). Developing mathematics understanding and abstraction: The case of equivalence in the elementary years. *Mathematics Education Research Journal, 21*(2), 76–95. https://doi-org.libproxy.nie.edu.sg/10.1007/BF03217546

Warne, R. T., & Burningham, C. (2019). Spearman's g found in 31 non-Western nations: Strong evidence that g is a universal phenomenon. *Psychological Bulletin, 145*(3), 237–272. https://doi.org/10.1037/bul0000184.

# Evidence of Young Students' Critical Mathematical Thinking

Chrissy Monteleone
*Australian Catholic University*
chrissy.monteleone@acu.edu.au

Jodie Miller
*The University of Queensland*
jodie.miller@uq.edu.au

In this study, the authors investigate the ways in which young students demonstrate their critical mathematical thinking (CMT). Students aged 5-6 who are beginning their first formal year of education participated in the study. Data is presented from individual clinical interviews undertaken with 16 students. These interviews were analysed using the Critical Mathematical Thinking for Young Students (CMTFYS) framework to identify common patterns in the responses. The findings suggest that these young students beginning school, most often rely on providing explanations and more specifically, justifying, to demonstrate their CMT.

Critical thinking has been identified as an essential skill both for education and future employability, as established by several studies, policy, and curriculum directives (ACARA, 2016; Urib-Enciso et al., 2017). It is a globally recognised term that is emphasised across various subjects, including mathematics, and is considered a crucial skill for preparing students for the 21st century (Urib-Enciso et al., 2017). However, a study conducted in 2018, evidenced that only 50.9% of Australian teachers of middle school classes (104/355) help students to think critically in mathematics lessons (Dix et al., 2018). This is not an easy result to interpret because at present there are no generally accepted definitions of what constitutes critical thinking, especially for young students, nor are there clear practices that can support teachers to develop critical thinking in mathematics. Despite the lack of a clear definition, it is well acknowledged that the development of critical thinking is important for all learners, and it is essential to start developing these skills at the start of formal schooling.

The process of critical thinking involves analysing, evaluating, and making informed judgments or decisions about information or ideas (Urib-Enciso et al., 2017). It can be argued that some of these processes also form part of mathematical thinking, which involves the application of logical reasoning and problem-solving skills to comprehend mathematical concepts and solve problems (Wood et al., 2006). While there are similarities between the two terms, there are also discrepancies. To better understand this intersection, an analysis was undertaken of both sets of literature as part of a larger study to establish the term, Critical Mathematical Thinking (CMT) (Monteleone, 2021) and develop a conceptual framework Critical Mathematical Thinking for Young Students (CMTFYS) that supports the definition and conceptualisation of how young students evidence CMT (Monteleone, 2021).

Despite the push in curriculum direction for teachers to engage students in critical thinking, it is unclear how much this is supported in mathematics education for young students. With little literature to draw on in CMT, it is important to examine the plethora of studies focusing on developing mathematical thinking for primary school students. Examining the research, it appears there are five approaches that can support teachers to guide young students to engage in mathematical thinking: (i) students engaging with strategies that support sense making (Wood et al., 2006); (ii) students displaying reasoning and justifying during learning experiences (Warren et al., 2013); (iii) students making known connections to mathematical ideas and transferring their thinking (Clements & Sarama, 2007); (iv) students progressing in trajectories and displaying their mathematical thinking (Siemon et al., 2017), and (v) students engaging in problem solving (Wood et al., 2006). It is important to note that, the majority of these studies have been conducted with students that have already been attending formal schooling, therefore, little is still known about how young students entering formal schooling, demonstrate their CMT.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 363–370). Newcastle: MERGA.

The purpose of this paper is to address this problem by examining what CMT young students display as they enter formal schooling, underpinned by the CMTYS framework.

## Critical Mathematical Thinking for Young Students Framework

Critical Mathematical Thinking (CMT) is a term that focuses on the application of critical thinking within a mathematical context (Monteleone, 2022; Monteleone et al., 2023). As mentioned above, CMT was conceptualised as part of a larger study, through a review of seminal literature pertaining to the broad areas of critical thinking and mathematical thinking. This led to the development of a conceptual framework titled, Critical Mathematical Thinking Framework for Young Students (CMTFYS) (Monteleone, 2021). The identified themes and sub-themes that underpin the CMTFYS framework emerged from both the critical thinking (Ellerton, 2018; Facione, 1990) and mathematical thinking literature (Cengiz et al., 2011; Wood et al., 2006). Figure 1 presents the CMTFYS themes and sub-themes.

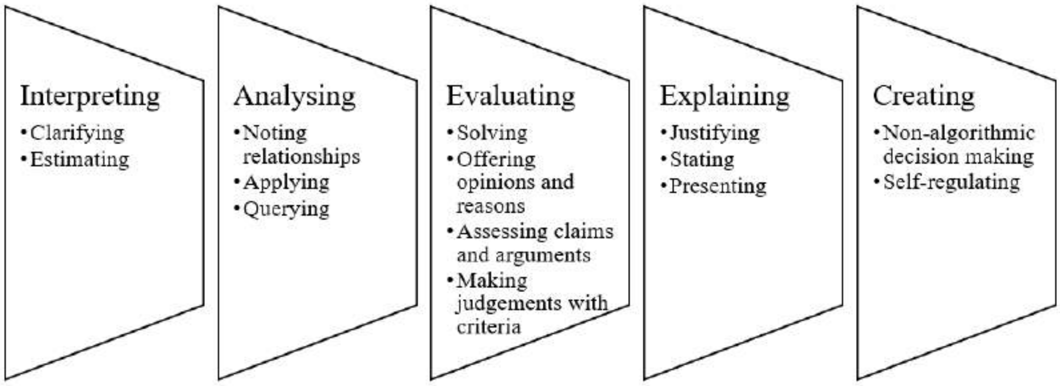



Figure 1. Critical mathematical thinking conceptual framework for young students (CMTFYS).

### *The Five CMTFYS Themes*

The following presents the literature for the five themes presented in the framework.

*Interpreting* is considered an essential part of critical thinking, as it involves the formation of logical judgments or conclusions (Ellerton, 2018). According to Facione (2011), critical thinkers who make decisions may also engage in interpretation. The literature on interpreting that is part of the CMTFYS include clarifying (Facione, 2011) and estimating (Lipman, 2003).

*Analysing* is recognised as an important component of critical thinking, with Facione (2011) incorporating it as a core skill in his definition. He describes analysing as both a cognitive skill and an affective disposition. The sub-themes that represent analysing in critical thinking, emerging from the American Philosophical Association's systematic review on critical thinking (Facione, 1990), include applying, questioning, and noting relationships.

*Evaluating* claims and thought processes has been identified as an essential practice for promoting mathematical thinking (Cengiz et al., 2011). Similarly, Williams (2000) and Wood et al. (2006), present a series of increasingly complex categories that enable students to evaluate their mathematical thinking. The literature on mathematical thinking also identifies sub-themes of evaluating, including making judgments based on criteria (Cengiz et al., 2011), solving problems (Francisco & Maher, 2005), and providing opinions supported by reasoning (Cengiz et al., 2011). The critical thinking literature identifies assessing claims and making judgements (Facione, 1990) as a common disposition found when individuals evaluate.

*Explaining* is when an individual provides reasons for decisions made, as well as depth and detail of the explanation (Halpern, 2013). To better understand how explaining fits within critical thinking, sub-themes were identified primarily from the seminal literature of Facione (1990; 2011). These sub-themes, such as stating, presenting, and justifying, help individuals develop their critical thinking skills by enabling them to explain their thought processes and how they arrived at their judgments (Facione, 2011).

*Creating* involves generating new and innovative ideas as noted by Lipman (1995). Sub-themes associated with creating and critical thinking are related to evaluation and decision-making. One such sub-theme is self-regulation, which involves an individual's ability to evaluate their own inferences. Non-algorithmic decision-making, which involves mental processes, strategies, and representations that people use to solve problems and make decisions, is also identified by Sternberg (1986) as a critical thinking element.

Thus, to ascertain evidence of CMT presented by young learners, the study was underpinned by the following research question:

- What CMT capabilities are evidenced by young students as they begin formal schooling?

## Research Design

The findings presented in this paper are from a larger study (Monteleone, 2021) that utilised an explanatory mixed methods design (Creswell, 2013) to investigate how young students elicit their CMT. This design involved collecting and analysing both quantitative and qualitative data to provide a comprehensive understanding of the research topic. The focus of this paper is on the qualitative interview data collected as part of the study.

### *Participants and Context of the Study*

The larger study involved a total of 161 Kindergarten students (5 years 1 month—6 years 8 months) who were in their first six months of formal schooling, from three urban primary schools located in New South Wales, Australia. All three participating schools had similar demographic features, with the Index of Community Socio-educational Advantage (ICSEA) levels ranging from 1092 to 1112. Additionally, the schools had similar above-average results in the National Assessment Program—Literacy and Numeracy (NAPLAN) assessments.

In total, 16 beginning Kindergarten students were selected to participate in the interviews, which included nine male and seven female students. These students were selected after a set of week-long classroom observations of all 161 students using a designed protocol based on the CMT framework, and analysis of quantitative measures (Raven's Progressive Matrices, Slosson Intelligence Test, and the Patterns and Structure Assessment). The 16 students selected represented each of the three participating schools, with 4 of the selected students coming from School A, 5 from School B, and 7 from School C.

### *Data Collection Methods*

All 16 students participated in individual video recorded task-based one-on-one clinical interviews, consisting of eight learning experiences (Table 1). This method follows Piaget's methode Clinique (Hunting & Doig, 1997), which aims to identify the cognitive capabilities of a child in a social learning context.

### *The Eight Learning Experiences*

In total, eight learning experiences were designed to identify young students' CMT. The learning experiences were designed to: (i) begin with an open-ended question, which allows for a wide range of possible responses and encourages students to think creatively and critically (Nicol & Bragg, 2009); (ii) provide multiple entry points for students, meaning that there were different ways for students to approach the learning experience depending on their prior knowledge (Jorgensen et al., 2010); (iii) use physical manipulatives (e.g., blocks or counters), to help students visualise and make sense of mathematical concepts (MacDonald & Lowrie, 2011); and, (iv) cover a range of mathematical content appropriate for the age group. Table 1 presents an overview of learning experiences from the interview including the types of tasks and questions that were used.

**Table 1**

*Example Learning Experiences (LE) from the Clinical Interview*

| LE | Description of the learning experience |
|---|---|
| LE1 | Framed photo finding the middle: This is a framed photograph of Joey (hold up frame). I would like to hang this frame in the middle of a wall. Now, imagine this piece of paper is a blank wall (hold up A3 paper) and this is the picture frame I need to hang (hold up smaller frame). How can I hang this frame in the middle of the wall? |
| LE2 | Counting unseen items: This is a mini bean bag (show mini bean bag). It is filled with little beans like these (show zip lock bag with some beans). It's too tricky to count them one by one. Can you think of another way to find out how many beans are in this mini bean bag? |
| LE3 | Why is 3 + 3 the same as 4 + 2?: Can you tell me why 3+ 3 is the same as 4+ 2? If appropriate, change the numbers to 2-digit numbers. Ask students to provide two reasons why they are equal. Can you tell me another way you can work this out |
| LE4 | Towers—identifying which tower is taller: Here are two towers that I built earlier (show readymade towers built with different sized blocks). Which tower do you think has more blocks? |
| LE5 | Teddy Bears—real like number sentences: I had some bears in my pocket. Emily gave me some more. I counted and found I have 11 bears altogether. How many did I start with and how many did Emily give me? |
| LE6 | Cubby house—identifying number of tiles required: I have just finished building a cubby house for my children at home (show picture of the cubby house). I would like to put these tiles down on the floor of the cubby house (show square tile). How can I work out how many tiles I need? |
| LE7 | Sandwich—cutting and sharing equally: How many different ways can you cut a sandwich in half? (Provide several pieces of paper shaped as a sandwich). |
| LE8 | Shapes—replicating: How many different ways, using the cut out shapes, can you re-create this shape? (Provide students with the cut out shapes). |

### *Data Analysis*

Deductive analysis, drawing on the CMTFYS framework, was undertaken on transcripts of the 16 video recorded interviews. The transcripts were analysed in iterative cycles focusing on a singular aspect of the CMT themes and sub-themes and coding the students dialogue, gestures (e.g., use of resources) and work samples. Each coded instance was discussed between the researchers to contest and critique during the analysis to ensure limited subjectivity. Table 2 displays an example of the coding undertaken for student 19 (S19) in learning experience six.

**Table 2**

*Example of Data Analysis and Coding for Learning Experience Six*

| Summary of student response | Speaker | Extract from transcript | CMTFYS |
|---|---|---|---|
| Student described using one tile, drawing around it to determine how many floor tiles are required altogether. | S19 | You can count and measure the tile, you can buy one tile and then measure it and then draw around it and then do the same on the others and then count the squares. | Explaining—Justifying (student justified to determine the number of tiles required) |
| | R | You're saying to take a tile and draw around it, trace it and keep tracing to see how many tiles we need? | |

## Results: Evidence of Young Students CMT

Table 3 displays the occurrence of CMT themes across all eight learning experiences. The table is ordered to display the most frequent occurrence of CMT. Each number represents one student exhibiting CMT in that learning experience. For example, 10 students explained their thinking in learning experience one, while five students evaluated in learning experience six.

**Table 3**

*Young Students CMT Themes Across Learning Experiences*

| CMT Theme | Learning Experiences | | | | | | | | Theme Frequency |
|---|---|---|---|---|---|---|---|---|---|
| | LE1 | LE2 | LE3 | LE4 | LE5 | LE6 | LE7 | LE8 | |
| Explaining | 10 | 1 | 1 | 2 | | 4 | 4 | | 22 |
| Evaluating | 1 | | 3 | | | 5 | 2 | 1 | 12 |
| Analysing | 1 | 1 | 2 | | 4 | | | | 8 |
| Interpreting | 6 | 1 | 1 | | | | | | 8 |
| Creating | 1 | | 1 | | | 4 | | | 6 |
| | 19 | 3 | 8 | 2 | 4 | 13 | 6 | 1 | 56 |

From the analysis it is evident that these young students engaged in all forms of CMT as identified in the CMTFYS. Learning experience one and three appear to have provided opportunity for young students to display a range of CMT. Explaining appears to be the most frequent CMT displayed by these young students. This type of thinking was evident across almost all (6/8) learning experiences. To better understand the specific explaining sub-themes evidenced by the students, further analysis was conducted. Table 4 displays the number of occurrences of the explaining sub-themes across each learning experience in order of frequency.

**Table 4**

*Young Students' CMT of Explaining Sub-Themes Across Learning Experiences*

| Explaining Sub-themes | Learning Experiences | | | | | | | | Sub-Theme Frequency |
|---|---|---|---|---|---|---|---|---|---|
| | LE1 | LE2 | LE3 | LE4 | LE5 | LE6 | LE7 | LE8 | |
| Justifying | 6 | | | 1 | | 4 | 4 | | 15 |
| Stating | 2 | 1 | 1 | 1 | | | | | 5 |
| Presenting | 2 | | | | | | | | 2 |
| Frequency across LE | 10 | 1 | 1 | 2 | 0 | 4 | 4 | 0 | 22 |

Analysis of the interview transcripts revealed that for these 16 young students, all sub-themes within the explaining theme were displayed at various points of the interview. Explaining-justifying appeared to be the most commonly displayed CMT theme and sub-theme. This type of CMT (explaining-justifying) also occurred across a range of learning experiences (LE1, 4, 6 and 7). While specific data analysis was not conducted on the learning experiences, it is important to note that it appears that LE1 (framed photo finding the middle) and LE6 (cubby house—identifying the number of tiles required) provided opportunity for these young students to demonstrate more instances of CMT.

The following excerpts are offered to better illustrate the ways in which these young students engaged in explaining-justifying across the learning experiences in the interview.

*Excerpt One*. While participating in LE1 Student 9 (S9) determined where the midpoint of the paper was by drawing lines (intersecting: vertically, horizontally, and diagonally). The conversation between the researcher (R) and S9 included:

R: How do you know?

S9: You can't fold a wall so you can't fold this paper. I'll draw a line here and here and just to prove it to you. I will draw another line this way and another line this way, that is the middle.

S9 displayed the CMT of justifying by also using gestures to show where lines might go and drawing lines to justify the location of the middle.

*Excerpt Two*. While participating in LE4 Student 23 (S23) broke apart the two towers (connected blocks) to demonstrate how the blocks were different in size and that the height of the tower would differ due to the different sized blocks. The statement made during the conversation with the researcher that supported S23's justification included:

S23: They both have the same amount of blocks. What I'm thinking right now is, you know how these blocks are more thicker and taller? If I break one off, you'll see the difference. If I put these together it makes a long tower and you see, if I break all of these off, it's small.

S23 displayed the CMT of justifying by explaining that the difference in the towers was not the number of blocks but the length of the blocks.

*Excerpt Three*. While participating in LE6, Student 1 (S1) used one tile as a repeated unit of measure to determine the tiles required for the cubby house floor. The conversation between the researcher (R) and S1 included:

S1: You can measure and put the square. You can draw the squares.

R: Can you show me what you mean?

S1: You can put tiles from the floor. If you're missing one, you can put one more.

S1 displayed the CMT of justifying by providing reasons to support his strategy.

*Excerpt Four*. While participating in LE7, Student 11 (S11) considered the real life shape of a slice of bread to ensure two people receive the same amount of bread. The conversation between the researcher (R) and student demonstrates how S11 provided a justification for their response.

S23: If you wanted to have two pieces of toast, you could do this. Two for me and two for you.

R: Which two would you get?

S23: I'll get those two and you'll get those two. Let's see how... One, two, three, four.

S11's justification for the actions taken ensures that each person is to receive the exact same amount of bread.

## Discussion and Conclusion

The findings revealed that young students can evidence CMT across all themes of the CMTFYS framework. This shows that CMT is evident in early schooling and young students have CMT capability. If teachers can continue to develop CMT capabilities in students from a young age, this may equip students with the necessary skills for later education and future employability. However, to be able to do this, teachers will need support to understand how to foster CMT in their mathematics classroom (e.g., Dix et al., 2018). The CMTFYS framework may address this issue by supporting teachers to recognise CMT and consider how they provide opportunity for these types of thinking in their mathematics classrooms.

The most common CMT displayed by these young students was *explaining-justifying*. This finding is consistent with earlier research and may have been prevalent for three reasons: (i) research has shown that young students are more likely to explain and justify their mathematical reasoning (Warren et al., 2006); (ii) promoting the ability to explain thinking processes is important for young students and therefore the researcher's questions may have prompted the students to explain their thinking more often; and, (iii) explaining is crucial for learning and understanding mathematical concepts (Facione, 1990; Halpern, 2013) and therefore young students may have had more experience explaining mathematical ideas than engaging in other forms of CMT in prior to school settings. While evaluating, creating, interpreting, and analysing were observed less frequently, these CMT skills should not be ignored. Reasons for the lower occurrences may be that young students have had a lack of opportunity to engage and develop these forms of CMT, and the types of learning experiences in the interview may not have provided enough opportunity to display these forms of CMT.

The study aimed to examine how young students displayed CMT capabilities as they begin formal schooling and has been addressed through the results in alignment with the CMTFYS. The results presented in this paper can be used to inform the teaching strategies that facilitate the development of CMT in young learners. We note the limitations of the study including the small sample size, which may affect generalisability to learners of different ages and the lack of consideration of contextual factors such as teacher practices. Therefore, further research is required including; (i) a larger sample of students to continue to evidence the CMTFYS framework, (ii) an investigation of long-term development of CMT, and (iii) the teaching methods required to foster CMT in young learners.

## References

Australian Curriculum, Assessment and Reporting Authority. (2016). *Critical and creative thinking*. ACARA. https://www.australiancurriculum.edu.au/f-10-curriculum/general-capabilities/critical-and-creative-thinking/

Cengiz, N., Kline, K., & Grant, T. J. (2011). Extending students' mathematical thinking during whole-group discussions. *Journal of Mathematics Teacher Education, 14*(5), 355–374. https://doi.org/10.1007/s10857-011-9179-7

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the building blocks project. *Journal for Research in Mathematics Education, 38*(2), 136–163. https://doi.org/10.2307/30034954

Creswell, J. W. (2013). *Research design: Qualitative, quantitative, and mixed methods approaches.* (4th ed.). Sage publications.

Dix, K., Hollingsworth, H., & Carslake, T. (2018). *Thinking maths: Learning impact fund evaluation report*. ACER.

Ellerton, P. (2018). *On critical thinking and collaborative inquiry*. NSW Department of Education. https://education.nsw.gov.au/content/dam/main-education/teaching-and-learning/education-for-a-changing-world/media/documents/Peter-Ellerton_Critical-Thinking.pdf

Facione, P. A. (1990). *Critical thinking: A statement of expert consensus for purposes of educational assessment and instruction*. The California Academic Press.

Facione, P. A. (2011). *Critical thinking: What it is and why it counts*. The California Academic Press.

Francisco, J. M., & Maher, C. A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *The Journal of Mathematical Behavior*, *24*(3–4), 361–372. https://doi.org/10.1016/j.jmathb.2005.09.001

Halpern, D. F. (2013). *Thought and knowledge: An introduction to critical thinking*. Psychology Press. https://doi.org/10.4324/9781003025412

Hunting, R. P., & Doig, B. A. (1997). Clinical assessment in mathematics: Learning the craft*. Focus on Learning Problems in Mathematics, 19*(3), 29–48.

Jorgensen, R., Grootenboer, P., Niesche, R., & Lerman, S. (2010). Challenges for teacher education: The mismatch between beliefs and practice in remote Indigenous contexts. *Asia Pacific Journal of Teacher Education*, *32*(2), 161–175. https://doi.org/10.1080/13598661003677580

Lipman, M. (2003). *Thinking in education*. Cambridge University Press.

MacDonald, A., & Lowrie, T. (2011). Developing measurement concepts within context: Children's representations of length. *Mathematics Education Research Journal, 23*(1), 27–42. https://doi.org/10.1007/s13394-011-0002-7

Monteleone, C. (2021). *Critical mathematical thinking in young students* [Doctoral thesis, Australian Catholic University]. Australia. https://acuresearchbank.acu.edu.au/item/8x1z9/critical-mathematical-thinking-in-young-students

Monteleone, C. (2022). Teacher questioning to support young students to interpret and explain their critical mathematical thinking. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 402–409) MERGA.

Monteleone, C., Miller, J., & Warren, E. (2023). Conceptualising critical mathematical thinking in young students. *Mathematics Education Research Journal*, 1–21. Advance online first. https://doi.org/10.1007/s13394-023-00445-1

Nicol, C., & Bragg, L. (2009). Designing problems: What kinds of open-ended problems do preservice teachers pose? *Proceedings of the 33rd annual meeting of the International Group for the Psychology of Mathematics Education* (pp. 225–232). PME.

Siemon, D., Horne, M., Clements, D., Confrey, J., Maloney, A., Sarama, J.,Tzue, R., & Watson, A. (2017). Researching and using learning progressions (trajectories) in mathematics education*. Proceedings of the 41st conference of the International Group for the Psychology of Mathematics Education* (pp. 109–136). PME.

Sternberg, R. J. (1986). *Critical thinking: Its nature, measurement, and improvement*. National Institute of Education.

Uribe-Enciso, O. L., Enciso, D. S. U., & Daza, M. d. P. V. (2017). Critical thinking and its importance in education: Some reflections. *Rastros Rostros, 19*(34), 78–88.

Warren, E., Miller, J., & Cooper, T. J. (2013). Exploring young students' functional thinking. *PNA, 7*(2), 75–84. https://doi.org/10.30827/pna.v7i2.6131

Williams, G. (2000). Collaborative problem solving and discovered complexity. *Mathematics Education Beyond, 2*, 656–663.

Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education*, *37*(3), 222–255.

# Achieving Teacher Professional Growth Through a Focus on Making Students' Mathematical Thinking Visible

Tracey Muir
*University of Tasmania*
tracey.muir@utas.edu.au

Providing opportunities for students to demonstrate their mathematical thinking can be challenging. This paper reports on a case study conducted with a teacher and her class of Grade 3/4 students which investigated this phenomenon. Following collection of baseline data that showed her students were not demonstrating evidence of mathematical thinking, the teacher implemented teaching approaches designed to address this. The findings indicated the teaching approaches were effective and led to changes in the teacher's practice and professional growth. The study has implications for teachers who are looking to make changes to their practice based on evidence-informed teaching approaches.

The importance of students learning mathematics with conceptual understanding and being provided with opportunities to demonstrate their thinking is promoted in the research literature and curriculum documents (e.g., ACARA, 2022; National Research Council, 2001). While research has provided us with a shared understanding of what constitutes quality mathematics teaching (e.g., Sullivan, 2011), there is wide variability in how effective teaching approaches are implemented in mathematics classrooms (Callingham, et al., 2017). Evidence suggests that mathematics lessons that are dominated by teacher-led activities and practices do not facilitate student thinking, or deep learning (Hattie, et al., 2017).

Research has shown that effective teaching approaches for mathematics include provision of appropriately challenging tasks, a focus on purposeful discussion, and an expectation that students can explain, reason, and justify their thinking (e.g., Sullivan, 2011). However, even with the provision of suitably challenging tasks and appropriate teaching approaches as advocated by Sullivan (2018), there is no guarantee that students will engage in problem solving and thinking behaviours. What is required is a shift from traditional practices and institutional norms which can be addressed through teacher Professional Learning (PL). If teachers are to make changes to the ways in which they teach mathematics, they require opportunities to be exposed to alternative practices, together with support to implement and reflect on the effectiveness of the practices, and their impact on students' learning (e.g., Muir & Beswick, 2007). Researchers have also identified that changes in teachers' practice are more likely to occur if the PL occurs in teachers' immediate environments (Loucks-Horsley, 2003), involves the use of a knowledgeable other (Timperley, 2008), and results in salient outcomes (Clarke & Hollingsworth, 2002).

This paper reports on a case study conducted with a teacher who engaged in collaborative practitioner research, with a view to improving her practice. The case study was situated in a real educational context, and focused on the trial and evaluation of Liljedahl's (2021) teaching practices for enhancing learning. The study complements and builds on Liljedahl's (2021) research in that it provides an in-depth account of how one teacher worked collaboratively with a researcher to implement three specific practices in a multi-year class in Australia. Liljedahl's reported findings offer accounts of examples of implementation from 40 different classrooms in Canada, rather than in-depth accounts of the benefits and challenges experienced by individual teachers as they implemented the strategies over a two-week period. Specifically the research questions were:

- What teaching approaches facilitate mathematical thinking and make it visible?
- What changes in a teacher's professional growth can occur as an outcome of implementing these approaches?

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 371–378). Newcastle: MERGA.

## Background Literature

### *Studenting Behaviours*

The term ‘studenting’, first coined by Fenstermacher (1986), describes the behaviours that students do to help themselves learn, including paying attention to following instructions, seeking help, and studying. He later expanded the term to include other behaviours that students do that do not help them to learn, such as avoidance behaviours. Liljedahl (2021), who has conducted extensive research in this area, identified five common studenting behaviours, that were manifested when students undertook ‘now you try one’ activities:

*Slacking.* No attempt at task; talking; doing nothing.

*Stalling.* No real attempt at task; legitimate off-task behaviours such as pencil sharpening.

*Faking.* Pretending to do task, but achieving nothing.

*Mimicking.* Essentially copying the teacher, others or worked examples; attempt and often complete task.

*Trying on own*. Use reasoning and understanding to work through task.

Liljedahl’s research found that the majority of students’ time was spent on ‘mimicking’ (53%) and that there was little evidence of actual thinking behaviour observed across the 40 classrooms in which his research was conducted.

### *Practices that Generate Thinking*

In order to address the prevalence of studenting non-thinking behaviours, Liljedahl (2021) identified 14 teaching practices that facilitated student thinking. These included types of tasks, where students work, forming collaborative groups, and when, where, and how tasks are given (for more see Liljedahl, 2021). The 14 practices provided a means for teachers to replace their familiar patterns of teaching, with different, more effective, field-tested techniques. As a result, changes occurred in teachers’ practice not for ideological reasons, but because implementation of the practices led to increased student thinking in the classrooms studied (Liljedahl, 2021).

The types of mathematical tasks that teachers choose to use in their classrooms determines what content students learn, and also how they think about, use and make sense of mathematics (Stein, et al. 1996). Tasks should not focus on right answers, but instead provide opportunities for students to struggle with ideas and to develop a range of mathematical processes (Anthony & Walshaw, 2009). Research-informed practices (e.g., Sullivan, 2011) include the use of challenging tasks that go beyond a focus on procedural fluency, and require students to select their own strategies, develop persistence, and explain their thinking. When teaching using challenging tasks, teachers are encouraged to provide minimal guidance to students when introducing the task, and affirm positive behaviours such as persistence, effort, cooperation, and flexible thinking. (Sullivan, 2018).

## Theoretical Framework

### *Professional Growth Model*

Clarke & Hollingsworth’s (2002) model of professional growth (see Figure 1) illustrates how an external source of information or stimulus provides an opportunity for teachers to engage in professional experimentation related to their practice. Consistent with Guskey’s (1986) seminal change work, the model supports the premise that changes in teachers’ beliefs and attitudes are primarily derived from classroom experiences and likely to occur following evidence of improved student outcomes. As Figure 1 depicts, the model contains four domains, with each domain connected through the mediating processes of enactment and reflection. Professional growth occurs when changes in the personal domain (knowledge, beliefs and attitudes) occur as a result of external stimulus, leading to professional experimentation (such as implementing new teaching strategies) and experiencing salient student outcomes (such as improvements in students' mathematical dispositions). The model was used as a framework to guide the collaborative inquiry process. The researcher acted as an external source of information or stimulus, and the teacher engaged in professional experimentation as she trialled different approaches over the course of a term. Reflective conversations occurred around noting the impact of the approaches on student outcomes, which were often salient and both the researcher and teacher noted changes in knowledge, beliefs and attitudes.

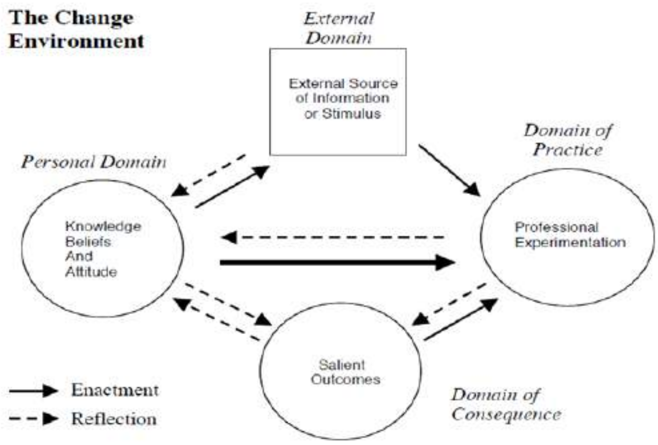



Figure 1. Model of professional growth (Clarke & Hollingsworth, 2002).

## Methodology

The study used a case study paradigm to explain and describe an event or phenomena in the everyday context in which they occurred (Yin, 2009). In this instance, the case study is used to understand and explain the impact of three different teaching approaches undertaken by a teacher involving her class of Grade 3/4 students. The adoption of specific teaching approaches constitutes the bounds of the case (Stake, 1995).

### *Context*

The study occurred in an Australian urban primary school with a Grade 3/4 teacher and her class of 25 students. The teacher 'Shandi' (pseudonym) had eight years of teaching experience, four years at that school and four years of teaching that grade level. She was motivated to participate in the study because she wanted to improve student engagement, and her experience of mathematics PL was limited to the occasional conference and in-school sessions. Full ethical approval was obtained, and consent provided for all participants.

### *Procedure and Data Collection*

The study took place in Term 4 in 2021, over the course of nine weeks. Initial base-line data were collected in two lesson observations to record evidence of studenting behaviours using a time-scale checklist. Unlike Liljedahl's research, the lesson observations were not restricted to 'now you try one' experiences. Entries were made at 5 minute intervals, with a different student observed at each interval. A total of 11 lesson observations occurred throughout the term, with each observation including a completed checklist and field notes. Teacher interviews were conducted following each lesson observation whereby professional conversation and teacher reflections on the lesson occurred, along with identification of when and how to implement future targeted teaching strategies. Following the initial two-week observation period, the iterative cycle involved the selection of an intervention (teaching strategy), implementation of the strategy over two weeks (approximately 5 lessons) and evaluation and reflection on the impact of the strategy. This cycle was repeated three times, with a total of 11teacher interviews being conducted.

### *Data Analysis*

Data from the teacher interviews were analysed using both an inductive and deductive approach (Braun & Clarke, 2022). Firstly, the interviews were fully transcribed and entered into NVivo. Open coding was initially applied, and instances of references to studenting behaviours and evidence of aspects of application of the four elements in the Model of Professional Growth were deductively noted. It was therefore possible for multiple codes to be applied to sections of the data. For example, the following excerpt was coded to salient outcomes; student behaviour—trying; challenging tasks:

> I'm pretty impressed with what I got out them today. So not necessarily if they got the answer right or wrong—just getting things down. Having a go. Being challenged and being okay with being challenged.

## Results and Discussion

In this section the findings from the teacher conversations are presented and discussed. The findings are structured according to the four elements of the model, beginning with detailing the external stimulus provided by the researcher and the student checklist results, and culminating in evidence of shifts in the teacher's beliefs and attitudes. The section concludes with a brief discussion that addresses the research questions.

### *External Source of Information or Stimulus*

The researcher was the primary external source of information which included familiarising Shandi with Lildjedahl's (2021) research into studenting behaviours and recommended teaching practices, and Sullivan's (2018) challenging task approach. Base-line data collected from two initial lesson observations were shared with her and professional conversation took place about the presence and regularity of non-thinking student behaviours. Checklist data showed that 63% of student behaviours observed were non-thinking behaviours, and 26% 'trying' behaviours. Interestingly, another category was generated as 11% of behaviours were best described as 'watching' as any mathematical thinking was not visible. The most prevalent studenting behaviours were slacking and stalling (54%) and included rubbing out, sharpening pencils, and playing with counters. These behaviours were especially dominant in unsupervised group work. In summary, lesson observations indicated there was evidence of high levels of non-thinking student behaviour occurring in mathematics lessons. The next step was then to identify which approaches to trial to address this issue.

### *Professional Experimentation and Salient Outcomes*

Professional experimentation is situated within the Domain of Practice and involves the trialling of an intervention or strategy. Shandi initially selected the provision of challenging tasks to trial as she felt a lot of her practice was text-book based and dominated by a focus on rules and procedures. Subsequent teaching strategies included random groupings and use of vertical writing surfaces (how, when and where tasks are given). Professional conversations tended to link the strategies with impact upon students' behaviour, hence this section is structured around Shandi's reflections on each teaching strategy trialled.

### *Challenging Tasks*

Shandi was "a little familiar with challenging tasks—I've had a little time working with them but not to a great extent" [initial tr interview]. Following a lesson where students were asked to individually record how many different picture frames they could depict that had a perimeter of 24 cm, Shandi explained her approach and what she noticed about the students' thinking behaviour:

> I chose the middle question from the open-ended questions book. I didn't give them any hints—I love that! So they do know this without us telling them so it's getting them thinking.

Shandi continued to source most of her challenging tasks from Sullivan's teacher resource books (Sullivan, 2018; Sullivan & Lilburn, 2006), and then complemented them with Liljedahl's teaching strategies. Towards the end of the term, Shandi shared that in the past she avoided giving students open-ended tasks as:

> They'd be under desks or leaving the room or having full meltdowns, because they would be like, no can't do it. Don't know what the question is—can't do it. Whereas now they don't even ask. They know it's open ended. And I think that's a big thing as well—setting them up for that. And they're definitely more willing to have a go.

### *Random Groupings*

Prior to participating in the study, Shandi's approach to group work was to carefully select group members based on their personality and ability level. She would typically have a teaching group of four or five students of similar ability, with the remainder of the class allocated to small groups who had different rotating activities to complete. As noted earlier, studenting non-thinking behaviours were especially prevalent in groups that were working independently from the teacher on set activities. In her first post-lesson conversation Shandi expressed her preference for similar ability grouping:

> I like the fact that I can work with each group based on ability. I know that ability grouping sometimes is not always the best, and I've tried to do some different strategies for that, but I think, for that explicit teaching, I do find it beneficial having the ability groups for that.

In the same conversation she noted that:

> Group work is not this group's strong point—not at all. They're OK with pairs, but … lots of tantrums when it comes to group work.

Shandi was initially hesitant about the concept of random grouping, mainly due to behavioural management considerations. When trialling this strategy, she used playing cards or random name generators to create groups of three. She provided the following observations regarding her professional experimentation and outcomes for students as they experienced random groupings:

> I think yesterday when we first did the groups, they found it a little bit challenging working with each other. But then today, it was a different kettle of fish. They were fantastic.
>
> I like the fact that they were random, and it gave kids who don't normally work together, or wouldn't even remotely think of joining a group together, the chance to do so. And Jack actually said that he didn't know too much about the two people he worked with and so he had to listen to what they were saying because their thinking was different to his—so I'm thinking, wow!
>
> I think, last week, three times in a row, I had three boys together, just at random, it wasn't anyone's doing. And they actually worked really well together. Whereas normally I would be 'No—they're not working together'. So yeah, that's good—but it has been hard to let that go.

### *Use of Vertical Writing Surfaces*

Once the random groups of three were created, the students were given a challenging task to collaboratively solve and record their thinking on vertical writing surfaces. As the classroom was not equipped with vertical writing surfaces, other than a small whiteboard, Shandi improvised and used A3 sheets of paper taped to walls and windows for the groups to record on. Students worked collaboratively on tasks, with the restriction that the student recording the answers could only record others' thinking, not their own. Shandi was particularly enthusiastic about the impact of this strategy:

> I think it's fantastic. I thought it was really good. It gets them all standing. So they don't have the ability to slack off as much, I think, because they all have a turn of the pen. The people that don't have the pen have to do the thinking. So it's a real group effort. Everyone has something to do, which I think in normal groupings, or activities that I've done, you have the people that will do the slacking or the avoiding, and not engaging

> with what's happening, with what they're doing. But this sort of takes that away. So yeah, I think that has worked really well.
>
> I think them standing and standing up looking at the process, as they do in the vertical way keeps them super engaged, because they don't have a … chance to go and wander around because they have to be with their group of three. Yes, there's a lot more responsibility on them to participate.

Shandi also felt that the combined use of small groupings and the vertical surfaces had a positive impact upon making students' thinking visible:

> I think because they have to articulate to each other their thinking instead of just the one person scribing, they have to actually be able to talk about the maths—because there's no way around it. And then having them change the pen—they're all accountable. So I think that's really improved that discussion, and the articulation of what's actually happening with their mathematical thinking.

For Shandi, the most salient outcome was students' increased resilience. In her last post lesson interview she stated that:

> [The biggest impact on students] has been their resilience. So their resilience to actually work through 'I don't know how to do this' to think' right—how else can I do this?' I think that's been massive, huge.

### *Knowledge, Beliefs and Attitudes*

In her first interview, Shandi revealed that when she reflected on her practice ("what I'm doing, and how I'm doing it"), she believed, "this is definitely not working" and asked herself, "What can I do? How am I going to change? How am I going to help them [students] out?" In the same interview, she self-reflected that:

> When I'm doing that explicit teaching I think sometimes I lose a few, so less me talking and more them doing, which is something I've always had trouble with, and letting go a little bit more. I think that in writing it's all 'I do, we do, you do' whereas in maths I need them to be doing it from the get go.

Shandi's comments indicated that there were aspects of her practice that she wanted to improve and was therefore receptive to trialling new approaches. Evidence of a shift in her knowledge, beliefs, and attitudes was apparent in ongoing post-lesson interviews when she reflected positively upon the students' engagement in the lessons, students' improved resilience, and the increase in thinking and on task behaviours. Furthermore, in her final interview, she stated that the biggest impact on her was:

> Letting go a bit more and not having such a tight hold on everything. I've had to go outside my comfort zone and be OK with the groups not being perfect. I've learned to be OK with them talking and being a bit louder whereas I think for me before, they were nice and quiet and relaxed, whereas this is a little bit more hectic. But for me, I've got to let that go, and it's OK because I know they're on task.

Participation in collaborative inquiry not only led to a shift in Shandi's beliefs, it also broadened her knowledge of teaching strategies. In summary, Shandi trialled the use of challenging tasks, and associated pedagogical approach, random groupings, and use of vertical writing surfaces. In her final interview, she confirmed that she would continue to implement the strategies and "that they could all be incorporated in some way, shape or form throughout all units of maths and in other areas as well". She questioned her previous approaches, such as "doing all that whiteboard stuff on the floor and them just copying what I'm doing" and the use of worksheets as "not showing that thinking—it's just copying or following the formula". Shandi identified that there would still be a place for "explicit teaching because sometimes you want to be able to copy to get it in some situations—it's a tricky one—but I think I'm relying on that too much".

### *Teaching Approaches That Facilitate Mathematical Thinking*

The 14 practices that were identified by Liljedahl (2021) as generating more thinking than institutional normative practices were subjected to trials and iterations until they became 'optimal practices for thinking' (p. 16). Like the teachers in Liljedahl's (2021) research, Shandi trialled each approach for two weeks, but in contrast to his teachers, she engaged in regular post lesson interviews with the researcher. The findings from Shandi's case study indicate that the three practices that she trialled with her class did facilitate mathematical thinking, and also developed students' dispositions to attempt tasks, persist and demonstrate resilience, likely influenced by the implementing challenging tasks approach (Sullivan, 2018).

The use of challenging tasks reduced the tendency for the students to copy or mimic the teacher—a dominant non-thinking studenting behaviour in both Liljedahl's research and consistently referred to by Shandi. While mimicking may be useful for teaching students how to replicate routines, it tends to happen not alongside of, but instead of, thinking (Liljedahl, 2021).

Visible random groupings facilitated thinking behaviour in that the mismatch that often occurs between the teacher and the students' goals when groups are strategically or self-selected, was removed (Liljedahl, 2021). Instead students did not enter into group work 'feeling like they were going to be a follower rather than a leader (Liljedahl, 2021, p. 41), provided the students believed that the groups were genuinely random. Consistent with Liljedahl's findings, groups of three were the optimal group size.

Liljedahl's (2021) research found that students standing and working on vertical surfaces had several advantages, including making their work visible, removing students' sense of anonymity, and increased accountability. In Shandi's classroom, this practice was optimised when students not only took turns having the pen, but were only allowed to record others' ideas. This seemed to result in an even distribution of contributions and an expectation that group members worked collaboratively.

While student mathematical outcomes were not evaluated, classroom observations reinforced Shandi's perceptions that students were on task, with less evidence of non-thinking behaviours than earlier in the term. This observation was reinforced from the final lesson observation where there were no instances of mimicking or stalling behaviour; instances of slacking behaviour (24%) were noted however. As indicated earlier, an additional category was added to the sudenting behaviours to accurately depict that 'watching' was commonly observed, which was not easily attributable to slacking, stalling or faking behaviour.

### *Teacher Professional Growth*

Clarke & Hollingsworth's (2002) model proved useful in interpreting Shandi's professional growth. Evidence from Shandi's interviews demonstrate that she was open to professional experimentation of teaching strategies provided through an external stimulus. Furthermore, she noted improvements in students' dispositions to undertake tasks and to work collaboratively, and a decline in non-thinking behaviours. These salient outcomes likely influenced her commitment to continue with trialling subsequent approaches, reflecting on their effectiveness, and identifying future teaching approaches. Consistent with effective professional learning principles, teacher change is more likely to occur and be sustained if it is situated within realistic contexts and teachers are given multiple opportunities to trial new approaches and evaluate the impact of these approaches (Timperley, 2008). Shandi's engagement with, and commitment to the process, including opportunity to engage in post-lesson reflections with the researcher, created the optimum conditions for professional growth to occur.

## Conclusions and Implications

The case study presented in this paper provides evidence to support Liljedahl's (2021) assertion that non-thinking studenting behaviours are prevalent in today's mathematics classrooms. It is possible, however, to reduce the prevalence of these behaviours through the use of teaching practices that enhance and promote student thinking. Through being exposed to these practices through the input of an external stimulus (the researcher), and supported to engage in professional experimentation through trialling selected practices, Shandi demonstrated professional growth in her teaching of mathematics. Like Ms Duo (Liljedahl & Allan, 2013), Shandi changed aspects of her practice, with her involvement in the collaborative inquiry process a powerful catalyst for initiating teacher change. She was motivated to sustain these changes as she observed and reflected upon the positive impact of the trialled teaching strategies on her students. In this sense, Shandi experienced a form of very individualised, supportive professional learning that occurred in the realistic context of her classroom.

This case study has implications for school leaders and policy makers in terms of valuing what constitutes effective professional learning and resourcing it accordingly. While involving teachers in collaborative classroom research may be time consuming and resource intensive, the benefits are likely to be worth the investment. For teachers, the practical implications are that reflecting on practice with an external colleague or educator, along with observing salient student outcomes, provides the impetus to continue to engage in professional experimentation. Implementing some of Liljedahl's (2021) 14 teaching strategies could form the basis of that experimentation given that this case study has added to the research evidence that the strategies are effective in increasing students' thinking behaviours. Future studies could examine the impact and effectiveness of other strategies, along with trialling how collaborative research into practice can be upscaled to develop teacher capacity and engage teachers in authentic individualised professional learning experiences.

## References

Australian Curriculum, Assessment and Reporting Authority. (2022). The Australian curriculum (version 9.0). ACARA. https://v9.australiancurriculum.edu.au/

Anthony, G., & Walshaw, M. (2009). *Effective pedagogy in mathematics*. International Academy of Education.

Braun, V., & Clarke, V. (2022). *Thematic analysis*. Sage.

Callingham, R., Beswick, K., Carmichael, C., Geiger, V., Goos. M., Hurrell, D., Hurst, C., & Muir, T. (2017). *Nothing left to chance: Characteristics of schools successful in mathematics*. Office of the Chief Scientist, Department of Industry, Innovation and Science.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teaching professional growth. *Teaching and Teacher Education, 18*(8), 947–967. https://doi.org/10.1016/S0742-051X(02)00053-7

Fenstermacher, G. (1986). Philosophy of research on teaching: Three aspects. In M. C. Whittock (Ed.), *Handbook of research on teaching* (3rd ed.), (pp.37–49). Macmillan.

Guskey, T. R. (1986). Staff development and the process of teacher change. *Educational Researcher*, *15*(5), 5–12. https://doi.org/10.3102/0013189X015005005

Hattie, J., Fisher, D., & Frey, N. (2017). *Visible learning for mathematics*. Corwin.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. The National Academies Press. https://doi.org/10.17226/9822

Liljedahl, P. (2021). *Building thinking classrooms*. Corwin Press.

Liljedahl, P., & Allan, D. (2013). Studenting: The case of "now you try one". In A. M. Lindmeier, & A. Heinze (Eds.), *Proceedings of the 37th conference of the International Group for the Psychology of Mathematics Education* (Vol 3, pp. 257–264). PME.

Loucks-Horsley, S., Love, N., Stiles, K., Mundry, S., & Hewson, P. (2003). *Designing professional development for teachers of science and mathematics.* Corwin Press.

Muir, T., & Beswick, K. (2007). Stimulating reflection on practice: Using the supportive classroom reflection process. *Mathematics Teacher Education & Development*, *8*, 74–93.

Stake, R. (1995). *The art of case study research*. Thousand Oaks.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455–488. https://doi.org/10.3102/00028312033002455

Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies*. ACER Press.

Sullivan, P. (2018). *Challenging mathematical tasks*. Oxford.

Sullivan, P., & Lilburn, P. (2004). *Open-ended maths activities* (2nd ed). Oxford.

Timperley, H. (2008). *Teacher professional learning and development. Educational Practices Series-18*. International Academy of Education & International Bureau of Education.

Yin, R. K. (2009). *Case study research, design, and method*. Sage Publications, Ltd.

# Peer Collaboration in Early Years Mathematics: A Linguistic Analysis

Carol Murphy
*University of Tasmania*
carol.murphy@utas.edu.au

Damon Thomas
*University of Queensland*
damon.thomas@uq.edu.au

Tracey Muir
*University of Tasmania*
tracey.muir@utas.edu.au

In this study, we aim to investigate the role of language in small group peer collaboration. Based on the notion of symmetrical scaffolding, we draw on theories of dialogic space and functional linguistics to analyse a transcript of three six-year-old students as they explore a shared understanding of ‘two more than’. Using Martin and White’s (2005) engagement framework as an analytical tool, we identify how the three students used language resources in contractive and expansive ways to move each other’s learning forward. These findings provide a perspective of symmetrical scaffolding closely focused on students’ language choices that support engagement with each other’s thinking. We suggest that a focus on use of language with young students is valuable in identifying the type of mathematical discourse that will support peer collaboration in problem solving and mathematical reasoning.

Small group work is often seen to benefit students in learning mathematics where there is potential for students to develop mathematical reasoning and problem-solving abilities. Studies that analyse small group work in mathematics are still few but key elements of such studies modelled how students share their mathematical point of view (Schoenfeld, 1992), investigate socially mediated metacognition in relation to Vygotsky’s Zone of Proximal Development (ZPD) (Goos et al, 2002), or identify types of productive talk (Mercer & Sams, 2006). Despite the acknowledgement of the role of language in such studies, direct analysis of students’ choice of language to share mathematical points of view are less well known.

In this paper, we base analysis on a functional use of language to identify how students’ choice of words play a role in symmetrical scaffolding in learning mathematics. Whilst scaffolding has been defined as “the process that enables a child or novice to solve a problem, carry out a task, or achieve a goal which would be beyond his unassisted efforts” (Wood, Bruner, & Ross, 1976, p. 90), we bear in mind that the basis of this definition stems from studies on how adults help children learn language through playful games. Hence, we propose that use of language is particularly salient with students in early mathematics classrooms who are in nascent stages of engaging in the discourse of mathematical reasoning and problem solving. To illustrate this premise, we present an example of analysis based on functional linguistics from a group work of three Grade 1 (six-year-old) students as they work together to share mathematical points of view in finding ‘two more than’.

## Key Literature

Scaffolding is more often used with reference to teacher and student interactions and Vygotsky’s Zone of Proximal Development (ZPD) has often been generalised to support from “more knowledgeable others”. Both of these suggest an asymmetrical relationship. However, several studies have related scaffolding and ZPD to peer collaborative learning. Forman (1989) investigated how students’ different perspectives of a problem are coordinated through group work as they explored each other’s reasoning, attended to another’s viewpoint, and potentially changed another’s mind. Goos et al’s (2002) study took a socially mediated metacognitive approach in relation to symmetrical scaffolding and suggested that, in peer collaboration, all students had some knowledge. Whilst the knowledge may be incomplete and relatively equal, the contribution of each group can

(2023). In B. Reid-O’Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 379–386). Newcastle: MERGA.

move knowledge forward. Forman also referred to this as one student's response 'pulling' another student into the zone.

Such perspectives are useful in modelling a symmetrical perspective of ZPD, but they do not focus on the way language is used to attend to and potentially change another's mind. Teasley and Roschelle (1993) once stated that collaboration is "a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of the problem" (p. 235). This notion of shared conception resonates with Schoenfeld's (1992) shared mathematical point of view. A shared conception or point of view is a fundamental characteristic of a dialogic space, a space where multiple ideas are explored and in which students "think and act collectively" (Wegerif, 2018, p.2). Within such a space, there is a process of "mutual attunement" or resonance of ideas (Wegerif, 2013). Participants are open to each other's ideas. Littleton and Mercer (2013) further defined dialogic space as a co-constituted linguistic process where students are engaged in iterative negotiations of shared meanings. As such, the function of language within a dialogic space is not just for co-location of individuals, but for shared meaning.

These notions suggest that the exploration of multiple ideas requires the opening of a dialogic space in which students can "think and act collectively" (Wegerif, 2018, p.2), and that language is intrinsic to this exploration. Hence, we build on Littleton and Mercer's assertion that the process of opening a dialogic space is linguistic and we assert that the opening of a space to explore meaning relies on students' choice of language. Some choices of language could open a space and others close it down. What is not known is how students' linguistic choices influence the opening and expanding of dialogic spaces. Finding a way to analyse the opening and closing of dialogic spaces, could help determine the effectiveness of talk to share thinking within group work.

Hence, we ask the question: In what ways can a linguistic analytical approach determine how students use language in small group work to share meaning in mathematics? To answer this question, we bring together facets from scaffolding and ZPD with the notion of dialogic space in examining the functional use of language in small group collaborative work. The aim is to explore how language is used by the students to 'pull' each other into the zone.

## Methodology

Halliday and Matthiessen's (2004) work on systemic functional linguistics (SFL) sees language as meaning oriented and embedded in a social context. Martin and White's (2005) engagement framework, based on SFL, is used in our study to analyse the semantics of young students' language within small group work. Martin and White's framework draws on Bakhtinian notions of heteroglossia and monoglossia. Monoglossia refers to talk that deals with one voice and one big idea (Bakhtin, 1984) and relates to discourse that is one-sided, rationale, and singular (Roth, 2009). Terms that are monoglossic do not seek to engage, they state facts (e.g., *so, 6 add 2 is 4* or *that accounts for…*). These terms do not leave space for negotiation. Terms that are heteroglossic seek to engage in multiple or possible ideas in a way that confronts, challenges, and transforms learners' ideas (e.g., *I think…* or *it could be...*). Our premise is that talk in small peer groups enables multiple ideas to be explored. Rather than "expounding already found, ready-made irrefutable truth" (Roth, 2009, p. 94), students with equal and incomplete knowledge are still determining "the truth," that is, a shared meaning regarding a mathematical idea.

Martin and White's engagement framework refers to monogloss and heterogloss in determining how language choices expand or contract the dialogic space. In this study, we focus on part of the framework that deals with linguistic terms that are heteroglossic in nature (Figure 1). Whilst dealing with multiple ideas, heteroglossic terms can be contractive in that they disclaim by denying (*no, never*) or countering (*yet, although*) or proclaim by concurring (*of course*), pronouncing (*indeed*) or endorsing one idea (*this proves that*). Heteroglossic terms can also be expansive in that they entertain alternative ideas (*it's possible that*) or attribute to existing knowledge (*the report states that*). It would seem that linguistic terms that are expansive are those most likely to open dialogic spaces and allow for students to share mathematical points of view and, hence, pull a student into the zone.

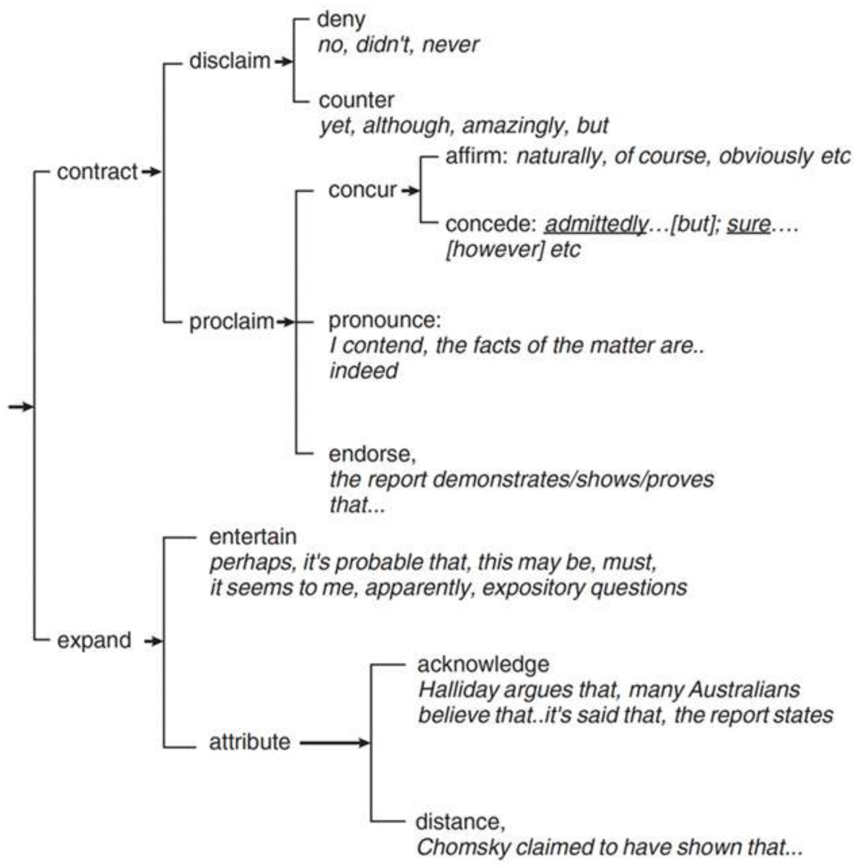



Figure 1. The engagement framework: Heteroglossia (from Martin & White, 2005, p. 134).

In this paper we present an extract of a transcript taken from a study in New Zealand where the researcher worked with one teacher over a year to introduce tasks to encourage collaborative talk amongst Grade 1 (six- to seven-year-old) students. The teacher and researcher had determined that, whilst the students were able to answer 3 + 2 confidently, they were not confident in the notion of 'two more than three'. Such lack of confidence is not unsurprising and suggests that the students are working within an operational view rather than a relational view of arithmetic (Stephens, 2006). The transcript is taken from a video recording early in the study towards the end of the first term focusing on one group of three Grade 1 students, Kim, Emma, and Helen (pseudonyms). Initial attempts to encourage the three students to work on such problems had not elicited any talk or agreement. It was decided to re-introduce the problem with visual representations, in this case ten frames (Figure 2). Ethical approval was obtained based on the university's ethical protocol. Teacher and parental consent were obtained as well as oral assent from the students involved.

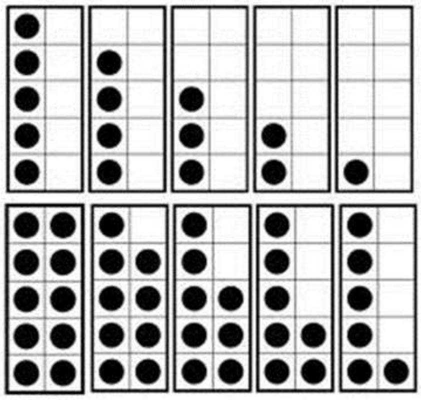


Figure 2. Set of ten frames.

## Analysis and Results

The three students, Kim, Emma, and Helen were asked by the researcher to order a set of ten-frames from one to ten. The three children were successful in ordering the ten-frames and were able to state that each card had one more dot. The researcher then introduced a task to order the ten frames so that there are two more each time. To start, the researcher placed the one ten-frame on the table and asked the students which ten-frame would have two more than one. Helen and Emma thought the two ten-frame would be next, but Kim thought it should be the three ten-frame. The researcher asked Kim to explain why she thought it should be three and not two. The remaining dialogue is presented as a transcript below and is analysed using an adaptation of Martin and White's framework (Table 1).

**Table 1**

*Adaptation of Martin and White's Engagement Framework*

| Heteroglossic terms | | Examples |
|---|---|---|
| Contract | Disclaim | Deny: No; No, it has to be… |
| | | Counter: But it has to… |
| | Concur | Affirm: I think she's right |
| | Proclaim | Pronounce: I've got… |
| Expand | Entertain | I think it might be…; See, I've got… |

| | | |
|---|---|---|
| 1 | Kim: | But it *(points to the 2 ten-frame)* has got one more on it, not two more on it. **DISCLAIM: COUNTER**<br>You have to say two more added, **PROCLAIM: PRONOUNCE**<br>like you're skipping one, like you're counting in twos. **ENTERTAIN** |
| 2 | Helen: | I think she's right because she's just skipping the two and it's going onto that one *(points from the one to the three ten-frame)*. **CONCUR: AFFIRM** |
| 3 | Emma: | I think it's right. **CONCUR: AFFIRM** |
| 4 | Researcher: | If that's got two more than one *(pointing to the 3 ten-frame)*, what's going to come next? *(pointing to the space following the ten-frame)* **ENTERTAIN.** It's got to have two more on it. *(Moves away from the table.)* **PROCLAIM: PRONOUNCE** |

5 Helen: *(Reaches over to 2 ten-frame but does not pick it up.)* Two. **MONOGLOSS**
*(Emma Picks up 2 ten-frame and then puts it down in front of her.)*

6 Helen: *(Looks at other ten-frames in front of her.)* No. **DISCLAIM: DENY**
Four, four, four. I've got four, I've got four. **PROCLAIM: PRONOUNCE**
*(Emma picks up 4 ten-frame and hands it to Helen (Figure 3).)*

Figure 3. Emma hands 4 ten-frame to Helen.

7 Kim: *(Moves 5 ten-frame in line next to 3 ten-frame and counts the dots on the 5 ten-frame (Figure 4).)* I can put two more on it. See one, two, three and then two more. Five. **ENTERTAIN**

Figure 4. Kim counts dots on 5 ten-frame.

8 Helen: Yes, that's five. **CONCUR: AFFIRM**

9 Researcher: You've all got to agree which ten-frame goes next. (*To Kim.*) You have to persuade them which ten-frame goes next.
*(Kim looks at Emma. Emma has 4 ten-frame in her hand and looks at it. Kim moves the 5 ten-frame away from the line but keeps it under her hand. Emma places the 4 ten-frame on table, near the line but not next in line (Figure 5).)*

Figure 5. Emma places 4 ten-frame next to the 3 ten-frame.

10 Kim: (*Points to the 3 ten-frame and then the 5 ten-frame.)* There's three on this line and three on this one but two more on it. *(Kim and Emma look at each other. Kim puts the 5 ten-frame next to the 3 ten-frame in line (Figure 6).)* **PROCLAIM: PRONOUNCE**

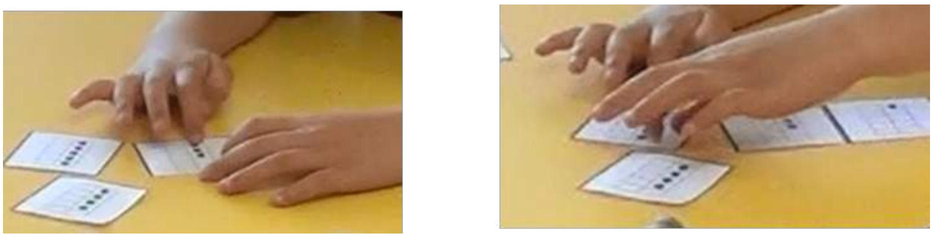


Figure 6. Kim puts the 5 ten-frame next to the 3 ten-frame in line.

At this point the children are silent. Their eyes look down towards the table or around the room. This lasted for forty seconds. From the observer's perspective, it is not clear if the students are still considering whether 5 is two more than 3, but then Helen entertains an idea to continue the task of 'two more than' without being asked.

| | | |
|---|---|---|
| 11 | Helen: | *(Moves forward in her chair and looks towards Kim.)* I think seven. **ENTERTAIN** |
| 12 | Kim: | *(Looks towards Helen.)* Uh… ha? **ENTERTAIN** |
| 13 | Emma: | *(Looks at Helen.)* I think seven too. **ENTERTAIN** |
| 14 | Helen: | Now we have to figure out which one is seven. *(All three children look through the ten-frames to find the 7.)* **INSTRUCTIONS** |
| 15 | Emma: | Seven *(picks up the ten-frame and hands it to Helen, who then hands it to Kim)* **MONOGLOSS** |
| 16 | Helen: | Seven, do you all agree? Kim do you agree? **ENTERTAIN** Emma? **ENTERTAIN** *(Emma nods head).* **CONCUR: AFFIRM** I think it's right *(Helen turns to look at Researcher)* **ENTERTAIN** |
| 17 | Researcher: | Ok, so what do you think will come after seven then? **ENTERTAIN** |
| 18 | Helen: | Nine **MONOGLOSS** |
| 19 | Kim: | Yes nine (Helen places the 9 ten-frame.), **CONCUR: AFFIRM** and then eleven, …if we had an eleven. **ENTERTAIN** |

In determining the results, we present a descriptive commentary on the students' (and the researcher's) dialogue. Initially (lines 1 to 3), students use dialogical contractions in the form of disclaim and concur. Kim denies the ideas of the other two students (*But it has got one more on it, not two more on it*) and then uses the modal term 'have to' in proclaiming and pronouncing that two have to be added. She then illustrates this with entertaining the notion of 'skipping one.' Kim's entertainment of skipping one appears to pull Helen into a shared idea and she uses similar terms regarding 'skipping' and 'going on to' to establish a shared understanding of two more than. Emma nods in agreement but at this point it is not clear how well her gesture of concurrence is related to a shared understanding.

The students then move to find two more than three. Despite what seemed like a shared understanding, when asked to find two more than three Helen states two (line 5) almost as a fact and Emma follows by placing the two ten-frame. However, Helen than disclaims her own thinking and pronounces 'I've got four'. Emma follows the pronouncement and hands the four ten-frame to Helen. Kim (line 7) then moves the five ten-frame. Kim's use of modality 'I can put two more on it' suggests she is entertaining an idea to be shared with the group. This is then affirmed by Helen (line

9). Whilst not saying anything, Emma still holds the four ten-frame, possibly suggesting that she does not concur with the Kim and Helen, and the researcher prompts Kim to share her thinking further with Emma. In line 10, Kim contends that five is two more than three, pronouncing 'there's three on this…' whilst pointing to and comparing the five ten-frame with the three ten-frame indicating the two more with her fingers.

It is not clear if Emma concurs with the shared viewpoint, but Kim seems to have pulled Helen's thinking forward to the extent that Helen now takes the lead with a leap to further entertain seven being next in the continued sequence (line 11), albeit after a gap in the dialogue. This leap was not instigated by the researcher or the other students, but it seems that the relational structure of 'two more than' is generalised further and Helen now pulls Kim (*Uh… ha* line 12) forward into the shared thinking and possibly Emma as well *(I think seven too* line 14). It now seems that all three students are entertaining a shared idea. The researcher then asks what would be next after seven and nine is given as a fact or a truth now known by the three students, however the students continue to entertain ideas beyond the known fact suggesting a secure shift in thinking towards relational thinking.

## Concluding Remarks

The analysis of the transcript based on Martin and White's (2005) engagement framework has enabled the determination of language choices by young students that support peer symmetrical scaffolding within small groups. Whilst confirming previous findings that symmetrical scaffolding can support learning in mathematics, such as Goos et al (2002), the analysis provides a fine-grained examination of how language is used to share the equal but incomplete knowledge of peers, and then used to draw each other into the zone or shared meaning.

The dialogue started from a basis where one student, Kim, had some understanding of the relational thinking. Through her use of language (and gestures) she pulled the other two students into the zone by using language that entertained ideas, to the extent that Helen then took the lead. Use of Martin and White's (2005) framework helped to understand how both expansive and contractive terms were used to create a dialogic space. It seems that most of the expansive dialogue occurred when students moved towards agreement, first with five as two more than three and then with seven as two more than five, as they entertained with each other a shared meaning of two more than and checked they are thinking the same. The contractive terms suggested that students were concurring and pronouncing how they now shared the relational meaning. Such analysis suggests that young students are capable of using language to share multiple ideas and arrive at a common understanding. The focus on SFL provided an alternative way to understand the symmetrical scaffolding that is taking place.

Further studies of young students' dialogue within shared group work are needed to establish this premise. In addition, the role of the researcher in instigating the students' dialogue is relevant and raises the question, how can a teacher instigate such dialogue in a classroom? What are the prompts and probes that encourage young students to use both expansive and contractive terms in productive ways? We note that the adaptation of Martin and White's engagement framework suggested a more limited choice of engagement language by these young students. Further research is needed to determine if extended use of prompts and probes modelled by teachers would develop the students' language choices to include more engagement terms. Also of note is the use of gestures, such as the nod of a head, a quizzical look, or directing attention by pointing and modelling using representations. Whilst SFL and the engagement framework focus on language and use of terms, the use of gestures seemed an intrinsic part of the dialogue and further work is needed to explore their role in expansive and contractive ways.

# References

Bakhtin, M. M. (1984). *Problems of Dostoevsky's poetics.* University of Minnesota Press.

Forman, E. A. (1989). The role of peer interaction in the social construction of mathematical knowledge. *International Journal of Educational Research 13*, 55–70.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics, 49*, 193–223.

Halliday, M. A. K., & Matthiessen, C. M. I. (2004). *An introduction to functional grammar.* Routledge.

Littleton, K., & Mercer, N. (2013), *Interthinking: Putting talk to work*. Routledge.

Martin, J. R., & White, P. R. R. (2005). *The language of evaluation: Appraisal in English.* Palgrave Macmillan.

Mercer, N., & Sams, C. (2006). Teaching children how to use language to solve maths problems. *Language and Education, 20*(6), 507–528.

Rojas-Drummond, S., & Mercer, N. (2003). Scaffolding the development of effective collaboration and learning. *International Journal of Educational Research, 39*(1), 99–111.

Roth, W-M. (2009). *Dialogism: A Bakhtinian perspective on science and learning.* Sense Publishers.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). Macmillan.

Stephens, M. (2006). Describing and exploring the power of relational thinking. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures, and learning spaces, Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia* (pp. 479–486). MERGA.

Teasley, S. D., & Roschelle, J. (1993). Constructing a joint problem space: The computer as a tool for sharing knowledge. In S. P. Lajoie and S. J. Derry (Eds.), *Computers as cognitive tools* (pp. 229–258). Erlbaum.

Wegerif, R. (2013), *Dialogic: Education for the internet age*. Routledge.

Wegerif, R. (2018). A dialogic theory of teaching thinking. In L. Kerslake, & R. Wegerif (Eds.), *The theory of teaching thinking* (pp. 98–114). Routledge.

Wood, D., Bruner, J. S., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry, 17*, 89–100.

# Data Interpretation and Representation in Middle Primary: Two Case Studies

Gabrielle Oslington
*Macquarie University*
gabrielle.oslington@mq.edu.au

Joanne Mulligan
*Macquarie University*
joanne.mulligan@mq.edu.au

Two case studies of Australian primary school students tracked changes in their data interpretation and representation over three years. Students were engaged in predictive reasoning tasks based on their interpretation of a data table showing temperature change over time. Students' explanations and graphical representations were collected at the beginning of Years 3 and 4 and the end of Years 4 and 5. The first case study was a student mathematically weaker than her peers while case study two was within the average range for her year. Despite differences in starting points, both case studies followed a similar developmental sequence of predicting, interpreting and representing, with the first case generally lagging one stage behind the second case. Similarities and contrasts between the two students are discussed.

Providing rich and complex contexts for data exploration has proved a valuable means for developing statistical literacy in primary school students. Through structured inquiry tasks with a range of possible solutions, students can make predictions and engage in meaningful investigations by making sense of the information provided (English, 2012; Fielding & Makar, 2022; Watson, 2018). In previous reports we have described changes over time of the predictions, interpretations and representations of 44 Australian primary school students when engaged in a single predictive task (Oslington, et al., 2020, 2021). In this paper we focus on the progress of two individual students attempting the same predictive task over a three-year period.

## *Conceptual Framework*

Based on our findings, we assert that as students move through their primary school years, they exhibit an increasing level of structure in their representations and reasoning about data. By increasing structure, we mean that students start to identify and explain general properties and relationships between data sub-sets and this relational understanding will be reflected through their observations and representations. In analysing statistical development, Konold et al. (2015) described students' data observations as moving through a "loose hierarchy" where the student's focal point—or data lens—changed with maturity. Data interpretation requires describing and generalising from aggregate features of data sets, however, younger students often see data as simply a collection of points. Konold et al. (2015) identified four distinct stages or perceptual units described as 'data lenses' through which sets may be viewed: (a) idiosyncratic or unrelated to the data set (b) a single data point or points only (c) similarities between groups of data and finally (d) a wholistic interpretation where the students observed aggregate and variable properties of the set which may include data range, modal clumps, data trends and aggregation. Our primary cohort study indicated shifts in students' focus from idiosyncratic observations towards describing aggregate properties as they moved through the middle primary years (Oslington et al., 2023). We hypothesise that these shifts in data lenses may also be apparent at the individual level when examining changes in students' data interpretations and how they represented similar data sets. Data representation is an important sense making process as it allows students to visualise the structure of the data and in the early years is closely aligned to students understanding of the meaning of the data (English, 2012; Leavy & Hourigan, 2018; Mulligan, 2015). Structural features might include graphing conventions such as collinearity, equal spacing, data sequencing and coordination of bivalent data. We predict that more of these features are likely to be present in the work samples of individual

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 387–394). Newcastle: MERGA.

students from the later primary years, compared with those they produced in earlier years. In this paper we focus on our research question:

- How does data interpretation and data representation change in individual students between Years 3 and 5?

## The Design Study

The first two iterations of this design study involving 44 primary school participants have been previously described (Oslington et al., 2020, 2021, in press). In this paper we provide a fine-grained analysis of students' shifts in data representation and interpretation through examination of two case studies. These case studies were selected post hoc as individuals who were considered representative of two contrasting ability levels: average and low achieving. The students attended an independent primary metropolitan school from the same year cohort. The school population had a high index of community socio-economic advantage (ICSEA), with 75% of families above the Australian average.

Iris (pseudonym) was 8y 0mo at the commencement of the study, and comparatively less able than her peers in mathematics and language arts. An individual learning assessment conducted at the end of Year 2 indicated that Iris was in the high-average range for IQ (Wechsler Intelligence Scale for Children—WISC-V), but approximately eight months behind peers for mathematics and 12 months behind peers in language arts (Wechsler Individual Achievement Test–WIAT-II). Iris received learning support through small group interventions with specialised teachers in Years 1, 2 and 3, and in-class learning assistance in Year 4. In the National Assessment and Literacy Program (NAPLAN) conducted in Year 3, Iris achieved mid-range Band 4 for mathematics which was slightly below the national average for other Year 3 students, and well below the school average at the top of Band 6. Sophia (pseudonym) was 7y 11mo at the commencement of Year 3. She was the more mathematically able than Iris, achieving low Band 6 for mathematics, slightly below the school NAPLAN average. Sophia had regular classroom placement for mathematics in all years and participated in literacy extension classes in Year 4 and Year 5. Neither student was achieving at the highest or lowest level for their year. Ethical consent for collection of digital recordings and work samples was obtained from participants, carers and teachers.

| **Highest daily temperature in the month** | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Max Temp** | **Jan** | **Feb** | **March** | **April** | **May** | **June** | **July** | **Aug** | **Sept** | **Oct** | **Nov** | **Dec** |
| 2010 | 41 | 38 | 31 | 31 | 27 | 21 | 21 | 25 | 27 | 29 | 30 | 31 |
| 2011 | 35 | 42 | 34 | 27 | 23 | 21 | 21 | 26 | 33 | 34 | 37 | 27 |
| 2012 | 30 | 29 | 27 | 26 | 23 | 20 | 20 | 21 | 24 | 26 | 27 | 29 |
| 2013 | 46 | 30 | 34 | 29 | 26 | 22 | 24 | 25 | 32 | 37 | 34 | 36 |
| 2014 | 37 | 32 | 30 | 30 | 27 | 23 | 25 | 23 | 34 | 34 | 37 | 32 |
| 2015 | 36 | 32 | 36 | 32 | 28 | 23 | 21 | 28 | 30 | 37 | 41 | 35 |
| 2016 | 39 | 33 | 31 | 34 | 28 | 22 | 26 | 25 | 25 | 34 | 33 | 38 |
| **2017** | | | | | | | | | | | | |

Figure 1. Temperature table provided to Year 3 students at the beginning of 2018.

At each of the four data collection points, Iris and Sophia were presented with a stimulus table containing temperatures from past years (Figure 1). They used this table, along with their own remembered experiences of temperature, to complete the blank row which was the year just gone. Students then graphed any aspect of the data they chose, and were subsequently interviewed about their graph, their data observations and their prediction strategies. Data were collected by the first author, as teacher-researcher. Interview prompts included: "tell me about your graph", "did you notice anything special about the numbers?" and "how did you choose your temperatures?" Data consisting of (a) predictions, (b) interpretations (from video) and (c) representations, were collected at four time points: February of Year 3 and Year 4, and November of Year 4 and Year 5. The stimulus table was the same each year, with the exception that each iteration added an additional past year's temperatures. Interviews were recorded using a handheld iPad, and students' responses probed for clarity. Video duration varied from 1min 55s (Iris, Year 3) to 7min 15s (Sophia, February Year 4).

## Results

Over the three-year span, Iris and Sophia became more accurate at data prediction, and increasingly used the statistical features of the temperature table when interpreting the data set. In addition, they also developed competence in graphing coordinate data. Table 1 lists the changes in predictions, interpretations and representations over the four data collection cycles. Predictions were considered reasonable if they fell within the 95th percentile range ever recorded for the relevant month, and the values reflected the number of reasonable predictions out of 12 months. Despite similarities, Sophia remained in advance of Iris at each iteration.

### *Early Year 3*

Iris and Sophia's earliest attempts did not draw upon either the column or row structure of the table when predicting. Nevertheless, six of Iris's predictions were reasonable when viewed in the context of the table, as were seven of Sophia's. The students differed in their interpretation and use of the table. While Iris's explanations demonstrated an awareness that months are clustered into seasons and of seasonal change, she held misconceptions about when winter occurred. She described the first few months of the year as the hottest, and the end of the year as the coldest. Her predictive strategies did not include reference to the data table, suggesting that her data interpretation was idiosyncratic. In contrast, Sophia drew upon the data table as a source for her predictions, but at interview revealed she viewed the table as a series of single or disconnected values. For each prediction she described selecting a temperature *not* already used in a column, but present elsewhere in the table. Both students created a grid structure for their representation: Iris represented a table without values—a focus on the gridlines themselves—while Sophia included data by copying the temperature table.

### *Early Year 4*

By early Year 4, both students incorporated data-based strategies when predicting, resulting in 11 (Iris) and 12 (Sophia) reasonable predictions. Iris observed data clusters through describing the vertical columns containing the monthly values "as being actually around the same amount". Sophia went further, describing modal values in columns as "the most common number that has happened". In addition, she utilised her knowledge of the seasons to identify cooling and warming trends across the years. The students' emerging understanding of the subtleties of the temperature table were reflected in their representations. Iris, like Sophia the year before, copied the data table—demonstrating a shift from her earlier focus on the gridlines—to a representation containing temperature data, though with transcription errors. Sophia's Year 4 representation was ambitious, consisting of a line graph of the months January to July with temperature variation on the $y$-axis and years 2010 to 2018 on the $x$-axis (Figure 2). This graph included many formal and accurate graphing elements, including a key, equal spacing on the $y$-axis and a temperature scale starting 20°C. Sophia competently organised the data in a coordinate arrangement demonstrating her understanding that variation increases with temperature. Despite being able to draw this graph, however, Sophia struggled with its interpretation. When comparing the lines for June and July (which charted months with similar, stable temperatures) with the hotter, more unstable months (November through to March), Sophia described *seasonal* changes rather than variation *within a month*.

**Table 1**

*Data Predictions, Explanations and Representations for Each Iteration*

| Year | Student | Predictions | Interpretation | Representation |
|---|---|---|---|---|
| Early 3 | Iris | 6 | Idiosyncratic memories and a false belief about the timing of winter. | Empty grid. |
| | Sophia | 7 | Sourced numbers from the data table without using the column structure. | Copy of data table. |
| Early 4 | Iris | 11 | Used the column structure to find common 10s while retaining false belief in the timing of winter. | Copy of data table. |
| | Sophia | 12 | Guided by modal temperature in columns and seasonal change. | Line graph of each month showing variability between months. |
| Late 4 | Iris | 12 | Used two previous years to predict values close to temperatures, while selecting numbers "a bit different" from ones in the column. | Bar graph of all values in the table January-July. |
| | Sophia | 12 | Included multiple features including mode, an average or representative figure and seasonal pattern (winter dip). | Bar graph of all years for two hottest and two coldest months. |
| Late 5 | Iris | 11 | Used two previous years to predict values close to temperatures. Predictions demonstrated continuity between seasons when compared with previous attempts. | Bar graph of two years used for temperature predictions. |
| | Sophia | 9 | Multiple features including seasons, and impact of bushfires, drought and stimulus values. Continuity between seasons when compared with previous attempts. | Line graph of 2010-2017. |

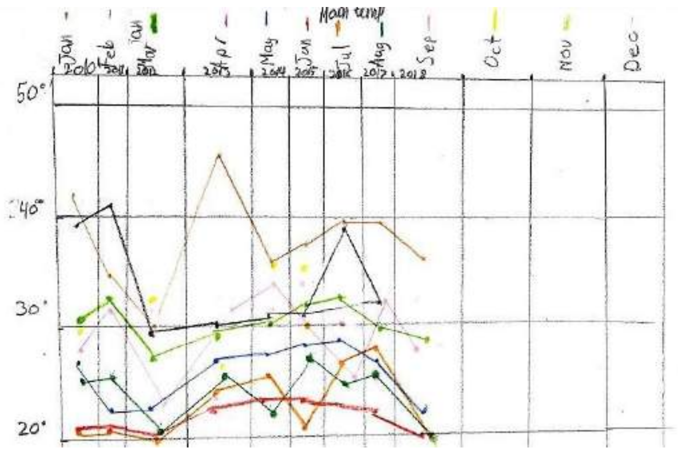



Figure 2. Line graph by Sophia early Year 4.

## *Late Year 4*

By the end of Year 4, both Iris and Sophia predicted reasonable values for every month. Iris's unit of analysis remained the column structure, and she described selecting temperatures similar to the previous two years. Consequently, all her predictions were all within one or two degrees of the previous two temperatures. She justified this strategy by referring to climate change i.e., only the past two years were reliable measures. Iris didn't draw upon the row structure of the graph, nor explicitly link predictions to seasonal change. In contrast, Sophia described multiple data features. At first, she looked for modes: "If there was any kind of a repeat…" it was preferred, and then explained looking for the "average". When probed to describe average, Sophia explained: "what most of them was closest to." Sophia also observed seasonal changes across the row structure of the table, describing a pattern as "…kind of making a dip. At first there are higher temperatures, then it goes lower, and then it goes up."

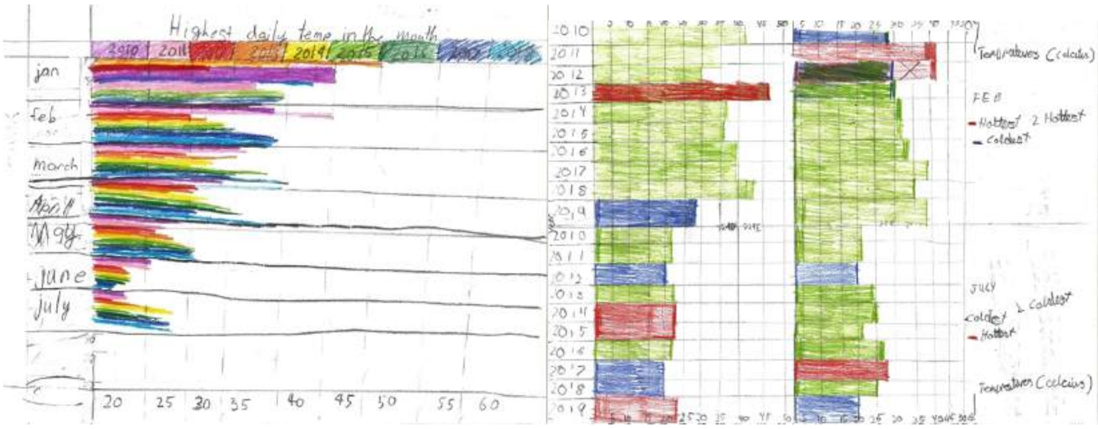



Figure 3. Graphs by Iris (left) and Sophia (right) from late Year 4.

Iris and Sophia both drew bar graphs for their late Year 4 representations, and it was Iris this time who attempted to include all data points. Her graph included formal features such as data labels, a key, a heading, appropriate colour coding, a scale focusing on the range of interest, and almost equal spacing (Figure 3, left). Iris described the graphing process as "confusing", and its planning and construction required making a graph different to any sort she had seen before. Sophia, in contrast, restricted her data representation to four temperature sequences, each one a single graph: January, February, June and July (Figure 3, right). Selecting some values over others led to a simpler, easier to interpret representation when compared to Sophia's former attempts. She included the formal graphing elements of key, labels, equal spacing and intervals of 5 degrees on the temperature scale. However, because Sophia again used the year, rather than the month as the independent variable, her graphs demonstrated differences between months and not seasonal change that occurred over the calendar year.

## *Late Year 5*

In Year 5, Iris's prediction strategy again referenced the two most recent years, also seen in her graph. This graph accurately representing the temperature changes, with a seasonal dip and a consistent scale confined to values above $10^{o}$ $^{C}$(Figure 4). Iris's predictions all fell within the historical range except for December which was $45^{o}C$. Sophia's predictions also included several overestimations relative to the historical data set, i.e., $48^{o}C$ for January and $43^{o}C$ for February. Despite these overestimations, the Year 5 predictions for both students showed continuity over the months i.e., they started as warm in January, declined to a winter dip, and then increased smoothly from August to a hot summer. Sophia accurately marked the seasons on her table and ensured that adjacent months followed appropriate seasonal trends. Eastern Australia experienced some very serious bushfires in January 2020, and Sophia's overestimations were directly linked to her memories of this event. She added that by March, the onset of Autumn was moderating maximum temperatures. The other monthly predictions were linked to their respective season, with the exception of October, where she also noted that her selection ($37^{o}C$) was also the mode.

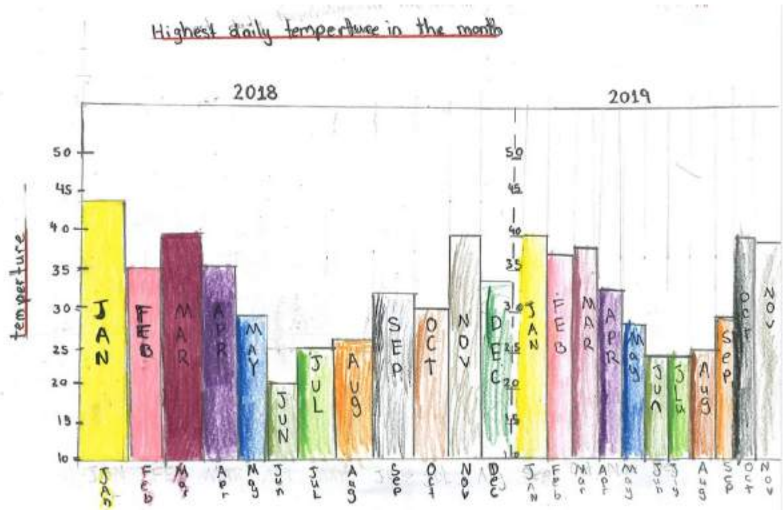



Figure 4. Graph of two previous years of maximum temperatures by Iris in Year 5.

When representing, Sophia returned to her early Year 4 strategy of using a line graph to represent every data point, although this time with more technical accuracy (Figure 5). Sophia chose a line graph "because it is easier to compare data than a bar or a column graph". While it was her intention to graph every year, she only completed 2010-2017 before running out of time. Her graph had an unusual orientation with temperature on the *x*-axis, and she started her scale from zero. She described her representation as "a line graph of the highest temperatures in each month. It shows the differences between the years in temperature, and it also showed the dip in temperature for when it becomes winter." This graph enabled Sophia to express an informal generalisation regarding variability in the monthly maximum temperatures: "They are basically different at the start and end of the year, but they all come quite close in winter".

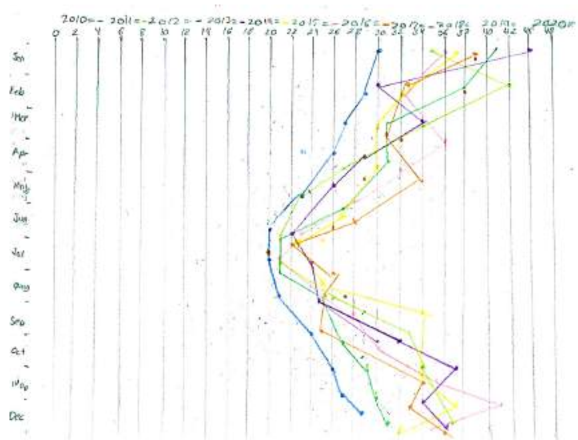


Figure 5. Line graph of monthly temperature changes by Sophia in Year 5.

## Discussion

Despite differences between the two cases, common elements emerged in their predictions, interpretations and representations. Iris's Year 3 interactions focused upon non-data components of the task, including her false belief about the timing of winter, even while remaining engaged in the task. Sophia from Year 3, used data-based strategies: i.e., sourcing values already in the temperature table and seeking missing temperatures in columns. The idea that all numbers "should" be included equally (making a flat data distribution) has been reported in other studies of young students, for example Year 2 students predicting lost milk teeth (Ben-Zvi & Sharett-Amir, 2005). It may be associated with the younger child's perception of fairness. Reasonableness of temperature predictions for both students improved significantly between Year 3 and the second data collection point in early Year 4. As they progressed through the years, their observations moved from "non-data based", or the first data lenses perspective (Iris only), to noticing "multiple and relational components" of the table (Konold et al., 2015). While there was a general trend towards more sophisticated data viewing, the lenses used for prediction were not always the same as those in the representations. For example, in early Year 4, Iris copied the whole data table paying attention to individual case attributes (second data lens). However, when she predicted, Iris noted the temperatures "as being actually around the same amount", i.e., recognising similarities between groups of data (third lens). Similarly, in late Year 4, Sophia observed relational components, i.e., seasonal change as a pattern moving from left to right demonstrating an aggregate data view. Her representation, however, was limited to the similarities in four sequences (third lens).

Similarly, for each data collection point, students' interpretations lagged the success of their predictions. Both Sophia and Iris identified strategies such as coldest temperatures at the end of the year, using the same 10s value or selecting a missing number in the table. These strategies were frequently applied inconsistently, and potentially moved from intuitive to explicit reasoning while the interview was in progress. Sophia articulated this in her late Year 4 interview where she explained that for some predictions, she "just got the sense of them". The students' representations followed a pattern, with Sophia approximately one year ahead of Iris at each point. Iris's Year 3 representation of the grid indicates she was paying attention to the physical grid lines, rather than the numerical values. Reading a data table, such as the stimulus table in the study, assumes that students have mastered the column and row construction and appreciate the construct as a spatial array of squares. Research by Battista et al. (1998) with Year 2 students demonstrated that this spatial array is not intuitive for many students. Sophia's focus in Year 3 and Iris's in Year 4 was on the whole data set. This reproduction of the table seems to stem from two factors. The first was simply not knowing what to graph. Prior experience with graphing included gathering information and organising into lists, tables, and picture graphs. For Sophia in Year 3 and Iris in Year 4, these prior experiences were not sufficiently flexible or ingrained to transfer when graphing something new. Second, neither knew what to include or exclude. English (2012) and Mulligan (2015) describe the challenge of data representation as a selection process, thus deciding which features to emphasise over others. The resistance to discarding information continued even as their graphing skills developed. Sophia's early Year 4 and Iris's later Year 4 representations both attempted to include all temperatures, resulting in messy graphs, difficult to interpret. In Sophia's case, her representation hindered, rather than supported her understanding of variation within months, and Iris was unable to read the basic feature of temperatures cooling in the winter. Sophia returned to an 'all values' representation again in Year 5, although this instance, her representation enhanced her interpretation of the data set.

This paper contributes to the growing awareness of students development of predictive reasoning and meta-representational competence in the middle years of primary school. The cases described here propose that students may move through stages when interpreting data tables and constructing graphs. By construction and visualisation of data sets through freehand drawings, students have the opportunity to notice and internalise key structural elements such as equal spacing, scale and coordinating axes. Developing meta-representational competence—or the capacity to represent and restructure data—prior to the introduction of formal graphing is recommended to avoid a procedural approach where students learn to graph without conceptual understanding. Further research into this process could inform educators of the optimal stage at which to intervene with more formal pedagogical approaches to developing statistical concepts and graphing.

# References


Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Van Auken Borrow, C. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education. 29*(5), 503–532

Ben-Zvi, D., & Sharett-Amir, Y. (2005). How do primary school students begin to reason about distributions? *Reasoning about distribution: A collection of current research studies. Proceedings of the 4th international research forum for statistical reasoning, thinking and literacy.* SRTL-4. Brisbane: University of Queensland. https://www.academia.edu/976792

English, L. (2012). Data modelling with first-grade students. *Educational Studies in Mathematics, 81*(1), 15–30.

Fielding, J., & Makar, K. (2022). Challenging conceptual understanding in a complex system: Supporting young students to address extended mathematical inquiry problems. *Instructional Science, 50*, 35–61.

Konold, C., Higgins, T., Russell, S. J., & Khalil, K. (2015). Data seen through different lenses. *Educational Studies in Mathematics, 88*(3), 305–325.

Leavy, A., & Hourigan, M. (2018). Inscriptional capacities and representations of young children engaged in data collection during a statistical investigation. In A. Leavy, & M. Meletiou-Mavrotheris (Eds.), *Statistics in early childhood and primary education* (pp. 89–108). Singapore: Springer. https://doi.org/10.1007/978-981-13-1044-7_6

Mulligan, J. (2015). Moving beyond basic numeracy: Data modeling in the early years of schooling. *ZDM Mathematics Education, 47*(4), 653–663.

Oslington, G., Mulligan, J., & Van Bergen, P. (2020). Third-graders' predictive reasoning strategies. *Educational Studies in Mathematics*, *104*(1)*,* 5–24.

Oslington, G., Mulligan, J., & Van Bergen, P. (2021). The development of predictive reasoning in Grades 3 through 4. In Y. H. Leong, B. Kaur, B. H. Choy, S. L. Chin, & J. B. Yeo (Eds.), *Excellence in mathematics education. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 305–312). Singapore: MERGA.

Oslington, G., Mulligan, J., & Van Bergen, P. (2023). Shifts in students' predictive reasoning from data tables in years 3 and 4. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-023-00460-2

Watson, J. (2018). Variation and expectation for six-year-olds. In A. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in early childhood and primary education: Supporting early statistical and probabilistic thinking* (pp. 55–73). Springer.

# Managing the Ongoing Impact of Colonialism on Mathematics Education

Kay Owens
*Charles Sturt University*
kowens@csu.edu.au

This paper is a brief summary of a large historic research project in Papua New Guinea (PNG). The project aimed to document and analyse the nature of mathematics education from tens of thousands of years ago to the present. Data sources varied from first contact and later records, archaeology, oral histories, language analyses, lived experiences, memoirs, government documents, field studies, and previous research especially doctoral studies. The impacts of colonisation, post-colonial aid and globalisation on mathematics education have been analysed and an understanding of the current status of mathematics education established as neocolonial. Managing neocolonial education policies may minimise the loss of cultural ways of thinking.

Papua New Guinean societies existed from at least 40 000 years ago with several migrations from the north or west. Around 5 000 years ago, a major new wave of migration occurred and the Austronesian Oceanic languages developed in East New Britain before spreading around the coast and to Island Melanesia as far as Fiji (Addison & Matisoo-Smith, 2010). Groups were relatively autonomous, managing with trade arrangements and intermarrying relationships to meet their needs. There was no central government. The 850 PNG cultures and languages remained with no influence from Europe or the Middle East until the 1800s. Not only was Australia a colony for which this study has some relevance but it also colonised Papua New Guinea (PNG).

## Research Aims and Methodology

The purpose of this research was to document and analyse the development of aspects of mathematics and mathematics education in Papua New Guinea from the past to the present. There were a couple of available bibliographies of education up until the mid-1970s (Cleverley & Wescombe, 1979; Smith, 1987) but no focus on mathematics education from the time before European contact and, despite on-going research within the country, there was little in the post colonial period on the development of mathematics education per se. The research involved an extensive use of first contact and later documents and memoirs, archaeological and linguistic research from a diversity of areas and language groups, oral histories, lived experiences, field visits to villages, large research studies on number systems, measurement practices, and mathematical words from different cultures across the country, research studies on mathematics education and teacher education, government documents especially major reports and plans recommending changes to education, syllabuses, and studies on the language of instruction.

Themes that emerged from these sources included the languages of mathematics in villages and in schools, the use of visuospatial reasoning in mathematical thinking, the valuing of both traditional mathematics for everyday life (once identified) and school mathematics for the dream of a job, and the dissonance of mathematics at home and at school. However, the key findings were (a) the depth and diversity of foundational/traditional mathematics, (b) the growth and sources of neocolonialism, (c) the limitations of neocolonialism, and (d) examples for overcoming neocolonialism.

## Historical Developments

In the late 1800s, a few anthropologists visited (e.g., Mikloucho-Maclay, 1975), European sailors navigated its waters, and a few German business men began plantations or recruiting for other plantations in the Pacific region (D'Entrecasteaux, 2001). Missionaries soon followed sharing the gospel of Jesus in the vernacular languages, often in a religious format but also in assisting villagers especially with health issues and education (Jinks et al., 1973).

(2023). In B. Reid-O’Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 395–402). Newcastle: MERGA.

Governments felt the need to set up administration and controls. The German government in the northern mainland and islands soon set up administrative centres laying claim to it as a colony in 1884. This prompted the British to lay claim to the southern side close to Australia, leaving the colony of Queensland and later Australia to administer. One issue of the early administrators was the exploitation of 'the natives' as they were called. This encouraged them to provide a basic education. Mostly it was through supporting the mission schools but then they began requesting that schooling be in English so that the administrators could converse with the natives. Money was attached. However, overall, little money was available to support the colonial administration. Interestingly, in early administrative reports, basic word lists of the local languages were recorded as new centres were set up.

In Port Moresby, Lawes (1890) and colleagues had written down the Motuan language and used it in the large schools for the local people (Owens et al., 2018). Other village languages were also used especially Dobu in the Papuan islands, Tolai in East New Britain, Bel in Madang area, and Kôte and Yambim in Morobe for church and school. Students completing Grade 6 would then be recruited as teacher assistants in schools. South Sea islanders also came as pastors and teachers. After World War I, the League of Nations passed the northern section to Australia as a Trust Territory. Gold mining was added to exploitation already occurring through plantations. This provoked many foot patrols into the virtually unknown, unpacified highland areas which were then opened up since aircraft were able to fly in. Not only the locals but also the Germans were required to have schools in English for government funding (Smith, 1987).

During and after World War II there was one administration. In the 1960s, the Prime Minister started to talk about autonomy for the Territories since more colonies became independent. However, education was very limited. Hurriedly, high schools were set up (Cleverley & Wescombe, 1979). By now many Australians, often quite young, were recruited as kiaps and teachers to remote areas as well as the towns and coastal centres. Teachers' colleges trained both Papua New Guineans and Australians. By 1966, the University of PNG was set up in Port Moresby and then PNG University of Technology in Lae. There were graduates by self-government in 1973. Research into education, particularly mathematics education, was strong and began influencing worldwide research (Owens et al., 2019; e.g. Bishop, 1988).

## Foundational/Traditional Mathematics Learning

Papua New Guinean societies were using mathematics in technology, trade, social relationships, and understanding natural sciences tens of thousands of years ago. Much of this knowledge is still passed on between generations today using Indigenous ways of learning and teaching (Paraide et al., 2022). Most foundational mathematics is learnt from older men or women who gather under relational connections to share knowledge in groups during everyday or special traditional activities (Paraide et al., 2022).

A few remarks might indicate the extent and depth of this knowledge. Seafarers had fishing and navigation skills, travelling over the horizon to distant places (Lewis, 1973). There were trading routes and reciprocity to negotiate with items often passed on to far distant places crossing many language groups (Swadling, 2010). Kinship patterns were extensive and again reciprocity was significant (Shaw, 1974). There were tools and processes for carrying, collecting, fishing, agriculture, food and materials preparation, creating, building, playing and celebrating (Paraide et al., 2022). There were designs of cultural significance, replication of objects like the curves and lines of canoes (Campbell, 2002), pots, drums, baskets, string figures (cats's cradles), shields, bows and arrows, axes, or house walls and roofs (Owens, 1999, 2012). There was extensive knowledge related to medicines (Kopi, 1997), spatial knowledge in recognising the place and the plant for gathering and then in knowing how and what malaise to treat with different medicines and processes.

Classification and sets in designs were sophisticated and related to culture (Owens, 2022). These are evident for the shapes on the various parts of canoe boards, house boards, shields, other carvings, leadership symbols (Were, 2010), food containers, and pots (Owens, 2015; Paraide et al., 2022). Actions, their order and links have been studied in string figures (Vandendriessche, 2015) but also in making other items like bilums. They are remembered but also reorganised to create new designs. Patterns occur in gambling practices (Pickles, 2013), weaving, and string bag making. Numeral systems are varied, some unique, and some shared with neighbours. Some are linked to collecting, measuring, trading or classifying (Owens, et al., 2018; Owens, 2020a).

## Colonial Impact on Education and Languages

Administration in colonising countries focussed on law and order, taxes, and keeping records of businesses and other groups such as churches (Megarrity, 2005). Initially in the early and mid-1900s, funds went to government schools and to missions if English was the language of instruction and proportional to achievement in English and mathematics examinations. The missions or churches have dominated education training so that all but one of the primary teachers' colleges are run by churches with the Institute of Education mostly concerned with early childhood education. The University of Goroka also provides Certificates and Degrees in Early Childhood Education and degrees (including Master's) in education for all sectors.

Before and after Independence, there were committees to advise on curricula for primary and secondary education and teacher education. The college staff were able to be in touch and share their ideas and strengths. Some overseas mission staff were in the country for many years while others came for short terms (Paraide et al., 2022; Quartermaine, 2001). Before self-government, Australia instigated a 6-month training program in Rabaul, mainly for Australians and then ASOPA in Sydney, assisting with some understanding of cultural diversity and respect for students undertaking school education, certificates and degrees (Paraide et al. 2022).

There have been schools using local languages for teaching, e.g. Tolai in East New Britain, Tok Ples or local church language in remote Morobe. School students from 1960 to 1985 reported they were punished for speaking languages other than English in both government and mission schools. Since the education system meant that students often left their village for a small centre, they were already beginning to use a non-home language. The children then went to high school, Senior High School, teachers' or another college or University where English and later Tok Pisin were the main languages between students. After years of education away from their village, teachers might or might not go back to their village area to teach. Many students struggled to keep their culture and vernacular language and to learn their village or family foundational knowledge. Despite this they still had strong connections and pride in their family and their family's foundational technological and mathematical knowledge (Owens, 1999; Owens & Kaleva, 2008). Was the loss irrevocable?

## An Indigenous Voice

Before Independence, a committee of educated Papua New Guineans chaired by Alkan Tololo prepared a report for education (Department of Education Papua New Guinea, 1974; Tololo, 1976). They recognised the importance of students valuing their culture, knowing how to live in their villages, and connecting village knowledge and school knowledge. However, there was still an Australian responsible for the Territories and he could not see how this report could be implemented so he went to the expatriate Dean of Education at the University who hurriedly prepared another education plan (Cleverley, 1976; Weeks, 1993). The opportunity to hear and develop the Indigenous voice was lost at this stage and indeed for 10 years until 1986 when another Indigenous committee, this time chaired by Paulius Matane wrote a report to which plans were made (National Department of Education Papua New Guinea, 1986). Importantly, with the support of a World Bank report, cultures and languages were to be recognised and used in education together with the unachieved goal of universal primary education (Weeks, 1993).

## Attempts to Educate Following the Indigenous Voice

The Reform period began. They were not just following externally dominated ideas. The desire for universal education meant elementary schools in villages would use the home language of the children (Paraide, 2002). Now there were more PNG educators with higher degrees and curriculum advisory committees had strong national representation from practicing fields, universities and schools. However, they were not necessarily meeting regularly as before 1990. In addition, funding was an issue. It was taken away from higher education but it did not reach the school sector. It was decided that villagers would provide the schools and teachers' houses while the government would provide salaries. Teachers first trained under the head teacher and were accredited upon inspection if they knew the local language, had a Grade 10 education and undertaken training Then they would be paid. However, for years training was often not available and inspectors found it difficult to visit. Many teachers received no or inadequate salary.

An Australian advisory team, whom it was said had too much say, was involved in teacher education for elementary schools (Weeks, 1993). The teacher education courses were set up as Self-Instruction Units with a short introductory workshop, often given as lectures to a large number of teachers in a village area. At first teacher education was by the travelling Institute staff and then by Provincial Education Officers of varying skills, training, and experiences. There was no full unit on teaching bilingually from vernacular language bridging to English language and there was no mathematics unit developed by the Institute using cultural mathematics. One book on Patterns incorporated many cultural materials. Later a research team did develop a Self-Instruction Unit that was given to the Institute of Education (Owens et al., 2015). Teachers who joined in remote workshops valued what they learnt but they were only 3 to 5 days. This was too little, too late. SIL was beginning to make good inroads into teaching teachers how to teach bilingually and to recognise cultural mathematics, at least their counting systems. However, the delay meant that after years of English (or Tok Pisin) education with limited understanding and loss of language, it was now difficult to arrest the language loss or valuing of language. The lack of teacher education meant that students were not adequately learning to read in Tok Ples or English and mathematics was just as poor. The Advisory team from Australian Aid (Curriculum Reform Implementation Project) introduced Outcomes-Based Education (OBE), then common around the world, with inadequate syllabuses for teachers to use and no initial or strong Indigenous voice. The Teachers' Guides and expensive books soon disappeared. OBE too was seen as a problem by the elite and others. The country was in a dilemma with its lack of funding and a new neocolonial curriculum. Like many of the reforms in mathematics education, even going back to the introduction of Dienes blocks, it was inadequately supported by teacher education or inservicing (Paraide et al., 2022).

## The End of Learning Cultural Mathematics in Home Language

In 2012, O'Neill was elected as Prime Minister with the promise that English would be the language of instruction from the start. This was despite so much research stating that learning mathematical and other concepts in one's home language and bridging later to English had so much strong support as the best educational approach although students were not doing well on Pacific standardised tests (Paraide et al., 2022). The elementary schools disappeared and were replaced by early childhood education centres for two years (having a play-based first year and picking up the pre-elementary syllabuses from the elementary schools) and then the students had to go to primary school for Grades 1 to 6. Mathematics was no longer called Cultural Mathematics. There were restructures yet again of the education school system (National Department of Education Papua New Guinea, 2016). In fact, instead of Australian colonialism, Japanese approaches to mathematics began. The English version of a Japanese textbook, was now available for teachers to buy if they did not receive it from the Department of Education. Standards-based assessment, following world trends again, was introduced. The initial syllabus supposedly had PNG Curriculum Advisors involved but there is some evidence there was little understanding of the Japanese approach (Paraide et al., 2022).

## A Possible Way Forward

In 2022, at the International Conference on Ethnomathematics 7, a plenary speaker from Papua New Guinea, Charly Muke, said that teachers and administrators now need to do something differently because they were stuck with English. If English is decreed the language of instruction from early childhood onwards, then there needs to be alternative ways forward. Voices like Charly's and Patricia Paraide's were being drowned out. Charly noted how he sat in primary school not understanding a word but for mathematics with concrete materials he figured out what was going on in his own language in his head. Patricia also from village parents had the good fortune of learning in her vernacular Tolai but was exceptionally good when she started school in English. Racial slurs on her background did not help her (Paraide et al., 2022). Notably, there was little work done on local language for mathematics outside of the counting words and systems. How this could be done would be costly and many skilled persons would be needed. Could the Teo Māori experience be repeated even in a small way? A list of mathematical terms for primary school were translated into local language in workshops by teachers and Elders in discussions but this was not ongoing and often only a few terms were explored in the short time available (Edmonds-Wathen et al., 2019). Perhaps, said Charly, we need to consider how Papua New Guineans think mathematically when they are doing cultural activities that often involve science, technology, engineering, and mathematics. Firstly, we know they think visuospatially in these contexts. They often call it 'in my head' or 'by eye'. How do they do this?

Charly also recommends the use of traditional games like their betting game with stones, cat's cradles, their counting systems and the links to cultural practices (see above). Work on the many 850 languages' orthographies, the involvement of Elders in the school curriculum and appropriate materials for schools needs funding. A generation had passed with poor local language or English education so rectifying this situation will not be easy. Teacher education for multilingual classes and cultural mathematics needs to be compulsory.

## Ethnomathematics and School Mathematics

There are sophisticated classification systems for counting, design, art (on cultural artifacts), gambling, and kinship (see above and cf. Almeida, 2022; Watson-Verran, 1992). Classification in school geometry is simplified by not having a spatial and cultural component. Knowledge of places and a mental map of large areas is held in people's heads as they traverse the forests or seas (Lewis, 1973). This knowledge involves position but also visuospatial knowledge of trees, soils, water movement, winds, reefs, fish, sharks, dugongs, shell fish and other creatures that inhabit the different areas. The interconnectivity of the mathematical aspects such as position, shape and vectors has purpose and purpose is a main driver for learning and remembering and making connectivity of mathematical ideas (Owens, 2015).

Knowledge of complex trade, intercultural relationships, and reciprocal agreements (Strathern, 1977) involves complex accounting systems covering many goods and money (PNG kinas or traditional money, e.g. shell *tabu*). Pairs, matching, equality and inequality, increase and decrease are central to these systems (Owens et al., 2018). All these are mathematics. Some mathematical knowledge is recorded, often on the body in some way or by objects and displays. Representations include tattoos, body parts, displays, *bilas* (body decorations), demarcation of land, and house size and design (Owens, 2015, 2020a, 2020b; Paraide et al., 2022). All cultures have mathematical thinking for activities—counting, measuring, designing, locating, playing and explaining

(Bishop,1988), understanding, interpreting, inventing, and reasoning (D'Ambrosio, 1985). There are techniques and modelling of cultural ways of thinking mathematically (Orey & Rosa, 2021; Vandendriessche & Pinxten, 2022).

## Implementing Ethnomathematics in Schools

Listening and working with Elders is essential. Money is needed for this. First the range of mathematical activities needs to be discussed, the mathematics teased out and represented as in mathematical modelling. The mathematics might not easily fit into the school curriculum but they can be used for patterns and relations. For example, string figures show algorithms and inventions, canoe boards show classifications and patterns. Designs e.g. *kapa* (round leadership symbols from hard shell and tortoise shell) have diverse symmetries, patterns, and angles. Ways of counting have systems, many can easily be coded (Kari, personal communication, 2003), others indicate intricacies related to cultural practices. Each basic counting system can be classified using the frame words (basic words from which others are made), and cycles (this indicates the systems of making high numbers and in most cases in PNG this is a more appropriate approach than using the term base. Most are digit-tally systems with (2, 5, 20) cycles (Owens, et al., 2018). Appropriate teacher education is essential (Tapo, 2004).

Besides the work on foundational/traditional mathematics given in the two chapters of Paraide et al. (2022), Owens (2022) discusses cultural implications for discussing large numbers, groupings, time and work patterns, transactions, classifications, art and design and Bino (2023) indicated mathematical thinking on model canoe building and sailing. In cultural practices, people discuss problems and situations that need resolving. They share their conceptual understandings which are generally associated with visuospatial reasoning which is a holistic way of presenting the problem. Concept, comparison, memories of the past related to the problem or object, patterns, parts, size and shape are all considered visuospatially and ecoculturally. The environmental supports and constraints are discussed including patterns of activities and diversity of responses. These sophisticated mental ways of thinking need to be expounded more by teachers, villagers, researchers, and curriculum writers. This idea of mental mathematical thinking which generally includes visuospatial reasoning (Owens, 2015, 2016) needs to be captured in mathematics and these thinking skills brought to the fore in school mathematics in PNG if neocolonial loss is to be overcome.

## Acknowledgments

I wish to acknowledge the many Elders, villagers, students, teachers, and academics and my co-researchers especially Lean, Paraide, Matang, Kaleva, Clarkson, Muke, Sondo, Bino, and Edmonds-Wathen who have contributed to my thinking to articulate these arguments.

## References

Addison, D. J., & Matisoo-Smith, E. (2010). Rethinking Polynesians origins: A West-Polynesia Triple-I Model. *Archaeology in Oceania*, *45*(1), 1–12.

Almeida, M. (2022). Indigenous mathematics in the Amazon: Kinship as algebra and geometry among the Cashinahua. In R. Pinxten, & Vandendriessche, E. (Eds.), *Indigenous knowledge and ethnomathematics* (pp. 221–242). Springer. https://doi.org/https://doi.org/10.1007/978-3-030-97482-4

Bino, V. (2023). Model outrigger canoe (*Asiasi*) race: Social equality and tolerance through ethnomathematics. *Proceedings of international conference on ethnomathematics 7—Ethnomathematics: Embracing diverse knowledge systems for social justice and peace* (pp. 39–43). Online—hosted by Philippines, Nepal, Indonesia, and Papua New Guinea, December 2022.

Bishop, A. (1988). *Mathematical enculturation: A cultural perspective on mathematics education*. Kluwer.

Campbell, S. (2002). *The art of Kula*. Berg, Oxford International Publishers.

Cleverley, J. (1976). Planning educational change in Papua New Guinea: A comparative study of the 1973 and 1974 five-year plans for education. *Comparative Education*, *12*(1), 55–65. https://doi.org/10.1080/0305006760120107

Cleverley, J., & Wescombe, C. (1979). *Papua New Guinea: Guide to sources in education.* Sydney University Press.

D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics*, *5*(1), 44–48.

D'Entrecasteaux, B. (2001). *Voyage to Australia and the Pacific 1791–1793* (E. Duyker, & M. Duyker; Eds. & Trans.). Melbourne University Press.

Department of Education Papua New Guinea. (1974). *Report of the Five-year Education Plan Committee, September, 1974* (Chairperson, A. Tololo). Department of Education.

Edmonds-Wathen, C., Owens, K., & Bino, V. (2019). Identifying vernacular language to use in mathematics teaching. *Language and Education*, *33*(1), 1–17. https://doi.org/10.1080/09500782.2018.1488863

Jinks, B., Biskup, P., & Nelson, H. (Eds.) (1973). *Readings in New Guinea history*. Angus and Robertson.

Kopi, S. (1997). *Traditional beliefs, illness and health among the Motuan people of Papua New Guinea* [Doctoral thesis, University of Sydney]. Australia.

Lawes, W. (1890). Comparative view of New Guinea dialects. In *Annual report on British New Guinea from 1889 to 1890* (pp. 158–167, Appendix X111).

Lewis, D. (1973). *We, the navigators*. University Press of Hawaii.

Megarrity, L. (2005). Indigenous education in colonial Papua New Guinea: Australian government policy (1945–1975). *History of Education Review*, *34*(2), 41–58. https://doi.org/10.1108/08198691200500009

Mikloucho-Maclay, N. (1975). *New Guinea diaries 1871–1883* (C. L. Sentinella, Trans.). Kristen Press.

National Department of Education Papua New Guinea. (1986). *A philosophy of education for Papua New Guinea* (chairperson, P. Matane). Government Printer.

National Department of Education Papua New Guinea. (2016). *National education plan 2015–2019*. Author.

Orey, D., & Rosa, M. (2021). From ethnomathematics to ethnomodelling. *Journal of Mathematics and Culture*, *15*(1), 148–168.

Owens, K. (1999). The role of culture and mathematics in a creative design activity in Papua New Guinea. In E. Ogena, & E. Golla (Eds.), *8th South-east Asia conference on mathematics education: Technical papers* (pp. 289–302). Southeast Asian Mathematical Society.

Owens, K. (2012). Papua New Guinea Indigenous knowledges about mathematical concepts. *Journal of Mathematics and Culture*, *6*(1), 15–50.

Owens, K. (2015). *Visuospatial reasoning: An ecocultural perspective for space, geometry and measurement education*. Springer. https://doi.org/10.1007/978-3-319-02463-9

Owens, K. (2016). The line and the number are not naked in Papua New Guinea. *International Journal for Research in Mathematics Education. Special issue: Ethnomathematics: Walking the mystical path with practical feet* *6*(1), 244–260. https://doi.org/http://sbem.iuri0094.hospedagemdesites.ws/ojs3_old/index.php/ripem/issue/view/99

Owens, K. (2020a). Indigenous knowledges: Re-evaluating mathematics and mathematics education *Quaderni di Ricerca in Didattica (Mathematics)*, *Quaderno numero speciale 7-Palermo*, 28–44. http://math.unipa.it/~grim/4%20CIEAEM%2071_Pproceedings_QRDM_Special_Issue%207_Plenaries.pdf

Owens, K. (2020b). Transforming the perceptions of visuospatial reasoning: Integrating an ecocultural perspective. *Mathematics Education Research Journal (Special issue on the relation of mathematics achievement and spatial reasoning)*, *32*(2), 257–282. https://doi.org/10.1007/s13394-020-00332-z

Owens, K. (2022). The tapestry of mathematics—connecting threads: A case study incorporating ecologies, languages and mathematical systems of Papua New Guinea. In R. Pinxten, & E. Vandendriessche (Eds.), *Indigenous mathematical knowledge and practices* (pp. 183–220). Springer. https://doi.org.10.1007/978-3-030-97482-4

Owens, K., Clarkson, P., Owens, C., & Muke, C. (2019). Change and continuity in mathematics education in Papua New Guinea. In J. Mack, & B. Vogeli (Eds.), *Mathematics and its teaching in the Asian/Pacific region* (pp. 69–112). World Scientific Press. *https*://*doi.org*/10.1142/11038

Owens, K., Edmonds-Wathen, C., & Bino, V. (2015). *Teaching mathematics: Self-instruction manual for elementary school teachers and trainers manual*. OREC.

Owens, K., & Kaleva, W. (2008). Case studies of mathematical thinking about area in Papua New Guinea. In O. Figueras, J. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Annual conference of the International Group for the Psychology of Mathematics Education (PME) and North America chapter of PME, PME32—PMENAXXX* (Vol. 4, pp. 73–80). PME.

Owens, K., Lean, G., with Paraide, P., & Muke, C. (2018). *The history of number: Evidence from Papua New Guinea and Oceania*. Springer. https://doi.org.10.1007/978-3-319-45483-2

Paraide, P. (2002). *Rediscovering our heritage*. Port Moresby, Papua New Guinea: National Research Institute.

Paraide, P., Owens, K., Muke, C., Clarkson, P., & Owens, C. (2022). *Mathematics education in a neocolonial country: The case of Papua New Guinea*. Springer Nature.

Pickles, A. (2013). *The pattern changes changes: Gambling value in highland Papua New Guinea*. University of St Andrews, ]. UK.

Quartermaine, P. (2001). *Teacher education in Papua New Guinea: Policy and practice.* [University of Tasmania]. Hobart, TAS, Australia.

Shaw, R. D. (Ed.). (1974). *Kinship studies in Papua New Guinea*. Summer Institute of Linguistics.

Smith, P. (1987). *Education and colonial control in Papua New Guinea: A documentary history*. Longman Cheshire.

Strathern, A. (1977). Mathematics in the *moka*. *Papua New Guinea Journal of Education, Special Edition The Indigenous Mathematics Project*, *13*(1), 16–20.

Swadling, P. (2010). The impact of a dynamic environmental past on trade routes and language distributions in the lower-middle Sepik. In J. Bowden, N. Himmelmann, & M. Ross (Eds.), *A journey through Austronesian and Papuan linguistic and cultural space: Papers in honour of Andrew Pawley* (Vol. 615, pp. 141–159). Canberra, Australia: Pacific Linguistics, ANU.

Tapo, M. (2004). *National standards / local implementation: Case studies of differing perceptions of national education standards in Papua New Guinea* Queensland University of Technology]. Brisbane, QLD, Australia.

Tololo, A. (1976). A consideration of some likely future trends in education in Papua New Guinea. In E. Barrington-Thomas (Ed.), *Papua New Guinea education* (pp. 221–225). Oxford University Press.

Vandendriessche, E. (2015). *String figures as mathematics: An anthropological approach to string figure-making in oral traditional societies*. Springer. https://doi.org.10.1007/978-3-030-97482-4

Vandendriessche, E., & Pinxten, R. (Eds.) (2022). *Indigenous knowledge and ethnomathematics.* Springer.

Watson-Verran, H. (1992). *We've heard that you teach maths through kinship: A Garma Maths course of study in the Yirrkala and Laynhapuy Schools community*. Yirrkala Literature Production Centre.

Weeks, S. (1993). Education in Papua New Guinea 1973–1993: The late-development effect? *Comparative Education*, *29*(3), 261–273.

Were, G. (2010). *Lines that connect: Rethinking pattern and mind in the Pacific*. University of Hawai'i Press.

# Identifying and Evaluating Upper Primary School Students' Mental Computation Strategies

Tracey Reader
*Griffith University*
tracey.reader@griffithuni.edu.au

Kevin Larkin
*Griffith University*
k.larkin@griffith.edu.au

Peter Grootenboer
*Griffith University*
p.grootenboer@griffith.edu.au

This conceptual paper discusses two frameworks, developed independently by the lead author, that will provide the conceptual foundation for the identification and evaluation of mental computation strategies students demonstrate during an upcoming research project entitled Mental Computation in Year 5. These frameworks will be used by the lead author during an intervention to investigate the application of mental computation strategies in problem solving tasks involving duration of time. It is an intended outcome of the project that the two frameworks will be useful for teachers and students in upper primary school to provide feedback regarding the teaching and learning of mental computation.

The research project, *Mental Computation in Year 5*, is a qualitative research study designed to investigate mental computation strategies used by Year 5 students when engaged in additive and multiplicative calculation tasks. The aim of this research is to contribute to educators' understanding of the mental computation strategies used by upper primary students. This research is important as mental computation is recognised in curriculum documents as an important component of Numeracy.

The research project involves documentation and analysis of the mental computation strategies used by Year 5 students in one school. The lead author will identify mental computation strategies by interviewing students individually to determine different types of strategies that are demonstrated by the students, the range of strategy types used by each student, and the ability of the students to be flexible in their use of mental computation. During these interviews, students will complete additive and multiplicative tasks using mental computation. The students will be invited to articulate their mental computation process by providing verbal reasoning, allowing the lead author to document the strategies that are being used.

To support the research, two frameworks have been designed by the lead author: *Mental Computation—Strategy Type Framework* (MC-STF) and *Mental Computation—Efficiency and Flexibility Framework* (MC-EFF). The MC-STF is designed to identify and name the strategies that students use and provides the foundation for coding student strategies identified in the research. The MC-EFF is designed to determine the effectiveness of the strategies identified, and to provide a simple mechanism, which supports provision of effective feedback to students. Both frameworks will be trialled by the lead author during an intervention where participants will develop their ability to use mental computation strategies to solve additive and multiplicative tasks and then use these strategies to solve problems involving duration of time.

In addition to the frameworks being used in the research project, we argue that both frameworks are useful for classroom teachers and will enhance the teaching and learning of mental computation in upper primary school and in the provision of feedback to students regarding their mental computation strategies. Hattie and Timperley (2007) indicate the importance of providing feedback to students in ways that help them identify their current knowledge (in this case their mental computation strategies) and then provide them with some concrete steps as to how their knowledge can be further developed. Therefore, the MC-EFF is an important pedagogical tool.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 403–410). Newcastle: MERGA.

In this paper we will establish a definition of mental computation, including strategies, using evidence from the literature. Next, we will discuss the two frameworks created for the research project and indicate how the frameworks provide a conceptually robust way to identify and evaluate the strategies used by the participants in the study. We conclude by identifying several pedagogical implications regarding the use of the two frameworks by classroom teachers to support the teaching and learning of mental computation.

## Defining Mental Computation

A significant advantage of mental computation is the development of conceptual understanding of number, as research has shown that students who engage in using flexible mental computation strategies develop greater number sense (May, 2020; Vincent, 2013). In addition, calculation is a mathematical skill used by adults in their daily life, 86% of which are done using mental strategies (Northcote & Marshall, 2016). Therefore, we argue that the teaching and learning of mental computation is an important part of the mathematics curriculum. Mental computation, or mental arithmetic as it is also commonly referred to, has been defined as the computation of numbers in the head (Heirdsfield, 2002; Maclellan, 2001), without the use of external aids such as calculators or pen and paper (Maclellan, 2001). Lemonidis (2016) claims that mental computation can also include some recording of symbolisation to assist with memory; however, for the purposes of this research, our definition of mental computation does not include any use of external aids. Establishing a clear definition of mental computation from the literature is difficult due to differences in the qualities that define the concept itself and the terminology used by various researchers in this domain (Ruiz & Balbi, 2019). In addition, different terms are used to refer to the same ideas or concepts and multiple interpretations are often given for the same term (Lemonidis, 2016). In this paper the following definition of mental computation is used—*Mental computation is the computation of numbers in the head using flexible strategies*.

To further complicate matters, researchers have also sought to identify the parameters of what constitutes or does not constitute mental computation. For example, Russo (2015) suggests that mental computation is not the recall of basic facts, that is, calculations involving single-digit numbers, which students learn to recall over time. Although, the quick-fire recall of these facts may assist with the mental computation process (Maclellan, 2001), we agree with Russo and do not consider the knowledge of simple basic facts as indicating an ability to complete mental computation tasks. Likewise, mental computation also does not include standard written algorithms, which Maclellan (2001) describes as being an example of an inflexible strategy given that they follow a standardised form in which numbers are treated as single digits, without any identification of their place value, and are then acted upon uniformly. Indeed, in our view, the use of written algorithms can be detrimental to the development of a range of flexible mental computation strategies.

Most current curriculums require students to use flexible mental computation strategies as well as encouraging students to invent and use their own strategies. For example, in the Australian context, The General Capabilities of the Australian Curriculum Version 9 describes a range of flexible strategies that students are expected to use (ACARA, 2023). Given the complexities of current classrooms, where students exhibit a wide range of mathematical abilities, and given the encouragement for students to develop and use flexible strategies, it can often be difficult for teachers to easily identify the steps that individual students are following when completing mental computation tasks.

## Mental Computation Strategies

As indicated earlier, mental computation, however defined, clearly involves the use of strategies (Lemonidis, 2016). However, due to the diversity in strategy names and processes, developing a similarly clear understanding of the range of strategies is difficult (Ruiz & Balbi, 2019).

Nevertheless, there are three broad categories of strategies evident in the literature: *Jump*, *Split*, and *Compensate* (Heinze et al., 2018; Lemonidis, 2016), which we will initially use to identify and describe students' mental computation strategies. General descriptions of each strategy, as presented in the literature, are provided below. These descriptions provide the basis for an initial identification and classification of the wide range of strategies that will likely be identified in the research project—*Mental Computation in Year 5*. The frameworks are an initial suggestion, based on the literature, as to the likely strategies that students will use in solving mental computation tasks. Should alternative strategies be identified during the project, the lead author will adapt and refine the frameworks. All three strategies apply to addition, subtraction, multiplication, and division, for the purposes of this research paper, we will only provide one example of each operation.

### *Jump Strategy*

The term *Jump* describes the strategy where the student jumps from one number to another. This strategy closely resembles a counting strategy. Over the last three decades this strategy has variously been labelled using the terms: sequential counting (Beishuizen et al., 1997); aggregation (Clark, 2008); bit by bit (Money, 2010); and stepwise (Csikos, 2016). Essentially, in each of the descriptions provided, the strategy involves starting the calculation process at one of the given numbers and jumping to the next number. Lemonidis (2016) provides examples of this strategy for multiplication and division. The expression 15 * 5 is solved by repeatedly adding (or jumping) 15, five times: 15, 30, 45, 60, 75. Repeated addition may also be used to solve the expression 75 ÷ 5 by repeatedly adding 15: 15, 30, 45, 60 then 75.

### *Split Strategy*

The term *Split* is used to describe the strategy where numbers are partitioned, or split, to make calculations more manageable. For the purposes of this research, splitting a number constitutes a change to that number. This strategy has been variously labelled using the terms: decomposition (Beishuizen et al., 1997; Torbeyns & Verschaffel, 2016); separation (Clark, 2008); break up numbers (Hartnett, 2008); place value right to left or place value left to right (Money, 2010); number splitting (Russo, 2015); and split 10s (Chesney, 2013). Numbers can be split using either standard or non-standard partitioning. Torbeyns and Verschaffel (2016) describe this strategy in relation to subtraction. The example they provide is 457 - 298 where both the minuend and the subtrahend are split according to place value (standard partitioning). The process follows 400 - 00 = 200; 50 - 90 = -40; 7 - 8 = -1. Therefore 200 - 40 - 1 is 159.

### *Compensate Strategy*

The final strategy, *Compensate*, again involves the manipulation of numbers to make the calculation more manageable. The strategy involves compensating for that manipulation or change to accurately complete the calculation. The compensate strategy has been variously labelled using the terms: varying strategies (Torbeyns & Verschaffel, 2016); holistic (Lemonidis, 2016); adjust and compensate (Hartnett, 2008); and round then compensate (Money, 2010). Heinze et al. (2018) demonstrates compensate for addition: 527 + 398. This strategy involves converting the number that is close to a multiple of 10 into a multiple of 10. In this instance, two is added to 398, treating it as 400. The new equation 527 + 400 is easy to do mentally, with a result of 927. The number then must be adjusted to compensate for the original change, so two is taken away from 927. Therefore 527 + 398 (+2) = 927 (-2) = 925.

## Mental Computation—Strategy Type Framework (MC-STF)

Due to the mathematical nature of the strategies, the effectiveness of the strategy differs according to the calculation task (Heinze et al., 2018). For example, the *Jump* strategy is the most efficient strategy when calculating amounts that bridge 10. *Compensate* is usually the most efficient strategy to use when calculating with a number near a multiple of 10. Torbeyns and Verschaffel, (2016) state that 963 - 499 is most efficiently calculated using the *Compensate* strategy because 499 is near 500.

As indicated earlier, many current curricular goals promote the development of students' ability to evaluate the characteristics of the task and determine which strategy will most effectively solve the equation (Torbeyns & Verschaffel, 2016). Students need to be able to compare different strategies and identify the one that would be most appropriate to use (Graven & Venkat, 2019). To achieve this, students need a common language to use. The MC-STF aims to provide teachers and students with a common language to use when discussing strategies used.

Although *Jump*, *Split*, and *Compensate* are three strategies that we expect to see used heavily by students in the research, as explained above, the identification and naming of strategies is diverse (Ruiz & Balbi, 2019) and it is expected that more than just these three strategies will be used by students. Therefore, the lead author has developed an initial conceptual framework, based on the Jump, Split and Compensate strategies, that will be an initial starting point for classifying students' strategies in a more fine-grained way. This approach considers that students will invent their own strategies, and that these invented strategies will likely involve a blending of the three strategies named above. The MC-STF will classify strategies, which use a combination of *Jump*, *Split* or *Compensate*, by linking the names. For example, a strategy that uses both *Jump* and *Split* would be named Jump-split.

For initial coding purposes, any blended strategies will also provide an indication of the major component of the strategy i.e., *Jump* or *Split* or *Compensate*. In any instance of a blended strategy, the major component of the strategy will be classified using an upper-case letter and the minor component (or components) will be classified using a lower-case letter(s). By way of example, the equation 38 + 17 may be done by partitioning 17 by place value (10 and 7), then starting the calculation at 38 and jumping 10, making 48, then jumping 7, making 55. This would be identified as Js. By recording the differences in strategies used by Year 5 students, the research aims to identify the range of strategies used, the effectiveness of these strategies and any common misconceptions that are evident.

## Mental Computation—Efficiency and Flexibility Framework (MC-EFF)

As we explored earlier, computational fluency is an essential skill in mathematics (NCTM, 2000) and is one of the proficiencies in the Australian Curriculum. ACARA (2023) states that fluency involves students carrying out procedures flexibly, accurately, efficiently, and appropriately; and that students are fluent when they "choose and use computational strategies efficiently" (ACARA, 2023, F-10 Curriculum Version 9: Mathematics). The definition of computational fluency used in this research aligns with the definition in the Australian Curriculum. Computational fluency describes the use of efficient strategies; and applying these strategies flexibly and with accuracy (Dole et al., 2018). Another component of computational fluency is students' ability to choose the most appropriate strategy. We now briefly define how flexibility, efficiency, accuracy, and appropriateness are defined in the literature.

Flexibility refers to the skill of using number sense knowledge and recall of basic facts to manipulate numbers to complete a calculation. Students require a rich understanding of number sense to be able to calculate flexibly (Graven & Venkat, 2019). This is seen in contrast to the use of inflexible strategies (i.e., algorithms), where students are merely following a set sequence of steps to do the calculation (Heirdsfield, 2002), with no requirement for flexible thinking. An efficient strategy is one that can be carried out easily by a student. This will be evident during the process as the student should manage the tracking of sub-problems (Russell, 2000), manage the cognitive load on their working memory, and navigate the changes they make to the numbers. Efficiency considers the number of performed solution steps and the mental effort needed to perform the solution steps (Heinze et al., 2018).

When developing computational fluency, significant emphasis is placed on accuracy. Most assessments of computation fluency are designed to only measure speed and accuracy (Hopkins et al., 2019), with little emphasis placed on the identification of strategies used. For the purposes of this research, the provision of accurate answers is critical to the results, and only the strategies that generated accurate calculations will be documented. The time taken to complete a calculation (i.e., speed), will not be measured in this project, but may be a consideration for future research. Finally, students need to be able to make appropriate decisions regarding the choice of strategies they use. They need to be able to recognize a variety of strategies to solve a computation (Dole et al., 2018), compare these strategies (Graven & Venkat, 2019), and then choose the most appropriate strategy (Heinze et al., 2018).

In this paper we suggest that the MC-EFF framework (See Fig 1) is a tool that researchers and teachers can use to evaluate the efficiency and flexibility of different strategies, and then provide feedback to students on the overall effectiveness of the strategy they are using. One of the aims of the project is to encourage teachers to teach, and students to learn, strategies that fall in the high flexibility-high efficiency quadrant. The MC-EFF will be used to measure the students' ability to choose the most appropriate strategy. In the research project, each strategy identified will also be coded according to their flexibility and efficiency and this will help determine the overall effectiveness of a strategy.

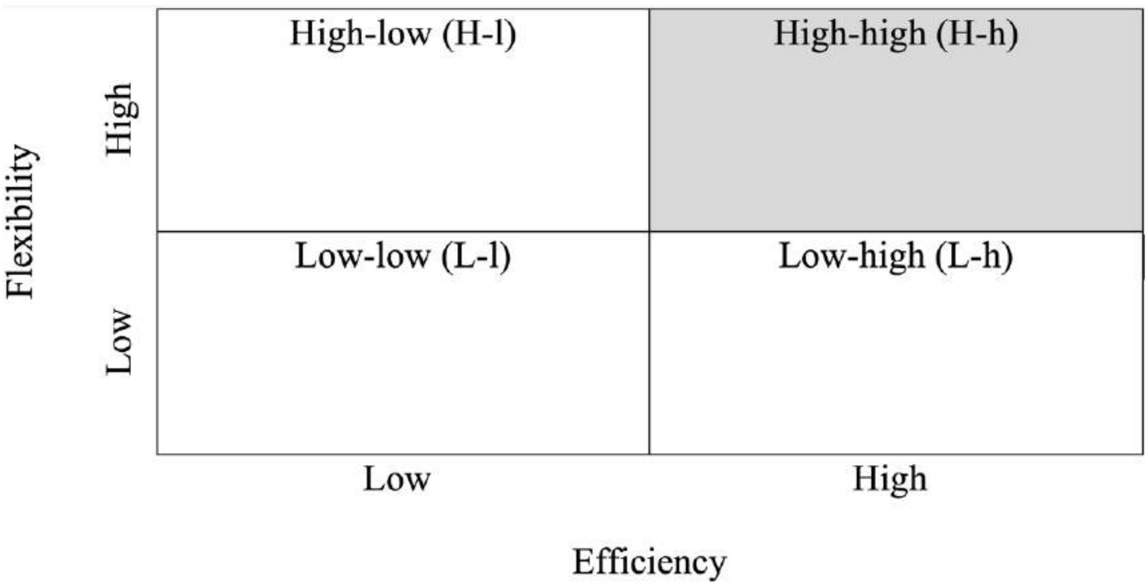



Figure 1. Mental computation—efficiency and flexibility framework. (MC-EFF) (Author 1)

For the purposes of this study, we have developed the following definitions of flexibility and efficiency. Flexibility involves modifying numbers, for example, standard partitioning, non-standard partitioning, or changing a number to make it more manageable. The number and types of changes used within the strategy determine its flexibility. If a strategy involves multiple changes to the numbers, and the use of different types of modifications, it will fall in the high flexibility category. Strategies using a low number of changes will fall in the low flexibility category. Efficiency involves the number of steps required to execute a strategy, for example, using a known fact to determine the calculation of an unknown equation, using a friendly number to assist with the calculation, and the amount of short-term memory required to complete the operation. Working memory is limited (Ding et al., 2021), so strategies that require less working memory are paramount. In this criterion, the minimum number of steps required, the use of known facts or friendly numbers, and having less short-term memory load, the higher the efficiency of the strategy. The number of steps and the types of modifications differs according to the complexity of the equation. For example, the steps used to efficiently solve 31 + 25 and 267 + 328 would differ.

Using the expression 28 + 13, we provide examples of the MC-EFF coding in use. These examples represent the possible mental strategy steps that a student may articulate when doing mental computation. Using one approach, determining the answer could follow these steps:

- 20 + 10 is 30 (both addends use the *Split* strategy)
- 8 + 3 is 11 (both addends use the *Split* strategy)
- 30 + 11 is 41

This approach to solving the expression would be identified as High-low (H-l)—demonstrating high flexibility, due to the multiple changes to the numbers, but low efficiency because there were 3 steps.

However, the same expression may be solved using the following approach:

- Move 2 from 13 to 28 to make 30 + 11 (2 numbers changed using the *Compensate* strategy)
- 30 + 11 is 41

This strategy would be identified as High-high (H-h)—demonstrating high flexibility because two numbers were changed and high efficiency as there were only two steps involved and thus putting less load on working memory.

Another approach could include the following steps:

- 13 - 2 = 11
- 28 + 2 = 30 (*Compensate* for the 2 removed from 13)
- 30 + 11 is 41

This strategy, like the first one, would be identified as High-low (H-l) demonstrating high flexibility due to multiple changes and low efficiency because of the higher number of steps.

## Implications for Pedagogy

Reasoning is the action of thinking about mental computation in a logical way. When students reason they build new knowledge as they create and validate their mathematical ideas (Herbert, 2019). However, teachers find it difficult to identify when students are reasoning in a mathematics lesson (Jazby & Widjaja, 2019) and these authors recommended that teachers plan carefully for reasoning, which includes designing appropriate tasks to increase the chance of students being able to reason. The MC-STF and MC-EFF frameworks, created by the lead author, have been designed to achieve this by assisting teachers and students in developing the knowledge and vocabulary needed to explain, analyse, and evaluate the strategies they use, identify the gaps or weaknesses in these strategies, identify more efficient ways of completing the computation (if possible), and then to justify why the strategy selection reflects the most efficient way to complete the mental computation. These are some of the skills identified by the Australian Curriculum when students are reasoning (ACARA, 2023).

Feedback is the information provided by a person or experience regarding someone's performance (Hattie & Timperley, 2007). It is proposed that both frameworks, but particularly the MC-EFF, can enhance the quality of feedback that teachers can provide their students as they complete mental computation tasks and as they make decisions about which strategies to use in future tasks. The structure of the MC-EFF provides teachers and students with the information they require to improve their strategy selection choice by increasing awareness of the flexibility and efficiency of their current mental computation strategies.

## Conclusion

The MC-STF and the MC-EFF frameworks, designed by the lead author, and explained in this paper, will initially be used to code the observations in the research project—*Mental Computation in Year 5*. The framework MC-STF will be used to name the strategies identified as students complete a variety of additive and multiplicative tasks and discuss their reasoning with the researcher. The MC-EFF will be used to identify the effectiveness of each strategy. At the completion of the project, both these frameworks will be available to assist teachers in identifying the mental computation strategies used by students and then assist them in the provision of feedback, that supports students identifying the strategies they are using and to determine their effectiveness in terms of flexibility and efficiency. This will add to our understanding of how best to support the development of fluency in mental computation.

## References

Australian Curriculum, Assessment and Reporting Authority (2023). *Foundation to Year 10 curriculum: Mathematics version 9.0*. ACARA. https://v9.australiancurriculum.edu.au/

Beishuizen, M., Van Putten, C. M., & Van Mulken, F. (1997). Mental arithmetic and strategy use with indirect number problems up to one hundred. *Learning and Instruction, 7*(1), 87–106. https://doi.org/10.1016/S0959-4752(96)00012-6

Chesney, M. (2013). Mental computation strategies for addition: There's more than one way to skin a cat. *Australian Primary Mathematics Classroom, 18*(1), 36–40. https://go.exlibris.link/y9DT1hFS

Clark, J. (2008). Year five students solving mental and written problems: What are they thinking? In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions. Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia* (pp. 131–137). Brisbane: MERGA.

Csikos, C. (2016). Strategies and performance in elementary students' three-digit mental addition. *Educational Studies in Mathematics, 91*(1), 123–139. https://doi.org/10.1007/s10649-015-9658-3

Ding, Y., Zhang, D., Liu, R.-D., Wang, J., & Xu, L. (2021). Effect of automaticity on mental addition: The moderating role of working memory. *The Journal of Experimental Education, 89*(1), 33–53. https://doi.org/10.1080/00220973.2019.1648232

Dole, S., Carmichael, P., Thiele, C., Simpson, J., & O'Toole, C. (2018). Fluency with number facts—Responding to the Australian Curriculum: Mathematics. In Hunter, J., Perger, P., & Darragh, L. (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 266–273). Auckland: MERGA.

Graven, M., & Venkat, H. (2019). Piloting national diagnostic assessment for strategic calculation. *Mathematics Education Research Journal, 33*(1), 23–42. https://doi.org/10.1007/s13394-019-00291-0

Hartnett, J. (2008). Capturing students' thinking about strategies used to solve mental computations by giving students access to a pedagogical framework. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions. Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia* (pp. 251–257). Brisbane: MERGA.

Hattie, J., & Timperley, H. (2007). The power of feedback. *American Educational Research Association, 77*(1), 81–112. https://doi.org/10.3102/003465430298487

Heinze, A., Arend, J., Gruessing, M., & Lipowsky, F. (2018). Instructional approaches to foster third graders' adaptive use of strategies: An experimental study on the effects of two learning environments on multi-digit addition and subtraction. *Instructional Science, 46*(6), 869–891. https://doi.org/10.1007/s11251-018-9457-1

Heirdsfield, A. (2002). Mental methods moving along. *Australian Primary Mathematics Classroom, 7*(1), 4–8. https://doi.org/https://search.informit.org/doi/10.3316/informit.405186314003848

Herbert, S. (2019). Challenges in assessing mathematical reasoning. In G. Hine, S. Blackley, & A. Cooke (Eds.), Mathematics education research: Impacting practice. *Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 348–355). Perth: MERGA.

Hopkins, S., Russo, J., & Downton, A. (2019). Mental computation fluency: Assessing flexibility, efficiency and accuracy. In M. Graven, H. Venkat, A. Essien, & P. Vale (Eds.), *Proceedings of the 43rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 376–383). South Africa.

Jazby, D., & Widjaja, W. (2019). Teacher noticing of primary students' mathematical reasoning in a problem-solving task. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 380–387). Perth: MERGA.

Lemonidis, C. (2016). *Mental Computation and Estimation*. Routledge.

Maclellan, E. (2001). Mental calculation: Its place in the development of numeracy. *Westminster Studies in Education*, 24(2), 145–154. https://doi.org/10.1080/0140672010240205

May, P. L. (2020). Number talks benefit fifth graders' numeracy. *International Journal of Instruction, 13*(4), 361. https://go.exlibris.link/zJzMsqbB

Money, R. (2010). Let's get mental. *Vinculum, 47*(4), 8–12. https://doi.org/https://search.informit.org/doi/10.3316/informit.424963157099788

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. (9780873534802;0873534808;). Reston, VA: National Council of Teachers of Mathematics, Inc.

Northcote, M., & Marshall, L. (2016). What mathematics calculations do adults do in their everyday lives? Part 1 of a report on the Everyday Mathematics project. *Australian Primary Mathematics Classroom, 21*(2), 8–17. https://doi.org/10.3316/informit.191396324989827

Ruiz, C., & Balbi, A. (2019). The effects of teaching mental calculation in the development of mathematical abilities. *The Journal of Educational Research, 112*(3), 315–326. https://doi.org/10.1080/00220671.2018.1519689

Russell, S. J. (2000). Developing computational fluency with whole numbers. *Teaching Children Mathematics, 7*(3), 154–158.

Russo, J. (2015). SURF's up: An outline of an innovative framework for teaching mental computation to students in the early years of schooling. *Australian Primary Mathematics Classroom, 20*(2), 34–40. https://search.informit.org/doi/10.3316/aeipt.211931

Torbeyns, J., & Verschaffel, L. (2016). Mental computation or standard algorithm? Children's strategy choices on multi-digit subtractions. *European Journal of Psychology of Education, 31*(2), 99–116. https://doi.org/10.1007/s10212-015-0255-8

Vincent, J. (2013). Thinking flexibly about numbers: Students developing their own written strategies. *Prime Number, 28*(4), 3–6. https://doi.org/https://search.informit.org/doi/10.3316/informit.617946414821821

# Mathematics Proficiency in F-6 in Version 9.0 of the Australian Curriculum

Bronwyn Reid O'Connor
*The University of Sydney*
bronwyn.reidoconnor@sydney.edu.au

Dung Tran
*Macquarie University*
dung.tran@mq.edu.au

Rebekah Strang
*Griffith University*
r.strang@griffith.edu.au

This paper reports on the representation of the four proficiency strands in the content descriptors in the latest iteration of the Australian Curriculum: Mathematics (version 9.0). Using a content analysis methodology, we documented the frequency of verbs used in the F-6 content descriptors and mapped these verbs to the proficiency strands. Fluency was found to be the dominant proficiency strand represented in all content descriptors, followed by understanding. Reasoning and problem solving were the lowest indicated proficiency strands. Results also indicated variation in the representation of the four proficiency strands across year levels and content strands. The implications for future curriculum versions are discussed.

In 2020, the Australian Curriculum underwent its first 6-yearly review and sought to refine, realign and declutter the curriculum content, resulting in the 2022 revised version 9.0 Australian Curriculum (ACARA, 2022). Whilst several revisions occurred, the four proficiency strands remained the consistent underpinning key ideas for the F-10 AC:M. The inclusion of the four proficiency strands stemmed from the work of Kilpatrick et al. (2001), who described five interrelated strands of proficiency (conceptual understanding, procedural fluency, adaptive reasoning, strategic competence, and productive dispositions). Despite Kilpatrick et al.'s (2001) inclusion of the affective strand of productive dispositions within their seminal work (a students' ability to envisage themselves as a capable learner of mathematics and perceive mathematics as useful), this strand is yet to be explicitly included in the Australian Curriculum proficiencies for mathematics. This exclusion has been considered problematic, and Woodward et al. (2017) argue for its inclusion to ensure Australian mathematics education emphasises the relevance and value of mathematics.

Mathematical proficiency remains an important outcome of school mathematics and describes what is necessary for success in learning mathematics (Kilpatrick et al., 2001). The phenomenon of using proficiency strands to drive mathematics curriculum development is not unique to Australia and has been seen in the United States NCTM Process Standards (NCTM, 2000), and in Singapore with the Ministry of Education's (MOE) Mathematics Curriculum Framework that highlights the development of students' mathematical abilities with a central focus on mathematical problem solving (2012). Additionally, the United Kingdom and Wales also follow a national curriculum with an emphasis on mathematical fluency, reasoning and competency (DOE, 2021).

Though critical in mathematics education, it can be challenging for teachers to understand how to incorporate each of the proficiencies into a balanced curriculum (Sullivan, 2011). Watson and Sullivan (2008) describe that for all proficiency strands to be present in a lesson, "teachers have to plan specifically for each and not merely offer tasks which tend towards one strand in the hopes that other strands will somehow develop automatically" (p. 112). It appears that the new AC:M has attempted to explicitly support teachers in doing this. A summary of the new curriculum revisions states that the new version has "embedded the proficiency strands into content descriptions"

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 411–418). Newcastle: MERGA.

(ACARA, 2022, p. 3). Therefore, this study aims to understand how the proficiency strands are represented in the content descriptors of the AC:M. Given this, we ask the research question:

- Is there variation in how the proficiency strands are represented in the AC:M across year levels, or across the content strands?

This study is also informed by research emphasising the critical role of curriculum analysis in education (e.g., Tran et al., 2016).

## Literature Review

### *Curriculum and Curriculum Analysis*

The curriculum is crucial in education as it impacts what teachers enact in classrooms, and subsequent student outcomes (Stein et al., 2007). Curriculum is "a plan for the experiences that learners will encounter, as well as the actual experiences they do encounter, that are designed to help them reach specified mathematics objectives" (Remillard & Heck, 2014, p. 707). As such, researchers have focused on investigating curriculum and curricular materials as primary data sources. Tran et al. (2016) argued that careful and systematic analyses of curriculum is crucial to inform public debate and support educational professionals in enhancing learning opportunities. Curriculum has been analysed for two purposes: evaluative and comparative (Tran et al., 2016). This study focuses upon comparative curriculum analysis that can assist school administrators and teachers to understand curriculum and in turn enhance learning opportunities. Different approaches in comparative curriculum analyses have included mapping content across curricula (e.g., Reys, 2006), and comparison of topics and performance expectations of one curriculum to another (e.g., Schmidt & Houang, 2012). Other researchers have focused on analysing the level of cognitive demands of content descriptions in the curriculum (Porter et al., 2011). Tran et al. (2016) suggest that researchers should focus on what mathematics (content) is emphasised and the level of cognitive demands or mathematical behaviours (processes) of the content descriptors to make curriculum content analysis meaningful. This suggestion is in line with the ways curriculum writers have emphasised the two aspects of content and process in mathematics through the description of key content underpinned by mathematical proficiencies (e.g., ACARA, 2022; CCSSI, 2010). Therefore, an analysis focusing on the process aspect of doing mathematics is helpful particularly in supporting teachers to interpret curriculum for their practice.

### *Mathematical Proficiency*

Mathematics educators and researchers have considered mathematical proficiency, the ability to develop increasingly sophisticated mathematical knowledge, skills, and understanding to a level that enables students to use mathematics effectively in everyday life (e.g., ACARA, 2022), to be crucial. Mathematical proficiency has been addressed in a range of terminology, including mathematical practices (CCSSI, 2010), mathematical proficiency (Kilpatrick et al., 2001), mathematical processes (NCTM, 2000), mathematical actions (Sullivan, 2011), and proficiency strands (ACARA, 2022). In the context of Australia, the curriculum emphasises the importance of students being able to apply their mathematical knowledge to real-world problems and communicate their thinking and reasoning clearly. There is also a strong focus on developing students' conceptual understanding of mathematical concepts, rather than just memorising procedures. Mathematical proficiency in the AC:M encompasses four strands, including fluency, understanding, reasoning, and problem solving. It has been noted that teachers can benefit from support in explicitly addressing mathematical proficiency (e.g., Selling, 2016), and proficiencies have currently received little attention in analyses (though some work has looked at specific aspects such as reasoning, e.g., Fowler et al., 2019; McCluskey et al., 2016). Tran et al. (2013) analysed how mathematical practices are addressed in teaching materials, but not the curriculum itself. Therefore, there is a need to examine how mathematical proficiency is represented in the curriculum, especially if the curriculum is used as a tool to reform teaching and learning.

## Methodology

The research design for this study involved a content analysis of the F-6 content descriptors in version 9.0 of the AC:M. Since content analysis methods involve analysing written, verbal, or visual communication in an objective and quantitative way to make valid inferences from texts (Berelson, 1952), it was appropriate to address the study aims and research questions.

The content of the AC:M is divided into six strands, including number, algebra, measurement, space, statistics and probability. In each year level from Foundation to Year 10, the AC:M includes overall year level achievement standards. Content descriptors for each strand describe what students learn, with further content elaborations providing examples of how the content can be taught. Version 9.0 of the AC:M also includes four mathematical processes: mathematical modelling, computational thinking, statistical investigation, and probability experiments and simulations; whilst important, these processes fell outside of the scope of this research. In conducting the content analysis, we attended to all strands' content descriptors for Foundation to Year 6. We chose to exclude analysis of content descriptors in Years 7 to 10, as these form the secondary years of schooling in Australia. A focus on F-6 is practical to fit within the scope of this study, and to ensure there is no confusion with the secondary curriculum which potentially has differing writers and aims. We only analysed the content descriptors, as they detail the intended student experience by specifying what teachers are expected to teach (ACARA, 2022). The curriculum authors also maintain that mathematical proficiency is embedded in content descriptors.

We sought to observe which verbs were used in the content descriptors and then identify which proficiency strand/s the verbs were addressing. The verbs were the focus of analysis as they emphasise the mathematical actions (i.e., the mathematical proficiencies) associated with the content (Anderson, 2009; Burrows et al., 2020; Fowler et al., 2019). This method of analysis reveals how the proficiency strands are represented in the content descriptors and provides insight into what proficiency strands teachers might attend to when interpreting the content descriptors. Prior to analysing and classifying verbs in the content descriptors, for reliable coding we firstly developed an analysis guide that allowed commonly used verbs to be matched to each proficiency strand. This analysis guide was informed by information from ACARA and existing literature. To develop this guide, we firstly analysed the descriptions of each proficiency strand provided by ACARA (2022) and identified the verbs. For each proficiency strand, the following key verbs (bolded) were noted from ACARA's definitions of each strand:

- *Understanding*: *connect* ideas, *adapt* and *transfer* understanding, *represent* concepts, *identify* commonalities and differences, *describe* thinking, *interpret* information.
- *Fluency*: *practice* and *consolidate* skills, *choose* appropriate procedures/strategies/representations, *carry out* procedures, *apply* knowledge and understanding of concepts, *manipulate* mathematical objects.
- *Reasoning*: develop logical thought and actions, analysing, proving, experimenting, modelling, evaluating, explaining, inferring, justifying, generalising, deducing, adapt the known to the unknown, transfer learning, compare and contrast ideas, reflect upon choices.
- *Problem Solving*: *solve* problems from mathematical and real-world contexts, *planning*, *apply* strategies and heuristics to find solutions, *reviewing* and *analysing* solutions, *evaluate*, *interpret*, and *communicate* solutions, *justify* the reasonableness of approaches, *identify* problems, *formulate* situations mathematically, *apply* mathematical understanding/fluency/reasoning to obtain solutions.

To conduct the content analysis, the key verbs in each content descriptor were then identified. The verbs were then mapped to one or more proficiency strands by interpreting how the verbs in the descriptions aligned with the verbs in analysis guide list. As well as consulting the analysis guide, we also took into consideration the context of the whole content descriptor and its intended meaning to ensure accurate coding. The three authors coded together for reliability and reached consensus for each descriptor. Therefore, the results represent 100% agreement between researchers. Once coding was complete, we also reviewed for consistency across year levels for each strand.

As an example of the coding process, we describe the coding of the Year 1 Number content descriptor: "recognise, represent and order numbers to at least 120 using physical and virtual materials, numerals, number lines and charts (AC9M1N01)". The three key verbs identified were *recognise*, *represent* and *order*. These verbs indicate skills related to what students do with numbers, including carrying out the procedures or manipulating mathematical objects. Therefore, this descriptor was coded as *fluency*. In addition, this descriptor is also coded as *understanding* as students need to connect the numbers to representations (e.g., number lines). Therefore, two proficiency strands were coded for this content descriptor.

## Findings

Table 1 summarises the proportion of content descriptors that denote each proficiency strand (through their use of verbs) for each year level. For example, 66.7% of Foundation descriptors had indications for understanding, and only 8.3% indicated problem solving.

**Table 1**

*Proportion of Content Descriptors Addressing Each Proficiency Strand Across F-6*

| | Understanding | Fluency | Reasoning | Problem Solving |
|---|---|---|---|---|
| F (12 content descriptors) | 66.7% | 100.0% | 25.0% | 8.3% |
| 1 (15 content descriptors) | 73.3% | 100.0% | 6.7% | 20.0% |
| 2 (18 content descriptors) | 94.4% | 100.0% | 22.2% | 16.7% |
| 3 (23 content descriptors) | 82.6% | 100.0% | 21.7% | 17.4% |
| 4 (23 content descriptors) | 100.0% | 100.0% | 26.1% | 26.1% |
| 5 (24 content descriptors) | 95.8% | 100.0% | 12.5% | 37.5% |
| 6 (24 content descriptors) | 100.0% | 100.0% | 29.2% | 37.5% |

It was found that fluency was present in all content descriptors across F-6, indicating that fluency was the basis of the development of content descriptors in the AC:M. That is, content descriptors usually included a verb that requires students to either carry out a procedure (e.g., count, compare, measure) or manipulate a mathematical object (e.g., symbols, numbers). Analysis of the verbs used to indicate fluency (e.g., name, partition, represent, copy, continue, quantify, choose) revealed that fluency could involve the application of skills, as well as the choice of appropriate strategies. Therefore, it was significant to observe that only seven content descriptors across F-6 addressed the choice of strategies or representations (e.g., Year 3 number—formulate problems using number sentences and choose calculation strategies).

Understanding was the second most frequent proficiency strand across all year levels. When addressing understanding, descriptors generally required students to represent mathematical objects, transfer between representations, make interpretations, or make connections among different representations or topics. It was significant to find that not all content descriptors clearly indicated understanding. Most missed opportunities for understanding occurred in Foundation and Year 1, including descriptors such as "acquire and record data for categorical variables in various ways including using digital tools, objects, images, drawings, lists, tally marks and symbols".

Reasoning and problem solving were the lowest indicated proficiency strands across F-6. For problem solving, descriptors often included mention of "problems", or "investigations" for students to address. For Foundation, only one descriptor explicitly indicated problem solving, and in Years 1 and 2 only three descriptors did so. In the upper grades, whilst there was an increased mention of solving "problems", arguably, this can be interpreted differently by teachers. For example, "solve problems involving division" (Year 5 Number descriptor) could involve straightforward exercise problems, worded questions, or richer modelling and investigating experiences.

For those indicating reasoning, most content descriptors addressed analysing something, making inferences, explaining, or had specific mention of using reasoning. In conducting the content analysis, it was noted that many descriptors had missed opportunities for reasoning across all grades. For example, the Year 4 Probability descriptor "describe possible everyday events and the possible outcomes of chance experiments and order outcomes or events based on their likelihood of occurring; identify independent or dependent events" could involve explaining reasoning for choices in ordering outcomes or justification for independent/dependent even classification.

**Table 2**

*Proportion of Content Descriptors Addressing Each Proficiency Strand Across Strands*

| | Understanding | Fluency | Reasoning | Problem Solving |
|---|---|---|---|---|
| Number (53 descriptors) | 83.0% | 100.0% | 17.0% | 35.8% |
| Algebra (16 descriptors) | 100.0% | 100.0% | 50.0% | 0.0% |
| Measurement (30 descriptors) | 90.0% | 100.0% | 16.7% | 20.0% |
| Space (15 descriptors) | 100.0% | 100.0% | 13.3% | 0.0% |
| Statistics (17 descriptors) | 88.2% | 100.0% | 17.6% | 58.8% |
| Probability (8 descriptors) | 100.0% | 100.0% | 25.0% | 0.0% |

From the analysis by content strands, it was found that understanding was indicated least in the Number strand (see Table 2). Descriptors from the Algebra strand contained the most frequent indication of reasoning, as these descriptors often mentioned using knowledge to develop or extend number facts (i.e., adapting the known to the unknown), and explaining patterns or connections between operations. Interestingly, space had the lowest mention of reasoning, as space topics could involve reasoning about shape/object clarification choices (this is only mentioned in Year 3 and then is not mentioned again). Notably, there was no explicit indicator of problem solving in the Algebra, Space, or Probability strands. Problem solving was addressed most frequently in the Statistics strand, as there are several instances of conducting some investigation or working with a question of interest.

Across all year levels, only six content descriptors provided clear indications for all four proficiency strands. An example of a descriptor addressing all proficiency strands is: "check and explain the reasonableness of solutions to problems including financial contexts using estimation strategies appropriate to the context" (Year 5 Number).

## Discussion

In this study, we analysed how version 9.0 of the AC:M addresses mathematical proficiency through the verbs used in the content descriptors. This is important as the curriculum descriptors are key sources of information for teachers when designing learning experiences in mathematics

(ACARA, 2022). To address the study aim, we considered the verbs in the F-6 content descriptors and coded these verbs as relating to specific proficiency strands. We noted the proportion of content descriptors which addressed each proficiency strand across each year level and across each content strand. Despite the AC:M version 9.0 reporting that revisions ensured that all proficiency strands apply to each aspect of the content, it was found that this was not reflected in the current curriculum documents.

Results indicated that all content descriptors addressed fluency. This is an expected result when every content descriptor is focused on doing something mathematical. In several instances, the expectations for fluency required students to perform calculations. This finding is consistent with previous studies (cf., Beswick, 2005) which found that doing mathematics is often considered as carrying out calculations. However, teachers need to highlight the second aspect of fluency, which is the selection of procedures, with requires understanding of procedures and their use. This formulates a key component of strategic competence emphasised in Kilpatrick et al.'s (2001) mathematics proficiency strands. It was significant that selection of the procedures was minimally mentioned across F-6.

In 2011, Sullivan discussed that achieving a balanced representation of proficiency strands in mathematics classrooms was "made more difficult by the way in which fluency is disproportionately the focus of most externally set assessments, and therefore is emphasised by teachers especially in those years with external assessments, often to the detriment of the other mathematical actions" (p. 8). We argue that it is not just the case that assessments drive an unbalanced focus on fluency, given the AC:M also continues to centre of fluent application of procedures with no reference to understanding, reasoning, or problem solving at times.

Despite a focus on fluency, it was found that understanding was well emphasised in the AC:M. More than 2/3 of the content descriptors addressed understanding, where students are required to represent or interpret a mathematical object, or to transfer between different representations. This finding is positive as researchers have advocated for teaching for understanding (Hiebert, 1986) and it was found that understanding was generally well emphasised across F-6. However, equitable focus on reasoning and problem-solving is yet to be achieved across F-6, with these strands less frequently indicated in content descriptors.

If mathematics learning serves as a tool for students to develop problem-solving skills, this proficiency strand needs to be emphasised. It was found that a very low proportion of the content descriptors addressed problem solving (less than 1/4 on average). Problem solving was also less frequently observed in the lower grades, compared to Years 5 and 6. Whilst young students are developing foundational mathematical understanding and skills in early grades, problem solving that is developmentally appropriate is still important for learning (Downton et al., 2020). When looking at specific content strands, problem solving is most frequently addressed in the Statistics strand, where many content descriptors allow students to investigate a problem or question of interest. This is perhaps a result of the increased focused on mathematical processes such as statistical investigations in version 9.0 of the AC:M (ACARA, 2022). Though this is a positive finding for Statistics, we consider that there was scope for this to be better addressed across other strands. In contrast, much of what version 9.0 of the AC:M has drawn attention to in its revisions has centred on "a stronger focus on students mastering the essential mathematical facts, skills, concepts, and processes" (2022, p. 3), with no mention of problem solving or reasoning. Supported by the findings of this study, we argue that this is perhaps an area that could be improved in future revisions of the AC:M. It would be worth considering in future research how other mathematics syllabi that purport a focus on problem solving and/or reasoning address these in their content descriptions.

## Conclusion

Version 9.0 of the AC:M sought to embed the proficiency strands within content descriptors. We believe that the evidence presented in the findings of this study indicate that version 9.0 of the AC:M is yet to meet their aspiration of effectively embedding the proficiency strands across all content descriptors. This is particularly evident in the imbalance in the representation of reasoning and problem solving in content descriptors across F-6 and across each content strand. Curriculum is crucial in education, however it only comes alive through teachers who interpret, transform, and implement it in their own classrooms (Nguyen & Tran, 2022). Therefore, it is teachers' ability to effectively interpret the curriculum that reforms the learning experiences delivered to students. This is important to note, as if the curriculum intends to drive effective mathematics education through its focus on mathematical proficiencies, then the proficiencies must be clearly indicated for teachers in the content descriptors. It is important to consider what an aspirational curriculum that addresses mathematical proficiency in a balanced way might look like. We advocate that more thoughtful use of verbs in content descriptors may better support teachers in equitably embedding mathematical proficiency in their teaching. Whilst we have pointed out several possible opportunities to further improve the way in which the AC:M provides direction for the inclusion of mathematical proficiencies across content descriptors, it would be of interest for further research to consider whether Australia has made progress in the representation of the proficiencies since our first iteration of the AC:M. Perhaps it is true that we are moving in the right direction through each version of the AC:M. An important direction for further research would also be determining whether verb use in curriculum content descriptors does influence teachers' practices. Overall, this content analysis of the AC:M provides insight into future directions for attention, namely increasing the balance of proficiency strand representation in the lower primary years, and across content strands.

## Acknowledgements

The first two authors have contributed equally to the preparation of this paper.

## References

Anderson, J. (2009). Mathematics curriculum development and the role of problem solving. In *ACSA conference* (Vol. 1), Canberra.

Australian Curriculum, Assessment and Reporting Authority. (2022). *F-10 mathematics curriculum (version 9.0)*. ACARA. https://v9.australiancurriculum.edu.au/

Berelson, B. (1952). *Content analysis in communication research.* The Free Press.

Beswick, K. (2005). The beliefs/practice connection in broadly defined contexts. *Mathematics Education Research Journal, 17*(2), 39–68.

Common Core State Standards Initiative [CCSSI]. (2010). *Common core state standards for mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Department for Education, United Kingdom. (2021). *Statutory guidance. National curriculum in England: Mathematics programmes of study.* https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study#key-stage-4

Downton, A., MacDonald, A., Cheeseman, J., Russo, J., & McChesney, J. (2020). Mathematics learning and education from birth to eight years. *RiMEA 2016–2019*, 209–244.

Fowler, S., O'Keeffe, L., Cutting, C., & Leonard, S. (2019). The mathematics proficiencies: A doorway into spatial thinking. *Australian Primary Mathematics Classroom*, *24*(1), 36–40.

Hiebert, J. (1986). *Conceptual and procedural knowledge: The case of mathematics.* Routledge.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: The National Academies Press.

McCluskey, C., Mulligan, J., & Mitchelmore, M. (2016). The role of reasoning in the Australian curriculum: Mathematics. In White, B., Chinnappan, M., & Trenholm, S. (Eds.), *Opening up mathematics education research. Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia* (pp. 447–454). Adelaide: MERGA.

Ministry of Education. (2012). *Mathematics syllabus: Primary one to six.* Singapore. https://www.moe.gov.sg/-media/files/primary/mathematics_syllabus_primary_1 to 6.pdf

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* National Council of Teachers of Mathematics.

Nguyen, T. D., & Tran, D. (2022, online first). High school mathematics teachers' changes in belief and knowledge during lesson study. *Journal of Mathematics Teacher Education.*

Porter, A., McMaken, J., Hwang, J., & Yang, R. (2011). Common core standards: The new U.S. intended curriculum. *Educational Researcher, 40*(3), 103–116. doi:10.3102/0013189x11405038

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM, 46*(5), 705–718.

Reys, B. J. (Ed.). (2006). *The intended mathematics curriculum as represented in state-level curriculum standards: Consensus or confusion?* Information Age.

Schmidt, W. H., & Houang, R. T. (2012). Curricular coherence and the common core state standards for mathematics. *Educational Researcher, 41*(8), 294–308. doi:10.3102/0013189x12464517

Selling, S. K. (2016). Making mathematical practices explicit in urban middle and high school mathematics classrooms. *Journal for Research in Mathematics Education, 47*(5), 505–551.

Stein, M. K., Remillard, J., & Smith, M. M. (2007). How curriculum influences student learning. In F. K. Lester (Eds.), *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics (pp. 319–369).* Information Age Pub.

Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies.* Australian Council for Educational Research.

Tran, D., I, Y., Soria, V., Darrough, R. L. (2013). Conceptualizing CCSSM mathematical practices on tasks and empirical works. In M. V. Martinez, & A. C. Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 312–315). Chicago, IL.

Tran, D., Reys, B., Teuscher, D., Dingman, S., & Kasmer, L. (2016). Curriculum standards analysis: An important research area. *Journal for Research in Mathematics Education, 47*(2), 118–133.

Watson, A., & Sullivan, P. (2008). Teachers learning about tasks and lessons. In D. Tirosh, & T. Wood (Eds.), *Tools and processes in mathematics education* (pp. 109–134). Sense Publishers.

Woodward, A, Beswick, K., & Oates, G. (2017). The four proficiency strands plus one?: Productive disposition and the Australian Curriculum: Mathematics. In R. Seah, M. Horne, J. Ocean, & C. Orellana (Eds.), *Proceedings of the Mathematical Association of Victoria annual conference 2017* (pp. 18–24). The Mathematical Association of Victoria, Australia.

# Rethinking the Number Magnitude-Based Progression: An Analysis of Place Value Development in Years 3–6

Angela Rogers
*RMIT University*
angela.rogers@rmit.edu.au

Place value is one of the ‘big ideas’ in number and plays a critical role in helping students develop their number sense, problem solving and computation skills. Yet, the elegant simplicity of our place value system belies the abstract nature of the construct. This paper presents data from 606 Year 3-6 students (ages 8-12) from two metropolitan Melbourne primary schools who completed the Place Value Assessment Tool (PVAT). Each student’s place value knowledge was categorised according to the Place Value Developmental Progression (PVDP). The results highlight the wide range of understanding in each year level and challenge the efficacy of a number magnitude-based progression in place value.

In every class within Australia, students are at various points on their journey to understand place value. Some are just beginning to appreciate the idea that 10 ones are 1 ten, while others are confidently able to apply their knowledge to work with decimal place value. It is a teacher’s role to determine the level of understanding for each student and decide upon the next best steps. This sounds relatively simple, but most educators who have taught place value would appreciate the considerable challenge this presents.

To effectively teach place value, teachers firstly require access to a quality, research-based assessment tool to determine each student’s level of understanding. Next, teachers must determine the instructional tasks which will scaffold a student to take the ‘next step’ in their development. To do this efficiently a teacher must appreciate the progression students make when coming to understand place value. Across Australia, place value instruction is largely guided by the number magnitude-based progression presented in the current Australian Curriculum document (ACARA, 2023). In the curriculum, students are introduced to 2-digit numbers in Year 1, followed by 3-digit numbers in Year 2 and so on. The aim of this paper is to use student data and the Place Value Developmental Progression (PVDP) created by Rogers (2014) to highlight the potential issues teachers face when using a number magnitude-based progression to guide their instruction. The research question this paper addresses is:

- Do the Australian Curriculum descriptors adequately describe the progression Year 3-6 students make in place value?

## Literature

### *Big Ideas*

Research by Siemon and colleagues (2012) identified six ‘big ideas’ students must obtain to develop mastery in number. The six ‘big ideas’ are: Trusting the count, place value, multiplicative thinking, partitioning, proportional reasoning, and generalising. Siemon et al. (2012) contend that a focus on these ‘big ideas’ in number ‘strips back’ the often overwhelming mathematics curriculum to the ‘non-negotiables’. Hurst & Hurrell (2014) describe how the ‘big ideas’ encourage teachers to take a more global view of mathematics education, allowing them to feel confident to teach students wherever they are in their development. Charles (2005) defines a ‘big idea’ as a “statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole” (p.10). Yet within most mathematics curriculum documents it is not immediately apparent to teachers which content is central to student learning. There is no indication of the relevant importance of individual descriptors. Curriculum descriptors related to the ‘big ideas’ such as place value, require weeks if not months of targeted instruction (Rogers, 2014),

(2023). In B. Reid-O’Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 419–426). Newcastle: MERGA.

whereas other descriptors can be addressed in just a few lessons. In Version 9.0 of the Australian Mathematics Curriculum (ACARA, 2023), the Year 3 syllabus has 23 curriculum descriptors. Only two of these descriptors explicitly mention place value (AC9M3N01 and AC9M3N03). For teachers who do not fully appreciate the importance of place value, the relatively small number of descriptors may lead them to underestimate the depth of understanding and time required to fully address this critical concept. Using the 'big ideas' to guide instruction removes the 'guess work' for teachers.

### *Place Value: A Big Idea*

Place value understanding underpins almost every part of the mathematics curriculum. Counting, estimating, money, addition, subtraction, multiplication, division, converting units and percentage all relate to place value. A lack of understanding in place value has been shown to negatively impact a student's sense of number (McIntosh et al., 1992), understanding of decimals (Moloney & Stacey, 1997) and comprehension of multi-digit operations (Fuson, 1990a, 1990b). Place value is an integral part of the Primary Mathematics Curriculum. Siemon et al., (2012), identified place value as the second 'big idea' in number. Place value is first introduced in the Australian Curriculum in Year 1 and continues to be a focus through to Year 6 (ACARA, 2023). Place value is an abstract concept that takes years to develop. It was described by Major (2011) to be like the framework of a house, supporting students to build further mathematical learning. Siemon (2017) notes that students are at considerable risk of failing to understand the subsequent big ideas (including multiplicative thinking and partitioning) without developing a solid understanding of place value.

Place value can be thought of in two ways: the place value *system* and place value *content*. As adults we are very aware of the place value *system.* This includes understanding the recursive multiplicative base 10 relationship between the place value columns, appreciating the role played by zero, and knowing a digit's value can be determined by its place in the number (Ross, 2002, Silveira, 2021). Yet we cannot simply 'teach' the place value system. We must provide students with multiple opportunities (Department of Education and Training, 2020) to engage with content related to all aspects of place value.

As noted by Major (2011), place value is often an "ill-defined concept in terms of teaching components" (p.16). The Australian Curriculum (v.9.0) uses a variety of verbs throughout Year 1-6 achievement standards and descriptors to describe place value. These include: *partition, rearrange, regroup, rename, recognise, represent* and *order* (ACARA, 2023). Yet these verbs (several of which have very similar meanings) can be easily misinterpreted by teachers. To address this issue, in her doctoral research, Rogers (2014) used Rasch analysis (Rasch, 1960) to empirically show that place value can be broken down into, and defined by, six aspects: *Calculate, Count, Compare/Order, Make/Represent, Name/Record, Rename*. The six aspects provide teachers with structure and clarity around the assessment and teaching of place value.

### *Assessing the Big Ideas*

Using assessment data to guide teaching has been shown to be one of the most effective, empirically proven processes to improve student performance (Black & Wiliam, 1998; Hattie, 2012). Yet not all assessments are created equal. If the items are too easy or too difficult the teachers will gain an incomplete picture of their students' knowledge (Izard, 2002). If online assessments are used, a teacher's involvement in the assessment process is reduced (Rogers, 2021). The opportunity to observe firsthand student responses is lessened, and a greater reliance is placed on a teacher's ability to make inferences from the data generated by the platform (assuming the data collected is valid and reliable) (Popham, 2018). Furthermore, if an assessment does not comprehensively cover the construct it is designed to address, teachers may overlook omitted content and over-emphasise the content included. Rogers (2014) highlighted this through her audit of place value assessments commonly used in Australian schools. She observed that many assessments failed to include items that addressed all six aspects of place value. For example, the skill of renaming was often overlooked, leading to a lack of teaching and student understanding in this important aspect.

It is critical that teachers have access to comprehensive assessments addressing the ‘big ideas’. The Scaffolding Numeracy in the Middle Years (SNMY) project (Siemon et al., 2006), the Reframing Mathematical Futures II (RMFII) project (Siemon et al., 2018) and the place value work conducted by Rogers (2014), have all produced valid and reliable assessments for teachers and associated learning progressions that guide evidence-informed, research-based instruction.

### *Student Progression in Place Value*

Place value is taught in every classroom across Australia, yet very little empirical evidence has been gathered to map student’s natural progression through this critical construct. Work by Clements and Sarama (2009) has shown the value of teachers using their knowledge of learning trajectories or developmental progressions to guide instruction. Common practice within Australian classrooms sees teachers use the curriculum to make decisions around the content they cover in a particular year level. Yet, as noted by Daro et al., (2011) curriculum documents are not typically “deeply rooted in empirical studies of the ways children’s thinking and understanding in mathematics actually develop” (p.16) This is evident in Version 9.0 of the Australian Curriculum (ACARA, 2023), which consistent with previous versions of the Australian Curriculum, presents a number magnitude-based place value progression. The curriculum states students should explore 2-digit place value, followed by 3-digit place value, 4-digit place value, 5-digit place value and finally decimal place value.

An alternate progression was developed by Rogers (2014). Rasch analysis (Rasch, 1960) was used to create the Place Value Developmental Progression (PVDP). The PVDP provides teachers with an evidence-based description of the typical ‘stages’ students move through when coming to understand each of the six aspects of place value. The PVDP stages increase in competence from Stage 1 through to Stage 4 and provide teachers with a brief description of the type of thinking students typically display within each aspect. Importantly the PVDP progression is *not* related to number magnitude, but more to the skills and understandings students display related to the six aspects within place value (Rogers, 2014).

Table 1 shows a summary of the recommended foci Rogers (2014) identified for each stage within the six aspects of place value. These foci inform the analysis presented in the discussion of this paper.

**Table 1**

*Place Value Developmental Progression (PVDP) Teaching Foci (Rogers, 2014)*

| Aspect | Stage 1 | Stage 2 | Stage 3 | Stage 4 |
|---|---|---|---|---|
| Calculate | Simple addition and multiplication involving tens | Basic calculations-composite units | Calculations-multiples of ten | Conceptual meaning behind multiplication and division involving multiples of ten |
| Compare/Order | Order numbers up to five digits | Identify ‘between’ | Compare using composite units | Multiplicative comparison |
| Count | Before/after, less/more | Bridging over centuples | Link between renaming/ counting | Flexible counting in multiple place value parts |

| Aspect | Stage 1 | Stage 2 | Stage 3 | Stage 4 |
|---|---|---|---|---|
| Make/Represent | Word/block association for base ten models | Explain base ten blocks | Canonical/non-canonical using proportional/ non-proportional models | Representations using other bases |
| Name/Record | Read/ write to 100 | Read/ write to 1000 | Read/write–any magnitude | N/A |
| Rename | How many tens in one hundred | Recognise composite units | Understand 'altogether' | Identify decimal parts when renaming |

## Methodology

The data referred to in this paper was gathered from two Catholic Primary schools in metropolitan Melbourne, Australia in 2016. Both schools were considered to have slightly above average levels of educational advantage as defined by their Index of Community Socio-Educational Advantage (ICSEA) value. Both school's NAPLAN data showed their Year 3 and 5 cohorts to be average or slightly higher than the national average in terms of their level of proficiency in Numeracy (ACARA, 2016a; ACARA, 2016b). This suggests the sample of students at both schools could be considered to represent a relatively typical cohort of Australian students. Both schools were keen to measure the place value knowledge of their students, and each Year 3-6 classroom teacher agreed to administer the Place Value Assessment Tool (PVAT) to their class at the beginning of the Australian school year (February) during a regular numeracy session. The PVAT was developed in doctoral research by Rogers (2014). At School A, n=296 Year 3-6 students completed the PVAT, while n=310, Year 3-6 students completed the PVAT at School B.

The PVAT is a paper and pen test which addresses the 'big idea' of place value. The PVAT has two parallel forms—Form A and B which were proven to be valid, reliable and equal in difficulty through Rasch analysis (Rasch, 1960). This paper will only refer to the data gathered from Form A of the PVAT. Form A consists of 58 short answer questions. The dichotomous items address the 6 aspects of place value (Rogers, 2014) and cover a range of difficulty levels. The PVAT items are presented from least to most difficult, and students are encouraged to complete as many questions as they can in 60 minutes. The teachers at School A and School B marked the PVAT in accordance with the marking guide and provided de-identified PVAT data sets to the researcher. These sets included the student's year level, gender and PVAT Form A raw score. Using School A and B's data, the researcher translated each student's PVAT raw score into a corresponding stage on Rogers' (2014) Place Value Developmental Progression (PVDP). The process to develop the PVDP and the raw score translator is explained in much greater detail in Rogers (2014). The results below present the PVDP stages of the Year 3-6 students in Schools A and B determined from their PVAT raw scores.

## Results

School A and B's results show a developmental progression through the four PVDP stages and across the year levels (see Figure 1 and 2). This means that in both School A and B, there is a large percentage of Stage 1 students in Year 3 (60% at School A and 30% in School B) but this decreases to almost zero by Year 6. Conversely, in both schools, there are no students in Year 3 at Stage 4, but by Year 6 a substantial number of students have reached this stage (64% in School A and 38% in School B).

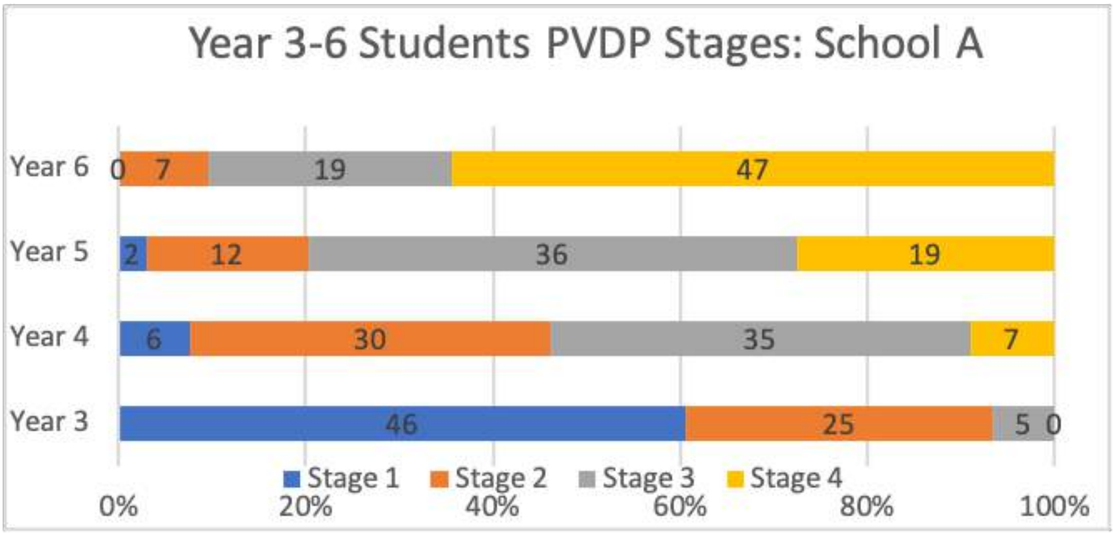



Figure 1. Place value developmental progression stages: School A.

The data also shows that within the eight cohorts of students there is a wide range of place value understanding. For example, Year 4 and 5 in School A and Years 4, 5 and 6 in School B have four stages of development amongst students, whilst Year 3 and 6 in School A, and Year 3 in School B have three stages present.

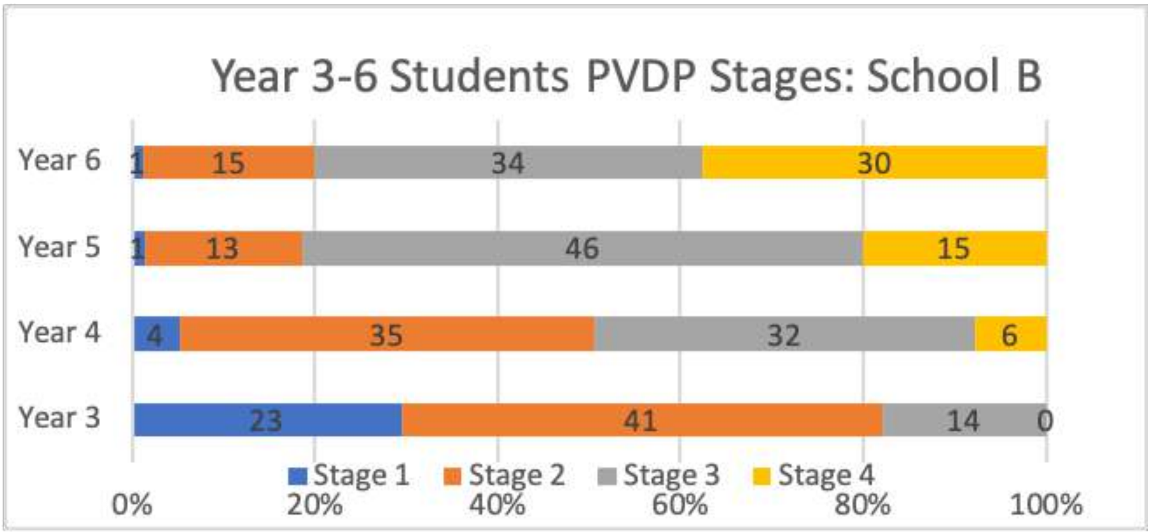



Figure 2. Place value developmental progression stages: School B.

## Discussion

The importance of providing instruction within each student's zone of proximal development has been well established in the literature (Clements & Sarama, 2009; Siemon et al., 2012; Vygotsky, 1978). Yet determining exactly what content is within reach of students is a challenge for teachers. For the most part, curriculum standards are informed by, and reflect research related to the progression students make when coming to understand mathematics. However, in the construct of place value, the progression presented in the Australian Curriculum (ACARA, 2023) contrasts with the PVDP developmental progression presented in Table 1. Two examples from the data presented above will be used to illustrate the potential issues faced by teachers using the number magnitude-based progression to guide instruction.

In Year 3, 93% of School A and 82% of School B students were found to be in PVDP Stage 1 or 2. Looking at Version 8.1 of the Australian curriculum, Year 3 students in 2016 (when this data was gathered) were required to: "Recognise, model, represent and order numbers to at least 10 000 (ACMNA052)" (ACARA, 2016). According to the PVDP foci presented in Table 1, in the aspect of 'compare/order', Stage 1 and 2 students would benefit from ordering 5-digit numbers and identifying the number 'between' two numbers. It would, therefore, be appropriate for these students to complete tasks involving ordering numbers of the magnitude suggested in the Year 3 curriculum descriptor (to at least 10,000). However, looking at the other five aspects within PVDP in Stages 1 and 2, it is not developmentally appropriate for students to count, calculate, rename, read, write or represent numbers of this magnitude. If a Year 3 teacher in School A or B was to provide place value instruction across these five place value aspects using 5-digit numbers or beyond, over 80% of the students in each class would fail to have the knowledge required to successfully engage with this content.

Looking at the data presented in Figure 1 and 2 we can see that a large proportion of Year 4 students at School A (91%) and School B (92%) display Stage 1-3 understanding in place value. The Australian Curriculum version 8.1 (ACARA, 2016) required decimals to be introduced to students in Year 4. Yet the Stage 1-3 PVDP foci presented in Table 1 suggest these students require work exclusively on whole number place value. It is therefore unrealistic and counterproductive to introduce decimal place value to students in PVDP Stages 1-3. As noted by Moloney and Stacey (1997), the concept of decimals relies on successfully integrating a thorough knowledge of the whole number place value system with the decimal system. The PVDP suggests Stage 1-3 students have not yet mastered whole number place value and introducing them to decimal place value places unnecessary pressure on both them and their teachers. This leads to superficial teaching and disengaged students. It is important to note that by PVDP Stage 4, students are considered developmentally ready to move to decimal place value. Thus, in School A and B, the introduction of decimals would be a more appropriate curriculum standard for the Year 6 cohort.

### *Implications: Teacher Education*

The two examples above show the importance of teachers being aware of the PVDP so as to refine their teaching of place value and better address the needs of their students. Place value is made up of six separate but interconnected aspects (Rogers, 2014). Each aspect requires a distinctive teaching and learning 'cadence'. For example, the rename aspect requires a sophisticated level of thinking underpinned by an appreciation of abstract composite units (Steffe et al., 1983). Renaming is multiplicative in nature, and multiplicative thinking has been shown to develop slowly in students. The PVDP indicates that renaming instruction needs to be slow and deep across the four stages. In contrast, ordering numbers appears in Stage 1 of the PVDP and is not considered a cognitively demanding skill. Being able to order 5-digit numbers does not indicate mastery in place value, yet success with a task involving numbers of this magnitude may provide teachers with an inflated opinion of a student's place value understanding. This is particularly true if the teachers are following a number magnitude-based progression, such as Australian Curriculum.

As Bednarz and Janvier (1982) noted, Year 3 and 4 children can easily compare numbers using a *digit-by-digit* procedure-based method. This means, just as placing words in alphabetical order does not require comprehension of the word's meaning, ordering numbers can be achieved without an appreciation of quantity. It is important for teachers to understand that ordering is a superficial place value skill that requires only a small amount of instruction time, while renaming is a complex skill requiring much more time. High-quality, strategic in-service and pre-service teacher education is required to ensure teachers understand the nuance required within each aspect. This knowledge will help teachers to see place value as a construct made up of smaller skills that must be taught at different rates to ensure success.

### Implications: Australian Curriculum Revision

Since the data presented in this paper was gathered, ACARA has released several updates of the Australian Curriculum. The most recent and significant update, Version 9.0, was released in 2022. It should be noted that Version 9.0 continues to present a number magnitude-based place value progression. While the author acknowledges the need for data to be gathered from current Year 3-6 classrooms to further validate its conclusions, it advocates for a revision of the place value descriptors in Version 9.0 of the Australian Curriculum. Currently one Year 3 descriptor states: "recognise, represent and order natural numbers using naming and writing conventions for numerals beyond 10 000 (AC9M3N01)" (ACARA, 2023). This descriptor requires students to work with numbers well beyond the capabilities of most Year 3 students at School A and B. It also fails to acknowledge the distinctive nature of the six aspects within place value. Similarly in Year 4, students are expected to explore decimal place value. Descriptor AC9M4N01 states: "recognise and extend the application of place value to tenths and hundredths and use the conventions of decimal notation to name and represent decimals." (ACARA, 2023). Again, this expectation was observed to be well beyond the ability of most Year 4 students at School A and B. Both these examples highlight the significant revision required to ensure the Australian Curriculum more closely reflects the six aspects of place value and the research-based PVDP progression. These revisions will assist teachers to be more accurately informed when identifying the next 'best step' for each student, and are an important first step in improving the teaching and learning of place value across Australia.

## Conclusion

This paper used the place value assessment data gathered from eight Year 3-6 classrooms and the Place Value Developmental Progression (PVDP) to emphasise three key points: the importance of teachers understanding student progression in place value through quality teacher education, the distinct role each of the six aspects play in this progression, and the necessity to reconsider the current number magnitude-based place value progression used in Version 9.0 of the Australian Curriculum.

## References


Australian Curriculum, Assessment and Reporting Authority. (2016a). *My school: "School A"*. ACARA. http://www.myschool.edu.au/

Australian Curriculum, Assessment and Reporting Authority. (2016b). *My school: "School B"*. ACARA. http://www.myschool.edu.au/

Australian Curriculum, Assessment and Reporting Authority. (2023). *Foundation to year 10 curriculum: Mathematics*. ACARA. https://v9.australiancurriculum.edu.au/

Australian Curriculum, Assessment and Reporting Authority. (2016). *The Australian curriculum: Mathematics,* version 8.1. ACARA. http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10

Bednarz, N., & Janvier, B. (1982). The understanding of numeration in primary school. *Educational Studies in Mathematics, 13*(1), 33–57.

Black, P., & Wiliam, D. (1998). Assessment and classroom learning: Principles, policy and practice. *Assessment in Education*, *5*(1), 7–74.

Charles, R. I. (2005). Big ideas and understandings as the foundation for early and middle school mathematics. *NCSM Journal of Educational Leadership*, *8*(1), 9–24

Clements, D., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. Routledge.

Daro, P., Mosher, F., & Corcoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction*. CPRE Research Reports. https://repository.upenn.edu/cpre_researchreports/60

Department of Education and Training (2020), *High impact teaching strategies—Excellence in teaching and learning.* https://www.education.vic.gov.au/Documents/school/teachers/support/high-impact-teaching-strategies.pdf

Fuson, K. (1990a). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and Instruction*, *7*(4), 343–403.

Fuson, K. (1990b). Issues in place-value and multidigit addition and subtraction learning and teaching. *Journal for Research in Mathematics Education*, *21*(4), 273–280.

Hattie, J. (2012). *Visible learning: A synthesis of over 800 meta-analyses relating to achievement*. Routledge.

Hurst, C., & Hurrell, D. (2014). Developing the big ideas of number. *International Journal of Educational Studies in Mathematics*, *1*(2), 1–18. http://doi.org/10.17278/ijesim.2014.02.001

Izard, J. (2002). *Describing student achievement in teacher-friendly ways: Implications for formative and summative assessment*. Ministry of Education.

Major, K. (2011). *Place value: Get it. Got it. Good enough?* (Master's thesis, University of Auckland]. Auckland: New Zealand.

McIntosh, A., Reys, B., & Reys, R. (1992). A proposed framework for examining basic number sense*. For the Learning of Mathematics*, *12*(3), 2–8.

Moloney, K., & Stacey, K. (1997). Changes with age in students' conception of decimal notation. *Mathematics Education Research Journal*, *9*(1), 25–38.

Popham, W. J. (2018). *Assessment literacy for educators in a hurry*. ASCD.

Rasch, G. (1960). *Probabilistic models for some intelligence and attainment tests*. Denmark's Paedagogiske Institut.

Rogers, A. (2014). *Investigating whole number place value assessment in years 3–6: Creating an evidence-based developmental progression* [Doctoral dissertation, RMIT University]. Melbourne: Australia.

Rogers, A. (2021). Computer based mathematics assessment: Is it the panacea? In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in mathematics education: foundations and pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia (*pp. 329–336). Singapore: MERGA.

Ross, S. (2002). Place value: problem solving and written assessment. *Teaching Children Mathematics*, 8(7), 419–423.

Siemon, D. (2017, February 23). Targeting 'big ideas' in mathematics. *Teacher Magazine*. https://www.teachermagazine.com/au_en/articles/targeting-big-ideas-in-mathematics

Siemon, D., Breed, M., Dole, S., Izard, J., & Virgona, J. (2006). *Scaffolding numeracy in the middle years: Project findings, material and resources*. Final Report. RMIT University.

Siemon, D., Bleckly, J., and Neal, D. (2012). Working with the big ideas in number and the Australian curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen, & D. Siemon, (Eds.), *Engaging the Australian National Curriculum: Mathematics—Perspectives from the field. Proceedings of the annual conference of the Mathematics Education Research Group of Australasia* (pp. 19–45).

Siemon, D., Callingham, R., Day, L., Horne, M., Seah, R., Stephens, M., & Watson, J. (2018). From research to practice: The case of mathematical reasoning. In *MERGA 41: Annual conference of the Mathematics Education Research Group of Australasia*. https://merga.net.au/Public/Publications/Annual_Conference_Proceedings/2018-MERGA-CP.aspx

Silveira, Everaldo (2021). A Study on the indications to the use of base ten blocks and green chips in mathematics textbooks in Brazil. *The Mathematics Enthusiast*, *18*(3), Article 6. DOI: https://doi.org/10.54870/1551-3440.1534

Steffe, L., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory and application.* Praeger.

Vygotsky, L. (1978). *Mind in society*. Harvard University Press.

# Barriers to Integration: A Case Study of STEM-Learning in Mathematics and Digital Technology

Emily Ross
*The University of Queensland*
emily.ross@uq.edu.au

Margaret Marshman
*University of the Sunshine Coast*
mmarshma@usc.edu.au

Teachers often perceive barriers to integrating technology with Mathematics. In the study, teachers participated in professional development about considerations for and examples of integrating Digital Technologies into Mathematics learning in primary schools meeting the *Australian Curriculum: Mathematics* and *Digital Technologies* requirements. This exploratory multiple case study included pre- and post-surveys, mentoring conversations and interviews to explore their beliefs and perceived barriers to integrating mathematics and digital technologies. This paper provides insights into their perceived barriers to integrating digital technologies.

When teaching STEM, many primary and junior secondary teachers have difficulty including mathematics at the correct curriculum level. The Queensland Government's review of STEM education in state schools found that STEM education could be strengthened by investment in primary teachers and junior secondary teachers (teaching out of field) by providing access to a "range of sustained and specialised professional development to support them to acquire STEM-specific content knowledge and pedagogical practices" (Department of Education, 2018). The review also described teacher confidence influenced the development and delivery of programs in STEM education, particularly in relation to integrated learning.

The *Australian Curriculum (AC): Technologies* is intended to be taught through integration with other learning areas. *AC: Mathematics* and *AC: Digital Technologies* share definitions and content in computational thinking and data collection, analysis, and representation, thus providing avenues for integration.

The project was designed to test the relevance of Ertmer's (1999) framework of first and second-order barriers to technology use in the classroom. The aim was to identify barriers that prevented teachers from implementing integrated mathematics and digital technologies lessons following professional development (PD) in a group of motivated teachers. Findings identified barriers for teachers interested in engaging in STEM-related activities in their classrooms and provided insights for developing and delivering future PD to understand the second-order barriers identified.

## Theoretical Perspectives

### *Curriculum Integration and STEM*

Curriculum integration is an authentic and engaging way of weaving across learning area boundaries to apply transdisciplinary skills for learning (Drake & Reid, 2020). An integrated approach to teaching is often complex (Kneen et al., 2020; Venville et al., 2002), and teachers spend hours planning the curriculum and resources to bring together different learning area threads (Nayler, 2014).

Advocacy for STEM education has momentum and support from business, industry, government, and education (Education Council Australia, 2015; Ross, 2022) as STEM is necessary to underpin Australia's economic growth. Knowledge from multiple disciplines is required to solve complex global challenges which are not derived from one area or discipline alone (Keeley & Knowles, 2016); hence students need to be educated in STEM-related areas to promote and foster students' career paths in STEM-related fields (Education Council Australia, 2015).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 427–434). Newcastle: MERGA.

Easton et al. (2020) stated that the mathematics component of STEM learning is perhaps the most important. Mathematics is a language that can cross the boundaries of all disciplines through problem-solving, logical reasoning and spatial thinking (Coad, 2016; Ferme, 2014). However, Mathematics is rarely foregrounded in STEM activities (English, 2016). Frequently, the focus of an integrated STEM unit has been derived from one discipline with identifiable connections to other disciplines (Ross, 2022). When these conceptual links are made, they are not at an age-appropriate level, but rather the concepts consolidate prior learning. The question is whether the learning experiences are drawn from mathematics concepts that further the learning expected at that year level or are more aligned to opportunities to consolidate prior learning or numeracy concepts. Mathematics is often the final element considered in the integration puzzle, which is the issue this research aims to address through connection with integrated digital technologies.

### *Barriers to Integrating Digital Technology*

Ertmer (1999) highlighted two categories of barriers to embedding digital technology in classrooms. First-order barriers refer to issues external to the teacher, such as access to the necessary technological devices or software, insufficient teaching time, and inadequate technical support. Second-order barriers are about the teacher, including beliefs about digital technology, appropriate pedagogy, and reluctance to engage with technology (Ertmer, 1999). Ertmer (1999) stated that both first- and second-order barriers hinder teacher efforts to embed digital technology in the classroom. However, she suggested that second-order barriers can be more difficult to overcome. First-order barriers are typically about resourcing; thus, once sufficient resources have been received, the problem has generally been resolved. However, second-order barriers involve personal beliefs and are intrinsic to the teacher.

Ertmer's (1999) study focused on technology use in the classroom. Subsequent research has considered technology integration in the classroom. An & Reigeluth (2012) encountered more first-order than second-order barriers among the teachers surveyed. Attard (2013), considering the use of mobile technologies in primary mathematics classrooms, found that teachers are often provided with the technology and expected to integrate it into their classroom practice without the PD support necessary, meaning that teachers struggled to understand how the technology could be integrated into a primary mathematics classroom to enhance learning. PD that is centred in the school context allows teachers to enact the learning and validate ideas in their own classrooms leading to collaboration, experimentation and reflection, and boosting connectivity between the learning and enactment of the learning (Silver et al., 2007). Teachers in Perienen's study (2020) identified the need for further training to be able to integrate digital technology into their mathematics teaching whilst those in Baya'an et al.'s study (2019) demonstrated how establishing a community of practice enabled mathematics teachers to integrate ICT into their teaching.

Version 9 of the AC makes the connections between *AC: Mathematics* and *AC: Digital Technologies* more explicit to support the integration of the two learning areas. This project included PD that focused on STEM curriculum integration and technological content knowledge to demonstrate connections between the *AC: Mathematics* and *AC: Digital Technologies* in the primary years. The project aimed to determine which first- or second-order barriers existed with the teachers who participated and whether implementing integrated Technology and Mathematics activities in the classroom reduced the number of barriers teachers perceived for integrating technology in mathematics. The research question was:

- What barriers exist for primary teachers when implementing integrated Digital Technologies in Mathematics?

## Research Design

### *Context*

This 6-month study is part of a larger study. The smaller study focused on the experiences of two primary school teachers at the same school. Only data from these two teachers were included and pseudonyms used. The teachers attended a PD session where statistical and digital technologies concepts were integrated and used this learning to enact integrated lessons using digital technologies to teach and consolidate mathematics concepts. The teachers were offered two mentoring sessions after trialling integrated lessons and a second PD session which further developed their understanding of integration through geometric visualisation activities. An exploratory multiple-case study design (Yin, 2009) used qualitative methods for collecting data, including surveys and a semi-structured interview.

### *Participants*

The participants were from one independent, high socio-economic status primary school in the western suburbs of Brisbane. The teachers' classes included students from a broad spectrum of academic achievement levels. The school has a reputation for STEM learning and engaging students in advanced learning in digital technologies and robotics. Whilst all teachers in the primary school were invited to participate, only two teachers accepted the invitation.

Abigail had less than seven years of teaching experience. She is keen to use technology in the classroom and assists her peers with technological issues and questions. Abigail has taught mathematics and digital technologies in Australia and the United Kingdom. This is her first school since her return to Australia.

Hillary also had less than seven years of teaching experience. She believes she has only a basic understanding of how technology can be implemented in the classroom. She had not had experience teaching with digital technologies and felt intimidated by the technology available at the school when she first arrived. She recently moved to Australia, having trained and taught in schools in the United Kingdom.

### *Data Collection*

The teachers completed a pre-PD survey to ascertain their *starting* beliefs, and participated in two mentoring sessions, a post-project survey and a final semi-structured interview at the conclusion of the study. The survey included questions from the *Technology Beliefs and Barriers to Creating Technology-Enhanced, Learner-Centred Classrooms* sections of An and Reigeluth's (2012) survey. Additional questions were derived from those sections relating to the study's specific context, i.e., beliefs about and barriers to integrating *AC: Mathematics* and *AC: Digital Technologies*. The survey was distributed again at the end of the school term following all teacher lessons and project mentoring sessions. The mentoring session prompts were based on Rolfe et al.'s (2001) framework for reflective practice asking the teachers to:

- describe the integrated lesson the teacher had completed, including aspects that worked well and challenges (what)
- highlight aspects that the teacher would change, were interesting or surprising (so what)
- outline what they planned to do next (now what).

The mentoring sessions were held through Zoom to enable flexibility. A second PD session was held following the second round of teacher lessons to share practice and provide additional concepts and examples for integrating digital technologies in different conceptual areas of mathematics—visualisation. Final interviews were held with the teachers at the conclusion of the project. Interview prompts related to their beliefs about teaching and learning mathematics, teaching and learning technology, attitudes toward integration, and barriers they perceive to integration in the classroom.

### *Data Analysis*

Thematic analysis was used to analyse the data collected using Braun and Clarke's (2006) six phases of thematic analysis to identify themes. A comparative analysis considered each case's emerging enablers and barriers (themes) to integrating digital technologies.

## Emerging Enablers and Barriers

Key themes emerging from the data were perceived experience and confidence to teach mathematics, perceived experience and confidence to teach digital technologies, and school culture and organisational practices, including access to resources.

### *Experience and Confidence to Teach Mathematics*

While Abigail was educated as a teacher in Australia, most of her teaching experience has been in the United Kingdom but feels confident about teaching the *AC: Mathematics* and *Digital Technologies*. However, Abigail explained that mathematics was stressful for her, and she appreciated that mathematics provided "one right answer", unlike other learning areas.

> I don't think I'm particularly confident in my own ability in maths. It was always a thing that I felt like anxious about. But at the same time, my lack of confidence, I think in like this feeling of anxiety that even I as a person, I think a lot of people feel that maths anxiety around numbers…I've always really liked that there is an answer at the end of the tunnel. Whereas all of those others like, you know, HASS and science, there's no right or wrong answer. It's about where you come through, because I really, I enjoy that part of math. … I definitely, as a person feel math anxiety. (Interview: Abigail)

Hillary described feeling confident about teaching mathematics as the AC is like what she taught in the United Kingdom. Hillary expressed that she initially disliked mathematics, was intimidated by it, and was taught that mathematics was about getting the correct answer. However, she explained that that did not always bring success because having the correct answer does not mean that a student understands the reasoning behind mathematics.

> As a child, I really disliked it [Maths]. And I think it's because we have that, you know, sit down, this is what you need to learn. You repeat it in this way. And you'll have success because eventually, you'll get it right. But that success doesn't really come, and it doesn't last because there's no actual understanding of the reasoning behind it…So I felt very intimidated by maths growing up. And it wasn't until I became a teacher and trained as a teacher that I've realised that the different way to do it, how it should have been taught to myself as a young person. (Interview: Hillary)

Now she feels excited to teach mathematics, particularly when she sees one of her students' faces lights up when they reach understanding.

> And now maths makes me, I don't know, I feel excited, really excited by it. And it's exciting to watch…the children's faces…light up when there is that element of understanding because we use a different way of explaining it or we've got the manipulatives out or we've used a visualisation tool and I got it and there's nothing better than that feeling. (Interview: Hillary)

Both teachers described the importance of real-world, hands-on, problem-solving for developing students' understanding and critical thinking and their preference for pedagogy that reflected this.

> Maths is one of the ones in our curriculum that is connected completely to like real-world situations like money or time or accounts, like just general things like that. And I think so obviously means, like real-world context, I think it also means problem-solving, like critical thinking, and being able to like decode and understand situations, and work our way through it quite systematically. (Interview: Abigail)

> …sometimes I have to ensure that I am going back to my values as a teacher and going back to what I know is the best pedagogy and ensuring that I mix that with what the school's expectations are for how things are taught. (Interview: Hillary)

## *Experience and Confidence to Teach Digital Technologies*

The two teachers' technology beliefs were very similar and did not shift throughout the project. Both teachers supported using technology in the classroom, believing it necessary for student learning. They believed as classroom teachers they needed to keep up with new technologies and embed them into their teaching.

> I feel confident teaching it [digital technologies] and delivering it. (Interview: Abigail, 29 Nov 2022)

In her final interview, Abigail described feeling hampered by the confidence of other staff to teach with technologies in two opposing ways. In her cohort, Abigail felt that she was frequently the teacher who wanted to continue to push forward with technologies but that she needed to be mindful of the confidence and capacity of her colleagues. She described that she is often the person who is called upon to explain the new technology or to support her fellow cohort teachers in teaching the technologies.

But it's hard to do it, I guess in practice sometimes. Because it's not necessarily just me in that room, or me in that cohort, teaching it. So, I think, getting others to be on the same board and teaching them I find that stressful and difficult. Because everyone doesn't have the same kind of like self-efficacy with technology that I do. And then, I have to consider them and what works best in the situation. And I think digital technologies is something that's a challenge for a lot of people. (Interview: Abigail)

In contrast, Hillary initially lacked confidence to teach digital technologies. She stated that she had little experience using digital technologies, particularly as she had taught in low socio-economic schools in the United Kingdom.

> There's something I started off with very, very little confidence...It's something that coming from the UK, in the sort of schools I worked in, we worked in very low socio-economic areas, there just was not the access for the children to digital technologies. So, they weren't used and utilised within the classroom. Because just financially, they couldn't afford that equipment. (Interview: Hillary)

Hillary was initially intimidated by the students' overwhelming access to technology, but her competence was increasing with her emersion. Nevertheless, she needed ongoing support for it to be a natural inclusion.

> Where it's been coming here, each girl has a one-to-one laptop; it was quite intimidating to start with. I will say my confidence has definitely grown. (Interview: Hillary)

## *School Culture and Organisational Practices*

The teachers agreed on several barriers to creating technology-enhanced, learner-centred classrooms. In the survey, the teachers strongly agreed that *Subject culture (the general set of institutionalised practices and expectations which have grown up around a particular school subject)* was a barrier as well as time, assessment requirements and knowledge about ways to integrate technology into learner-centred instruction. Through analysis of the survey and interview data, themes emerged related to school culture and organisational practices.

Abigail and Hillary reported high levels of support for teaching technology at the school with good access to new technology and support for experimenting with its use. There was a technology specialist within the school, however, both teachers reflected on issues relating to this human resource.

> I think because she is so fantastic. And she's got all of those wonderful banks of ideas already. I wonder whether teachers sometimes take it upon themselves to try and learn new things to integrate into their classrooms, or whether there's kind of that reliance on using her to feed ideas into their own rooms (Interview: Hillary)

> We were really fortunate with the tech teacher. … I didn't feel that I could incorporate it [digital technologies] as much, I guess, into my classroom use because she's got it so set up. But we started to do that,

> I think through this programme and have that more confidence to bring it over and to utilise it in the classroom rather than it being quite segregated. So that's definitely good. (Interview: Abigail)

Both teachers agreed that they would like to try using the technology in a more integrated way. While they had almost unlimited access to digital technologies resources, they still felt that they could benefit from PD focussed on using them effectively in the classroom. Abigail explained that the primary school used a timetable to allow students a wide array of specialist teachers, including music, visual art, languages, digital technologies, health, and physical education. The students were provided with learning from subject-matter experts but to the detriment of making clear connections across programs or using integrated learning.

> I think it's been quite hard with a K to 12 school, which is very secondary in terms of the way they timetable. A lot of our subjects are quite siloed, … the primary, they encourage us, and they're willing, they're happy for us to kind of go and integrate because we are stuck to like at a time to go, I found it quite tricky. (Interview: Abigail)

Although the school was supportive of teacher innovation and integration, Abigail found that constraints and expectations of the school environment the lack of flexibility meant that she felt that she was not able to engage in "a big inquiry or big problem-solving" and there was an expectation for more traditional mathematics teaching styles.

Hillary described the volume of content to be taught as well as the expectation to use traditional didactic learning styles when teaching mathematics.

> I think because of the school being the way it is and the sort of girls that come into the school. There is that expectation that the work is that high, high pitch, so I think there's a lot of content to get through. And some of it is set quite a bit higher than the year group expectations. And it just puts a lot of pressure on getting through it all. And whereas I guess I would go slower and deeper is my gut feeling, and to do lots of reasoning and problem-solving, you know, beginning with that fluency and that understanding, but then having time within a lesson or within the next lesson to apply that. (Interview: Hillary)

Hillary expressed a preference for going *slower and deeper* to develop conceptual understanding and allow for application. Both teachers described the broader primary school staff's preference to teach according to the chosen textbook. Abigail described the textbook as quite text-dense, thereby creating issues for students with literacy issues. Abigail described feeling caught between wanting to use the text as a resource with other, more hands-on dialogic learning experiences and feeling the pressure of parent expectations given the purchase of the textbook. Hillary also spoke of the importance of more hands-on techniques supporting student needs, developing from her understanding of mathematics. Further, she described a preference for ongoing monitoring and formative assessment rather than the school's preferred assessment strategies focused on written tests to collect evidence. She wanted to allow the students to explain how and why they got their answer, not just the solution itself.

## Discussion

### *Second Order Barriers*

The teachers described in this paper expressed second-order or intrinsic barriers including knowledge of technologies, knowledge of pedagogy, and developing confidence.

Despite including two PD sessions and two mentoring sessions with the teachers, both claimed they needed further support to effectively use of digital technologies in the classroom and integrate technology into learner-centred instruction. As Attard (2013) observed, the teachers in this study were given access to technologies without the necessary PD support for effective inclusion in the classroom. Whilst the school invested in a specialist teacher to support the use of digital technologies they were used as much to deliver the classroom teaching, rather than to support teachers' learning about teaching of digital technologies.

Each teacher described past anxiety relating to the teaching of mathematics and that they perceived their school culture to support more traditional pedagogies and textbook use. As Ertmer (1999) describes, second-order barriers often relate to teachers underlying beliefs about teaching and learning, thus making them harder to shift as sometimes the teacher is unaware that they hold these deep-seated beliefs. During mentoring meetings, the teachers frequently doubted the veracity of their integrated tasks. They suggested that the tasks in their lessons were ill-conceived and did not focus on the areas they aimed to develop. However, with mentored reflection, the teachers realised that their integrated lessons provided mathematics learning opportunities beyond what they had intended and that the additional discussions and engagement for their students provided richer learning opportunities than they had expected.

### *First Order Barriers*

The survey data showed that both teachers identified first-order or external barriers relating to time, assessment, and subject culture barriers to technology integration. The interviews also revealed other first order barriers relating to school culture and organisational practices.

Both teachers described feeling that the culture of the school and expectations for teaching and resources hindered their capacity to engage in further integration of digital technologies in their mathematics classrooms. The requirement to complete high volumes of content contributed to a lack of time to engage in deeper problem-solving and integrated learning. Ertmer (1999) described the perception that first-order barriers were easy to overcome as they frequently referred to barriers emanating from resourcing. These two teachers described here believed they had access to almost unlimited digital technologies resources, were keen to integrate digital technologies in their lessons, and learn more about integration, however, for both teachers overcoming the barrier of school culture felt insurmountable. The teachers felt the pressures of a siloed approach to teaching the learning areas through school timetable restrictions, the expectation that the mathematics textbook would be used, and the perceived expectations to use a more didactic teaching style. A limitation of this study has been the focus on the teachers from the school without further input from the school administration to seek clarification as to the messaging to staff about issues pertaining to integration, timetabling, textbooks, pedagogy, and assessment.

## Conclusion

This paper provided a small snapshot of a larger study that explored teachers' perceived barriers to integrating mathematics and digital technologies. In this paper, two teachers' journeys were used to illustrate an exploration of Ertmer's (1999) first and second-order barriers to integration of digital technologies. The teachers, Abigail and Hillary, were similar in all but one aspect of their experience, their confidence to teach digital technologies. Yet both teachers described similar second order (intrinsic) barriers to teaching integrated mathematics and digital technologies in their school, including knowledge of technologies, knowledge of pedagogy, and developing confidence.

Although the study was limited to one school, lessons learned may be useful in considering future PD. The first order (extrinsic) barriers described in the case studies identified aspects of school culture, even within a technologically advanced school, as producing perceived barriers to further technology integration. This is important consideration for schools when new technology is introduced. However, further research incorporating the voices of the school leadership would be beneficial to provide clarity of expectation for teachers.

# References

An, Y. J., & Reigeluth, C. (2012). Creating technology-enhanced, learner-centered classrooms: K–12 teachers' beliefs, perceptions, barriers, and support needs. *Journal of Digital Learning in Teacher Education, 28*(2), 54–62.

Attard, C. (2013). Introducing iPads into primary mathematics pedagogies: An exploration of two teachers' experiences. *Paper presented at the 36th annual meeting of the Mathematics Education Research Group of Australasia* (MERGA). Melbourne, Victoria: Australia.

Baya'a, N. F., Daher, W. M., & Anabousy, A. A. (2019). The development of in-service mathematics teachers' integration of ICT in a community of practice: Teaching-in-context theory. *International Journal of Emerging Technologies in Learning, 14*(01), 125–139. https://doi.org/10.3991/ijet.v14i01.9134

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Coad, L. (2016). Opinion: The M in STEM: What is it really? *The Australian Mathematics Teacher, 72*(2), 3–6.

Department of Education. (2018). *Review of STEM education in Queensland state schools 2015–2017: Final report 2018*. https://education.qld.gov.au/curriculums/Documents/review-of-stem-education-queensland-state-schools.pdf

Drake, S. M., & Reid, J. L. (2020). 21st century competencies in light of the history of integrated curriculum. *Frontiers in Education (Lausanne), 5*. https://doi.org/10.3389/feduc.2020.00122

Easton, G., Wellington, A., Davis, J. P., & Yeh, A. (2020). Foregrounding mathematics in integrated STEM projects via digital technology. *The Australian Mathematics Education Journal, 2*(3), 10–15.

Education Council Australia. (2015). *National STEM school education strategy, 2016–2026.* www.educationcouncil.edu.au/site/DefaultSite/filesystem/documents/National%20STEM%20School%20Education%20Strategy.pdf

English, L. D. (2016). STEM education K–12: Perspectives on integration. *International Journal of STEM Education, 3*(3). https://doi.org/10.1186/s40594-016-0036-1

Ertmer, P. A. (1999). Addressing first- and second-order barriers to change: Strategies for technology integration. *Educational Technology, Research and Development, 47*(4), 47–61. https://doi.org/10.1007/bf02299597

Ferme, E. (2014). What can other areas teach us about numeracy? *The Australian Mathematics Teacher, 70*(4), 28–34.

Kneen, J., Breeze, T., Davies-Barnes, S., John, V., & Thayer, E. (2020). Curriculum integration: The challenges for primary and secondary schools in developing a new curriculum in the expressive arts. *Curriculum Journal, 31*(2), 258–275. https://doi.org/10.1002/curj.34

Nayler, J. (2014). *Enacting Australian curriculum: Making connections for quality learning—An issues paper prepared for the Queensland Studies Authority*. www.qcaa.qld.edu.au/downloads/p_10/ac_enact_ac_paper.pdf

Perienen, A. (2020). Frameworks for ICT integration in mathematics education—A teacher's perspective. Eurasia *Journal of Mathematics, Science and Technology Education, 16*(6), em1845. https://doi.org/10.29333/ejmste/7803

Rolfe, G., Freshwater, D., & Jasper, M. (2001). *Critical reflection for nursing and the helping professions: A user's guide*. Basingstoke, UK: Palgrave Macmillan.

Ross, E. (2022). Approaches to integrating STEM learning. *Australian Mathematics Education Journal, 4*(2), 15–20. https://doi.org/10.3316/informit.589970101384704

Silver, E. A., Clark, L. M., Ghousseini, H. N., Charalambos, C. Y, & Sealy, J. T. (2007). Where is the mathematics? Examining teachers' mathematical learning opportunities in practice-based professional learning tasks. *Journal of Mathematics Teacher Education, 10*(4), 261–277.

Venville, G. J., Wallace, J., Rennie, L. J., & Malone, J. A. (2002). Curriculum integration: Eroding the high ground of science as a school subject? *Studies in Science Education, 37*(1), 43–83.

Yin, R. K. (2009). *Case study research: Design and methods*. Thousand Oaks, CA: Sage Publications.

# Examining the Role of Mathematics in Primary School STEM Lessons: Insights from a Professional Development Course in Indonesia

Rumiati
*BBGP D.I. Yogyakarta, Indonesia*
rumiati1@yahoo.co.id

Wahyudi
*SEAMOLEC, Jakarta, Indonesia*
wahyudi@seamolec.org

*Balai Besar Guru Penggerak Daerah Istimewa Yogyakarta (BBGP DIY)* conducted a professional development course aimed at enhancing primary school teachers' capacity to develop and implement STEM lesson plans in their classrooms. As part of the course, teachers were asked to record their lessons, which were later analysed to identify the mathematical content involved and how teachers delivered this content during the STEM lessons. The findings indicated that, although mathematics appeared to be less pivotal than other STEM disciplines in the lesson plans, the STEM activities provided rich opportunities for developing students' mathematical content knowledge. Most of the teachers realised this and were able to deliver the mathematical content within the context of STEM education. This paper presents an insight into how primary school teachers in Indonesia deliver mathematical content in their STEM lessons and highlights the role of mathematics in STEM education.

STEM education has gained attention around the world since the last decade, including from those within the mathematics education research community in Australia. An examination of MERGA publications during the last five years showed a number of papers reporting STEM education in various aspects. Anderson et al. (2017) reported a professional learning program to support secondary school teachers to design and implement the most appropriate STEM program for their students. Gervasoni et al. (2017) argued that focus on practices were more aligned with the play-based and intentional teaching objectives in the early years learning framework, rather than the traditional thinking concerning the integration of discipline content knowledge. Symposia papers by Clements et al. (2019) reported the use of digital technology to inspire preschool children's curiosity and engagement in STEM concepts. A study by Ferme (2018) indicated that STEM teachers have greater confidence and attitude towards numeracy than those of non-STEM teachers. Mulligan et al. (2022) reported an interdisciplinary approach to STEM subjects, namely mathematics and science and the role of mathematics in such approach.

Similar to the neighbouring country, there was also growing interest in STEM education in Indonesia during the last decade. A systematic review by Zainal Arifin et al. (2021) showed that the trend of STEM education research in Indonesia began in 2016 and has experienced significant growth afterwards. A Scoping review by Farwati et al. (2021) to map all articles on STEM education implementation published online found that research on this topic has been conducted in 19 out of 38 provinces in Indonesia. Those studies were dominated by West and East Java, the two most populous provinces in the country. STEM education has also been studied in all levels of education, and high schools were the most widely used research contexts. Regular courses for teachers on STEM education have also been conducted by several institutions in Indonesia, namely SEAMEO QITEP in Science, SEAMEO QITEP in Math, BBGP DIY (Professional Development Centre for Educators in D.I. Yogyakarta Province) and BBGP Jawa Barat (similar institution in West Java).

BBGP DIY is an institution under the Ministry of Education, Culture and Research in Indonesia that has responsibility to develop and empower teachers and education personnel in Yogyakarta Province. One way to carry out the duty is by conducting professional development courses to increase teacher competency and their teaching quality. There were four courses offered by BBGP DIY in 2022, namely courses on STEM Education, Differentiated Instruction, Literacy and Numeracy, and Computational Thinking. This paper discusses one of the courses, the course on STEM education. The paper also examines the mathematical content in the lesson plans developed by the course participants and how these were delivered during the lessons. The paper highlights the

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 435–442). Newcastle: MERGA.

role of mathematics in STEM education and provides an insight on how teachers deliver mathematical content in their STEM lessons.

## The Professional Development Course on STEM Education

Ten Indonesian primary school teachers from five districts in Yogyakarta province completed an 82-hour professional development course on STEM Education in November 2022. The course aimed to introduce STEM education and equip teachers with skills to design and implement STEM lesson plans. The teachers, who had more than five years of experience, were generalist teachers teaching various primary school subjects except for physical and religious education. The course also encouraged knowledge and experience sharing among participants. The first author of this paper was the facilitator of the course.

Before the course, a survey was conducted to assess the teachers' understanding of STEM education. Results showed that none of the teachers had undergone professional development on STEM before, and half were not familiar with STEM at all. The other half had heard of STEM but were unsure of its meaning and how to incorporate it into their lessons. They had only encountered STEM through conversations with colleagues, news articles, or online searches.

The professional development course was divided into three phases: development, implementation and evaluation phase. The description of each phase was as follows.

### *Phase 1: Development of STEM Lesson Plans*

In the first phase, teachers participated in a three-day face to face workshop at BBGP DIY or equal to 30 learning hours (one hour was 45 minutes). Each day, the workshop started 7.30 am and finished at 5pm. The workshop covered a brief history of STEM education, the rationale of STEM education, the current trend of STEM education in Indonesia and other countries, and the definition of STEM according to a number of sources such as Tsupros et al. (2009), Bybee (2010), Brown et al. (2011) and Kelley and Knowles (2016). During the discussion, it was found that many participants were familiar with the traditional disciplines of science and mathematics, however they needed more time to discuss and established the definition of technology and engineering. Furthermore, teachers were introduced to the definition of STEM education according to Moore et al. (2014) who argues STEM education is an “effort to combine some or all of the four disciplines of science, technology, engineering and mathematics into one class, unit or lesson that is based on connections between the subjects and real world problems” (p.38).

The workshop introduced three approaches to teaching STEM: Silo, Embedded, and Integrated (Roberts & Cantu, 2012). With Silo, each discipline is taught independently, while Embedded uses one discipline as the anchor, with others providing support. Integrated treats STEM as a single subject, with a minimum of two disciplines involved. Participants shared that they previously taught math and science separately, so the facilitator encouraged them to adopt new approaches such as Embedded or Integrated with more than two disciplines, including Art or Reading, which could result in STEAM or STREAM education

In developing the lesson plans, teachers were free to choose any topic, any content, any real world problem or any situation for their lesson plan. They might choose teaching objectives in accordance with the curriculum used in their school, namely *Kurikulum 2013*, *Kurikulum Darurat* (emergency curriculum) or *Kurikulum Merdeka*. Teachers might use teaching methods such as problem-based learning, project-based learning or inquiry-based learning. Furthermore, to guide teachers to develop STEM lesson plans, the facilitator introduced a five step process which was a modification of the Engineering Design Process: *Empathy, Questions, Ideas, Prototype,* and *Experiment/Evaluation*.

*Empathy*. During this step, teachers presented situations or problems and discussed it with their students. Teachers might present the problem or situation in the form of news, videos, readings or free guided discussion with their students. Teachers were encouraged to foster students' empathy towards their surroundings or environment, understand other people's perspectives, care about others and cultivate students' willingness to help each other for the betterment of all.

*Questions*. This step was important to clarify the problems or situation and ask appropriate questions. During this step, teachers might guide students by asking questions such as "what do you notice?", "is there any problem?", "what happens if...". These questions might overlap with questions to collect ideas from the students.

*Ideas*. Teachers guide students to brainstorm ideas to solve the selected problem or situation. Teachers probed students with questions such as "what can we do in this situation?" "what can we do to solve this problem?" "what else has been done in this situation?' "can we solve this problem better than what other already did?". They might reflect on their experience and if possible undertake research to solve the problem. Several options might arise during the discussion, and teachers guide students to narrow down the options and finally agree on what they would do, for example creating a product, doing activities or conducting experiments.

*Prototype*. Teachers and students discussed tools and material that might be useful for creating the products, doing activities and conducting experiments. Because many teachers were afraid that STEM lessons might become an expensive lesson, they were encouraged to use tools and materials which were available and easy to find around them. This step was a time for them to realise their ideas. Teachers were encouraged to be open-minded and seriously listen to the students to accommodate their creativity as far as it is possible.

*Experiment/Evaluation*. Teachers guided students to test their solution. They might succeed or fail, but it was the opportunity for students to get a better perspective of the problem or situation, and to find better solutions in the future. Teachers guided students to present their work to the class. This was also an opportunity for students to improve their communication skills, one of the important skills needed in 21$^{st}$ century (Partnership for 21$^{st}$ century skills, 2007).

The product of the first phase was a draft of lesson plans which were ready to be implemented in the classroom during the second phase of the professional development course.

### *Phase 2: Implementation of the STEM Lesson Plans*

The second phase, which lasted for three weeks or 35 learning hours, involved teachers implementing the lesson plans in their schools while adhering to their usual teaching schedules. The teachers had the flexibility to create their own implementation schedules based on the lesson plans and school timetables. However, there was a weekly three-hour online meeting with the facilitator to discuss progress, challenges, and solutions to problems that arose during implementation. The online meetings also served as a platform for teachers to give and receive feedback from each other and the facilitator.

Teachers were asked to record the STEM lessons or activities using simple video recording such as mobile phone or any other recording device available in their school. Teachers were also asked to take pictures of students' worksheets and their products. Those recordings, pictures and products were reported to the facilitator and used to support the presentation and discussion at the third phase.

### *Phase 3: Evaluation of the STEM Lesson Plans*

In the third phase, a two-day face to face workshop or equal to 17 learning hours was conducted at BBGP DIY. This second workshop provided an opportunity for the participants to present their final lesson plans, to evaluate their implementation, to learn from other teachers' experiences and to reflect on what they have learnt during the professional development course. During the final workshop, teachers were also introduced to S-T-E-M Quartet Model (Tan et al., 2019; Tang Wee, et al., 2021), that is a framework that helps educators plan, develop, and assess their activities. This common framework provided structure to STEM education, it could improve professional conversations between teachers. In this study, the model was adopted as a tool to assist teachers to evaluate the levels of contribution and connection of each STEM disciplines in their lesson plans, whether it was low, moderate or high. However, due to the limited space for this paper, Table 1 only shows the science and mathematics components and their contribution in the STEM lesson plans identified by the participants.

**Table 1**

*Teachers' Ratings of the Science and Mathematics Components in the Lesson Plans and the Level of Their Contribution in Each Topic*

| Code | Project title | Grade | Science | Mathematics |
|---|---|---|---|---|
| A | Upgrading wasted goods | II | Understanding examples of solid material (Moderate) | Understanding 3D shapes (Moderate) |
| B | Water cycle diorama | III | Understanding the concept of changing the state of matter through rainwater cycle experiment (high) | Measuring the size of the material used for water cycle diorama (low) |
| C | Pretty pots from plastic waste | IV | Utilisation of the school waste, such as plastic bottles (moderate) | Measuring distance between pots, measuring distance of the rope used to hang the pots, frame of 3D shapes (low) |
| D | Fireball thrower | IV | Understanding type of force, elastic force and motion (high) | Measuring and comparing distance of the fireball after it is thrown (Moderate) |
| E | Changing shapes | IV | Understanding the change in state of an object when it is subjected to energy (high) | Measuring time (duration) needed for the shape to change (Moderate) |
| F | Elastic force rocket launcher | IV | Understanding type of force, elastic force and motion (high) | Measuring and comparing distance of the rocket after it is launched (Moderate) |
| G | Creative farming in a small garden | IV | Understanding generative /vegetative propagation of plants with hydroponic media or soil (high) | Understanding data collection by collecting data of the growth of plants, and presenting the data in table or diagram (Moderate) |
| H | Mini solar cell project | V | Understanding sustainable energy, application of series and parallel circuits (high) | Understanding nets of 3D shapes to create a solar cell container (Moderate) |
| I | Floating house | VI | Understanding the cause and effect of flood (high) | Measuring length and calculating area of 2D shapes (Moderate) |
| J | Lunar eclipse model | VI | Understanding the process of a lunar eclipse (high) | Measuring distance between the moon and earth, measuring angle (low) |

## The Research Questions and Data Analysis

The lesson plans, recordings, pictures of students' worksheets, students' products and facilitator's notes during the professional development course were the sources of data. The recordings were viewed multiple times and the other sources were examined to find common themes and differences as well. To reduce bias, the authors shared the data, its analytical process and the results with her colleagues. The data were analysed to answer the following research questions.

- What mathematical content is evident in the STEM lesson plans developed by the participating teachers?
- How do the participating teachers deliver mathematical content during the implementation of the STEM lesson?

## Results and Discussion

### *Mathematical Content in the STEM Lesson Plans Developed by the Participants*

Table 1 showed that measurement was a dominant mathematical topic that appeared in most (seven out of ten) of the STEM lessons. Two teachers used geometry content in their projects, *upgrading wasted goods* and *mini solar cell* project, and one teacher strengthened students' ability in data collection and presentation through the *creative farming in small garden* project. Study by Lasa et al. (2020) also showed that mathematical content in STEM activities is basic and utilitarian and being mostly related to the measurement.

Table 1 column 4 also showed the level of contribution of mathematics in the lesson plans according to the teachers. While eight out of ten teachers rated the level of contribution of science content in their projects as high, none of the teachers rated the contribution of mathematical content as the same. Three teachers rated the contribution of mathematical content as low, while the other seven rated as moderate. These data suggested that teachers put more emphasis in the acquisition of science rather than mathematical concepts during the STEM lessons. This may be because science sounds more closely related to STEM than other disciplines. One teacher associated science and STEM as follows "STEM would never be STEM without science". His opinion might be because science comes as the first letter in the STEM acronym and he was not too familiar with technology and engineering. The finding of this study is in line with study by Just and Siller (2022) that mathematics is often seen as minor matter or a means to an end in STEM secondary classrooms.

Although most teachers rated the contribution of mathematics as moderate or low in their STEM lessons, they still recognised the importance of mathematics as a discipline. They believed that STEM lessons provided rich contextual opportunities to enhance students' interest and understanding of mathematics and its value in daily life. According to one teacher, collaborative STEM lessons could promote a positive attitude towards mathematics, as students learn mathematics in a more friendly and less pressured way compared to traditional classroom settings. All teachers reported that their students were highly engaged and interested in their STEM activities, and they were proud of the products they created themselves.

### *Delivering Mathematical Content During the STEM Lessons*

Observation into STEM lesson plans and the lessons' recordings suggested that teachers used two different approaches in implementing the STEM lesson plans. The first approach (used by teachers A, C, G, H, and I) was started by presenting science related problems or situations, continued by guiding discussion to clarify the problems and then giving students a task to create a product to solve the problem. The second approach (used by teachers B, D, E, F and J) was started by presenting science concepts followed by a science experiment or building a model to enhance students' understanding of the concepts. Further examination of the recordings showed that even though mathematics was not the central role in those STEM lessons, there were opportunities to deliver mathematical content which appeared throughout the lessons and most teachers used those opportunities accordingly. The authors named the opportunity as a "mathematical moment". Figure 1 shows the mathematical moments that appeared in the STEM lessons.

Figure 1. Mathematical moments during STEM lessons.

Figure 1 number 1 showed the moment from *upgrading wasted goods* project. At the end of the project, students were asked to present their work. During the presentation, Teacher A deliberately asked her students questions, such as "what is the name of 3D shape that you made?". She saw the opportunity to strengthen students' mathematical knowledge of various 3D shapes and their properties and she used the moment to push mathematical content to the front. Similar moments were found during lessons by Teacher F (Figure 1 Numbers 2 and 3), Teacher C (Number 4), Teacher H (Number 5), Teacher J (Number 6) and Teacher I (Numbers 7 and 8).

Pushing mathematics to the front during the STEM activities was suggested by previous studies (Fitzallen, 2015; Nu'man, 2022) in order to strengthen mathematics during STEM lessons. The participants of the course used this suggestion. However, this effort has its weakness. It could interrupt students' activities, and the students might not give full attention to teachers' explanations. It remains a challenge for teachers to push mathematics content accordingly to gain maximum benefit.

Teacher G did not find mathematical moment similar to other teachers. She created mathematical moments to happen. Figure 1 Number 9 captures *creative farming in small garden*. Teacher G gave tasks to her fourth grade students to observe the growth of the plants. Students measured the height of the plants periodically and noted the results. The data collected were used in the mathematics learning spaces in which she guided the students to learn about presenting data in tables and diagrams. Teacher G strengthened mathematics in STEM lessons by integrating real world problems from STEM activities in the mathematics learning space.

## Conclusion, Limitations and Further Studies

This study found that mathematical content was not the main focus of the STEM lesson plans developed by the participants. However, STEM lessons still provided opportunities for teachers to enhance students' mathematical content knowledge by recognising and prioritising mathematical moments. Teachers could then bring those moments into the next mathematics lesson to demonstrate the connection between mathematics and the real world and to enhance students' interest in mathematics. Further professional development initiatives that can help teachers better integrate mathematical concepts and skills into their STEM lesson plans, as this can lead to more engaging and effective STEM instruction for students

The study's results may only be relevant to primary school teachers and their students, and may not be generalisable to other levels of education. The study did not address factors such as curriculum limitations, appropriate mathematical content, and students' developmental levels. Future research is needed to explore effective approaches for promoting mathematics within the framework of STEM education.

## Acknowledgments

The authors wish to thanks to teachers who participated in the professional development course and their students, Dr. Adi Wijaya, M.A, the head of BBGP DIY for his support during the professional development course and Rhonda Faragher for providing feedback during the writing process of the paper.

## References

Anderson, J., Holmes, K., Williams, G., & Tully, G. (2017). STEM professional learning: Evaluating secondary school teachers' and students' experiences. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning. Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 586–602). Adelaide: MERGA.

Brown, R., Brown J., Reardon, K., & Merrill, C. (2011). Understanding STEM: Current perceptions. *Technology & Engineering Teacher*, 70(6), 5–9.

Bybee, R. (2010). What is STEM education? *Science*, *329*, 996.

Clements, D., Lowrie, T., Larkin, K., Logan, T. (2019). STEM and digital technologies in play based environments: A new approach. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics Education Research: Impacting Practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 68–80). Perth: MERGA.

Farwati, R., Metafisika, K., Sari, I,, Sitinjak, D. S., Solikha, D. F., Solfarina, S. (2021). STEM education implementation in Indonesia: A scoping review. *International Journal of STEM Education for Sustainability*, *1*(1), 11–32

Ferme, E. (2018). A comparison of STEM and non-STEM teachers' confidence in and attitudes towards numeracy. In Hunter, J., Perger, P., & Darragh, L. (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 298–305). Auckland: MERGA.

Fitzallen, N., (2015). STEM education: What does mathematics have to offer?. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 237–244). Sunshine Coast: MERGA.

Gervasoni, A., Lowrie, T., Logan, T., Larkin, K., Bateup, C., & Kinny-Lewis, C. (2017). STEM practices: A reconceptualization of STEM in the early years. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning. Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 620–632). Adelaide: MERGA.

Just, J., & Siller, H-S. (2022). The role of mathematics in STEM secondary classrooms: A systematic literature review. *Educ. Sci*. *12*, 629. https://doi.org/10.3390/ educsci12090629

Kelley, T. R., & Knowles, J. G. (2016). A Conceptual framework for integrated STEM education. *International Journal of STEM Education*, *3*, 11. https://doi.org/10.1186/s40594-016-0046-z

Lasa, A., Abaurrea, J., & Iribas, H. (2020). Mathematical content of STEM activities. *Journal on Mathematics Education*, *11*(3), 333–346. http://doi.org/10.22342/jme.11.3.11327.333-346

Moore, T., Stohlmann, M., Wang, H., Tank, K., Glancy, A., & Roehrig, G. (2014). Implementation and integration of engineering in K–12 STEM education. In S. Purzer, J. Strobel, & M. Cardella (Eds.), *Engineering in pre-college settings: Synthesizing research, policy, and practices* (pp. 35–60). West Lafayette: Purdue University Press.

Mulligan, J., Tytler, R., Prain, V., White, P., Xu, L., & Kirk, M.(2022). The role of mathematics learning in the Interdisciplinary Mathematics and Science (IMS) project. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 410–417). Launceston: MERGA.

Nu'man, M., Retnawati, H., Sugiman, Jailani. (2022). Strengthening m(athematics) in the implementation of STEM education in Indonesia. *AIP Conference Proceedings 2566*, 080001 (2022); https://doi.org/10.1063/5.0116680.

Partnership for 21st Century Skills. (2007). *Framework for 21st-century learning*. http://www.p21.org/documents/P21_Framework_Definitions.pdf

Roberts, A., & Cantu, D. (2012). Applying STEM instructional strategies to design and technology curriculum. *Technology Education in the 21st Century*, 111–118.

Tan, T. L., Teo, T. W., Choy, B. H., Ong, Y. S. (2019). The STEM Quartet. *Innovation and Education* 1 (1), 1–14.

Tang Wee Teo, Aik Ling Tan, Yann Shiou Ong, Ban Heng Choy. (2021). Centricities of STEM curriculum frameworks: Variations of the S-T-E-M Quartet. *STEM Education*, *1*(3): 141–156. doi: 10.3934/steme.2021011

Tsupros, N., Kohler, R., & Hallinen, J. (2009). *STEM education: A project to identify the missing components.* Intermediate Unit 1: Center for STEM Education and Leonard Gelfand Center for Service Learning and Outreach, Carnegie Mellon University, Pennsylvania.

Zainal Arifin, Sukarmin, & Sarwanto. (2021). Research and trend on STEM education in Indonesia: A systematic review based on bibliometric mapping (2000–2020). *Psychology and Education*, *58*(5), 3235–3243.

# Gender Differences in How Students Solve the Most Difficult to Retrieve Single-Digit Addition Problems


James Russo
*Monash University*
james.russo@monash.edu

Sarah Hopkins
*Monash University*
s.hopkins@monash.edu


Despite curriculum expectations, many students, including a disproportionate number of girls, do not 'just know' (retrieve) single-digit addition facts by Year 3. The current study employed structured interviews to explore which strategies Year 3/4 students (n = 166) used when solving more difficult addition combinations. Results revealed that students preference the near-doubles strategy when the difference between the addends was one, the bridging-through-10 strategy when one of the addends was a nine, and the count-on-from-larger strategy when a derived strategy was more effortful. Moreover, whereas boys were more inclined to use derived strategies, girls were almost three times more likely to use the count-on-from-larger strategy.

In Australia, students are expected to learn their single-digit addition combinations using increasingly efficient strategies in Year 1 and Year 2, so they can fluently recall (i.e., 'just know') their addition facts by the end of Year 3 (ACARA, 2015). Moreover, this pathway towards fluent recall put forward in the Australian curriculum mathematics (Version 8.4) is consistent with learning trajectories of addition fact mastery that have been postulated in educational research (Baroody, 2006; Carpenter et al., 2015). For a problem such as 8 + 6, it might be expected that:

- a Year 1 student would count-on from the larger number (8, 9 10, 11, 12, 13, 14);
- a Year 2 student would use an efficient derived strategy, such as: near-doubles (6 + 6 + 2), bridging-through-10 (8 + 2 + 4 or 6 + 4 + 4), compensate—overshoot (6 + 10 - 2), compensate—equalise (6 + 8 = 7 + 7) and;
- a Year 3 student would simply recall that "8 plus 6 is 14", and could use this knowledge to solve more complex problems, such as recognising that 78 + 56 is equivalent to 120 and 14.

However, as many teachers observe, by the end of Year 3, large numbers of students do not 'just know' (i.e., retrieve) addition facts, or use efficient derived strategies, but continue to 'count-on'. Research confirms that over one-third of students continue to rely on accurately executed counting-based strategies into Year 3 and beyond (Gervasoni et al., 2017; Hopkins & Bayliss, 2017). Compared with students using more efficient strategies, students who continue to count-on perform more poorly on standardised mathematics assessments (Hopkins & Bayliss, 2017), and have lower levels of mental computation flexibility (Hopkins et al., 2022).

Although relying on counting-based strategies beyond the stage when it is developmentally appropriate to do so is generally considered problematic, little is known about how student strategy use varies across different types of single-digit addition problems. The purpose of the current paper is to examine how students solve the most difficult to retrieve single-digit addition problems, and to consider whether this strategy profile varies across gender.

## *Gender Differences in Strategy Use*

There are notable differences between boys and girls in the tendency to rely on counting-based strategies compared with retrieval (Bailey et al., 2012). Carr and Jessup (1997) interviewed 58 Year 1 students on three occasions throughout the school year to examine how they solved simple addition and subtraction problems. They found that gender differences in the use of counting-based strategies vis-à-vis retrieval increased across the year. By the end of the year, girls were 1.73 times more likely to use an overt counting-based strategy than boys, and 1.79 times more likely to correctly execute such a strategy. By contrast, boys were 1.48 times more likely to attempt to use a retrieval strategy,

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 443–450). Newcastle: MERGA.

and 1.58 times more likely to correctly execute a retrieval strategy. Similarly, Carr and Davis (2001) interviewed 84 Year 1 students about how they solved single-digit addition and subtraction problems and found that girls were more likely to attempt (1.83 times) and correctly execute (1.91 times) an overt counting-based strategy, whilst boys were more likely to attempt (1.75 times) and correctly execute (1.95 times) a retrieval strategy. Moreover, there is evidence that these gender differences in both preferences for using retrieval, and accuracy retrieving, persist throughout primary school (Bailey et al., 2012), and into secondary school (Hopkins & Bayliss, 2017).

Explanations as to why these gender differences exist and persist have not been explored in depth. After finding that girls were more than twice (2.26 times) as likely to be clustered into a strategy profile group defined by their propensity to accurately count-on from the larger number, Hopkins and Bayliss (2017) concluded that these differences possibly reflect girls setting "a higher threshold for determining confidence with retrieval" (p. 30). Bailey et al.'s (2012) study found these gender differences were not related to either differences in central executive function or intelligence, and instead postulated that boys' relative preference for "risk taking" in settings involving social evaluation and greater interest in competition, results in them developing a preference for using retrieval. This preference leads to more practice using retrieval-based strategies, which in turn results in superior retrieval performance for boys.

### *Gaps in the Research*

There are two limitations to the current suite of studies on single-digit addition that present gaps in our understanding of how students solve single-digit addition problems. First, generally single-digit addition (and subtraction) problems have been considered as a single, coherent category (e.g., Carr & Jeeup, 1997; Hopkins & Bayliss, 2017), and how students perform on a subset of single-digit addition problems that possess particular characteristics, such as those problems identified as difficult to retrieve, has tended to not be an explicit focus of prior research. This means that particular phenomenon that have been observed, such as girls being more inclined to use counting-based strategies, are perhaps not as well understood as they might be. For example, it may be that the magnitude of the gender differences from the Carr and Davis (2001) and Carr and Jessup (1997) studies are either masked or amplified by the specific number facts chosen for inclusion in these studies compared with, for example, gender differences on those single-digit addition facts that students find most difficult to retrieve. Second, prior studies have often not delineated the use of efficient derived strategies from covert counting-based strategies (e.g., Carr and Davis, 2001). Moreover, even when they do make this delineation, they generally do not distinguish between the various derived strategies, such as near-doubles or bridging-through-10, but rather collapse them all into a single category (e.g., Geary et al., 1996). This is problematic if one considers that exploring different strategies for solving single-digit addition problems, and the various derived strategies in particular, is a large focus of contemporary instruction in number in the first three years of school (e.g., ACARA, 2015). The current study seeks to address these two limitations by focusing on a subset of single-digit addition problems that students find most difficult to retrieve (see Russo & Hopkins, 2022), as well as distinguishing between the various derived strategies and reporting on these separately.

### *The Current Study*

Given that we know large numbers of students are not able to recall their single-digit addition combinations (Hopkins & Bayliss, 2017), we wondered which of these combinations students find most difficult to retrieve. Surprisingly from our perspective, the research literature relied on data that was over 80 years old to inform us on this question (see Wheeler, 1939). Moreover, this data was collected following an intervention in which the use of number sense strategies was actively discouraged, less they interfere with rote memorisation. Consequently, as part of a larger research project, we decided to investigate the issue ourselves (see Russo & Hopkins, 2022). We were interested in finding out which addition combinations students found most difficult to recall. We invited students in Years 3 and 4 to solve 36 single-digit addition problems (see Table 1) under two conditions (a strategy choice condition and a quick response condition) and used this data to create a composite measure designed to capture student difficulty retrieving addition facts. Using this composite measure, the 10 most difficult single-digit addition combinations for students to accurately recall are (in descending order): 6 + 9; 7 + 8; 7 + 9; 6 + 8; 5 + 9; 5 + 8; 6 + 7; 5 + 7; 8 + 9; 4 + 9. The current paper delves deeper into our data to explore which strategies students tended to rely on when solving these more difficult combinations under the strategy choice condition. An additional focus is to explore whether there were any notable differences between boys and girls, given research suggesting that boys are more likely to retrieve addition facts than girls, whilst girls are more likely to use counting-based strategies (Bailey et al., 2012; Carr & Davis, 2001). Our research questions include:

- Which strategies do students choose to use to solve difficult to retrieve addition combinations?
- Are there differences between boys and girls in terms of strategy choice when solving difficult to retrieve addition combinations?

Our study sits within a social cognitive perspective on how children develop computational strategies. This perspective contends that individual level factors (e.g., working memory), interactions with others (e.g., teachers) and contextual factors (e.g., problem type) coalesce to influence strategy choice. The key focus of the current paper is on how the individual level factor of gender interacts with the contextual factor of problem difficulty to shape student choice of strategy for solving addition problems.

## Method

Victorian primary school students in Years 3 and 4 (n = 166; girls = 84; boys = 82) from three different schools in metropolitan Melbourne solved 36 single-digit addition problems during an individual structured interview. The problems included all single-digit addition problems where the smaller addend is presented first, plus doubles, but excluding plus zero and plus one (see Russo & Hopkins, 2022 for more information about the study methodology). During the interview, single-digit addition problems were presented on a screen using the Fact Cat program. Students were instructed to call out the answer to the problem as soon as they worked it out. Student responses were recorded by the researcher, who also asked the student “How did you do it?” and recorded their strategy. For the purposes of the current paper, student strategy choices when solving the 10 most difficult to retrieve single-digit addition combinations were subsequently coded into SPSS v. 26 for analysis (see Table 1), and delineated from the 26 easier to retrieve single-digit addition problems.

## Results

### *Which Strategies do Students Choose to use to Solve Difficult to Retrieve Addition Combinations?*

Table 1 summarises the strategies that students used to solve the 10 most difficult to retrieve single digit addition problems that are the focus of the current paper. Overall, students used a variety of strategies to solve these problems and there are some notable differences in the average frequency of each strategy across problems.

The most frequently used strategies overall were bridging-through-ten (26%), near-doubles (25%), count-on-from-larger (22%), retrieval (knew-it) (10%), compensate-overshoot (6%) and compensate-equalise (2%). Collectively, these strategies accounted for 91% of all strategy choices when answering these most difficult problems. The remaining 9% of strategy choice are explained by students either using a relatively inefficient count-on-from-the-first-number strategy, a count-all strategy, a highly idiosyncratic strategy or those who responded 'don't know' when asked how they worked it out.

**Table 1**

*Strategy Choice for Solving Single-digit Addition Problems*

| Problem | Correct | Knew-it | Count-On | Near-doubles | Bridging-through-10 | Compensate—overshoot | Compensate—equalise |
|---|---|---|---|---|---|---|---|
| 4 + 9 | 92% | 14% | 23% | 0% | 40% | 10% | 1% |
| 5 + 7 | 93% | 13% | 23% | 33% | 19% | 0% | 5% |
| 5 + 8 | 93% | 9% | 32% | 18% | 28% | 4% | 1% |
| 5 + 9 | 87% | 8% | 20% | 12% | 35% | 15% | 1% |
| 6 + 7 | 91% | 12% | 17% | 48% | 14% | 1% | 0% |
| 6 + 8 | 91% | 7% | 29% | 23% | 27% | 1% | 3% |
| 6 + 9 | 89% | 13% | 23% | 11% | 32% | 8% | 1% |
| 7 + 8 | 90% | 7% | 18% | 50% | 15% | 2% | 0% |
| 7 + 9 | 90% | 7% | 22% | 11% | 34% | 11% | 4% |
| 8 + 9 | 92% | 8% | 16% | 43% | 16% | 8% | 0% |
| Total | 91% | 10% | 22% | 25% | 26% | 6% | 2% |

In terms of different strategy profiles across problems, near-doubles was the most frequently employed strategy for problems where the difference between the addends was one (6 + 7; 7 + 8; 8 + 9) and was typically used by just under half of students to solve these problems. However, there were two other problems for which near-doubles was utilised by at least one-fifth of students: 5 + 7 (i.e., 5 + 5 + 2 or 7 + 7 - 2) and 6 + 8 (i.e., 6 + 6 + 2 or 8 + 8 - 2). By contrast, bridging-through-10 was the most frequently employed strategy when one of the addends was 9 (4 + 9; 5 + 9; 6 + 9; 7 + 9), with the exception of the problem 8 + 9, where the two addends had a difference of one, and near-doubles was used most frequently (i.e., 8 + 8 + 1 or 9 + 9 - 1). Although compensate-overshoot was not as frequently used as bridging-through-10, it also tended to only be employed for problems where one of the addends was 9.

Given that counting-on from the larger number is a strategy always available to students, it is noteworthy that there were considerable differences in how often it was used across different problems. Consider the three problems that each summed to 13: 4 + 9; 5 + 8; 6 + 7. Whereas approximately one-third of students counted-on to solve 5 + 8, less than one quarter counted-on to solve 4 + 9 and only around one-sixth counted-on to solve 6 + 7. Count-on does seem more likely to be used when a derived strategy is more effortful, specifically for those problems where the difference between the addends is at least two and where 9 is not one of the addends.

### *Are There Differences Between Boys and Girls in Terms of Strategy Choice When Solving Difficult to Retrieve Addition Combinations?*

Differences between strategy use of boys and girls on the 10 most difficult to retrieve problems are presented in Table 2.

**Table 2**

*Strategy Choice for Solving the 10 Most Difficult to Retrieve Single Addition Problems by Gender*

| | Correct | Knew-it | Count-On | Near-doubles | Bridging-through-10 | Compensate—equalise | Compensate—overshoot |
|---|---|---|---|---|---|---|---|
| Boys | 92% | 11% | 12% | 31% | 28% | 2% | 6% |
| Girls | 90% | 9% | 33% | 19% | 24% | 1% | 6% |

As is apparent from viewing the table, boys were more likely to use the near-doubles strategy (31% vs 19%), or to a lesser extent the bridging-through-10 strategy (28% vs 24%), on these most difficult problems whereas girls were more likely to use the count-on-from-larger strategy (33% vs 12%). Overall, boys used an efficient derived strategy on two-thirds of trials, whereas girls used an efficient derived strategy on half of trials.

Differences between boys and girls across problem type in their usage of the count-on-from-larger, near-doubles strategy and bridging-through-10 are displayed in Figures 1, 2 and 3 respectively. Figure 1 displays some stark differences in the propensity to count-on from the larger number between girls and boys. For example, whereas half of girls counted-on to solve 5 + 8, only one-seventh of boys used this strategy. Similarly, girls (37%) were four times as likely to employ the count-on-from-larger strategy to solve 5 + 7 compared with boys (9%). With regards to the near-doubles strategy (Figure 2), boys were consistently more likely to use this strategy than girls, however the differences across individual problems were somewhat less dramatic than when comparing the count-on-from-larger strategy. However, it is clear that boys were notably more likely to use near-doubles strategy regardless of the difference between the addends compared with girls. Specifically, with the exception of 4 + 9, for which the near-doubles strategy was not employed by any student, at least 15% of boys used the near-doubles strategy on each of the remaining nine problems, whereas the equivalent 'floor usage' of the near-doubles strategy for girls was only 5%. It is particularly striking that almost one-fifth of boys used the near-doubles strategy to solve the most difficult to retrieve problem in the data set, 6 + 9 (6 + 6 + 3 or 9 + 9 - 3), despite the bridging-through-10 or compensate-overshoot strategy appearing more efficient and less effortful. Regarding the bridging-through-10 strategy (Figure 3), gender differences on individual problems trend in the same general direction, with boys being more likely than girls to employ this strategy for all problems except for 6 + 8.

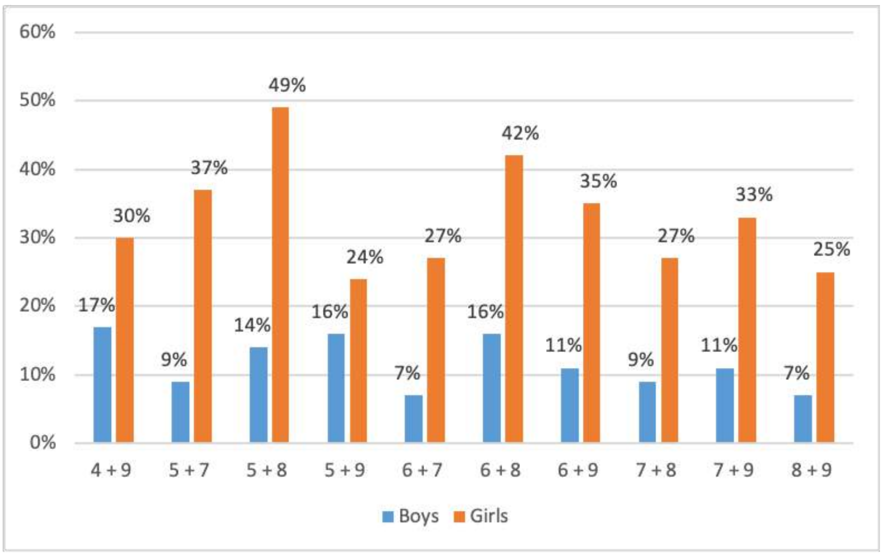



Figure 1. Count-on-from-larger strategy by problem type by gender (%).

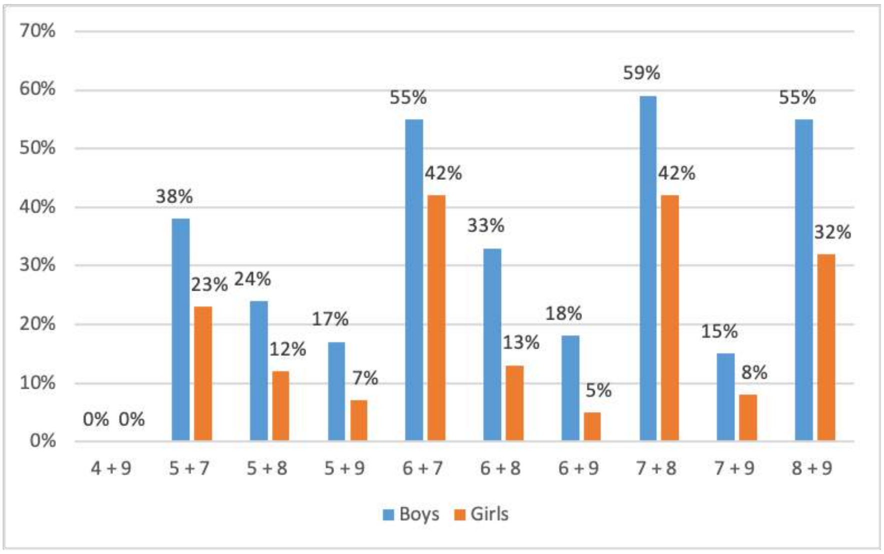



Figure 2. Near-doubles strategy by problem type by gender (%).

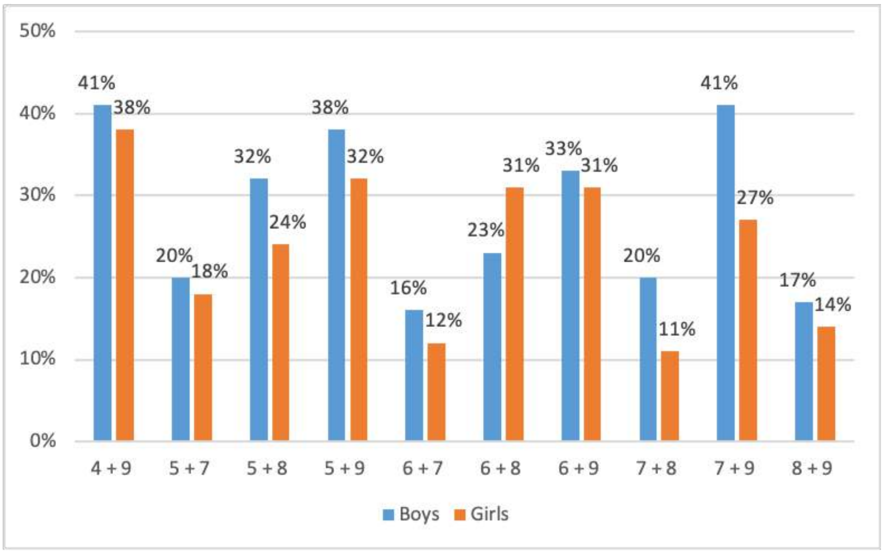



Figure 3. Bridging-through-10 strategy by problem type by gender (%).

### *Additional Analysis: Comparing the Propensity to Count-on Across Different Problem Types by Gender*

Given these substantial gender differences in the propensity to count-on from the larger number for comparatively difficult single-digit addition problems, it is worth comparing this finding with gender differences on the remaining comparatively easy problems. It is clear from Table 3 that girls are more likely to count-on when presented with a problem from the more difficult set than when presented with a problem from the easier set, whereas problem difficulty makes relatively little difference to whether boys use the count-on strategy or not. Consequently, the ratio of girls-to-boys who use the count-on strategy is far larger for the difficult problem set (2.81 times) compared with the easier problem set (1.58 times).

**Table 3**

*Mean Propensity to Count-On-From-Larger Number by Problem Type by Gender*

| | 10 most difficult problems | 26 easier problems | Total |
|---|---|---|---|
| Boys (total) | 12% | 11% | 11% |
| Girls (total) | 33% | 18% | 22% |
| Ratio (Girls/ Boys) | 2.81 times | 1.58 times | 1.93 times |

## Discussion and Conclusions

Consistent with previous research (e.g., Hopkins & Bayliss, 2017; Gervasoni et al., 2017), our findings demonstrated that students use a variety of strategies to solve single-digit addition problems. Our study, however, builds on previous work by revealing how the propensity to use particular derived strategies differs substantially across problems. For example, although the bridging-through-10 strategy was potentially available for all 10 problems presented in Table 1 (given all problems summed to at least 12), the strategy was reported to be used by as few as 14% of students for the problem 6 + 7, and by as many as 40% of students for the problem 4 + 9. This suggests that students are flexible in choosing not only between counting-based strategies, derived strategies and retrieval, as has been demonstrated previously (Hopkins & Bayliss, 2017), but also in choosing amongst particular derived strategies.

Our findings are also consistent with previous research in that they revealed that girls are more inclined to use counting-based strategies than boys (Bailey et al., 2012; Carr & Davis, 2001; Carr & Jessup, 1997). However, rather than boys being more likely to use retrieval (e.g., reporting to 'just know' the answer) as has been reported previously (e.g., Carr & Davis, 2001), boys in our study were more likely to report employing derived strategies than girls, in particular the near-doubles strategy. Indeed, some boys consistently chose the near-doubles strategy when a more efficient alternative was available. Differences between our study and previous research may reflect our emphasis in seeking detailed information about the specific strategy employed from participants, and coding these strategies accordingly.

It is also notable that the magnitude of gender differences in the propensity to use counting-based strategies were larger in our study than reported previously. For example, in our study, girls were 2.81 times more likely to attempt to use the count-on-from-larger strategy than boys, notably highly than reported gender differences in the propensity to employ overt counting-based strategies in the Carr and Jessup (1997; 1.73 times) and Carr and Davis (2001; 1.83 times) studies. Further analysis revealed that these larger gender differences were likely a result of our study focussing on the 10 most difficult to retrieve problems, given that the remaining 26 easier addition problems for which data was also gathered showed substantially smaller gender differences in girls' relative use of the count-on-from-larger strategy (1.58 times more likely). The conclusion is that girls are somewhat more likely than boys to use the count-on-from-larger strategy when the problem is comparatively easy (e.g., 3 + 5), but far more likely to use the count-on-from-larger strategy when the problem is more difficult (e.g., 5 + 8).

Bailey et al.'s (2012) suggestion that boys' relative comfort (on average) with greater "risk taking" in a setting where their performance is being evaluated leads to them being less likely to favour counting-based strategies is consistent with our findings. Specifically, as problems become more difficult, more risk adverse students will be more inclined to utilise counting-based strategies to prioritise "not being incorrect". This is because, once mastered, the count-on-from-larger strategy becomes what is effectively a universal back-up strategy for solving addition problems that guarantees one arriving at the correct answer. By contrast, students who are less risk adverse will still be comparatively comfortable attempting to retrieve the answer or executing a retrieval-based strategy as problems become more difficult, even if this means a slightly higher likelihood of them making an error, in order to prioritise "being correct with minimal cognitive effort" and/ or "being correct quickly".

Given the magnitude of the gender differences reported in our study, we would emphasise that further research is necessary to examine whether these differences are replicable across larger, more diverse samples of students. Moreover, we are reluctant to conclude that gender differences reported here are necessarily highly problematic. For example, additional data collected from the same participants suggested that, although students who relied more on accurately executing the count-on-from-larger strategy had lower levels of mental computation flexibility than students who relied more on retrieval, girls did not have statistically significantly lower levels of mental computation flexibility on average than boys. It may be that there are different pathways to mental computation flexibility that are leveraged by boys and girls, as suggested by Bailey et al. (2012). However, boys' tendency to select and appropriately execute more efficient strategies may still have implications for gender differences in mathematical performance beyond mental computational flexibility, particularly if such differences persist throughout schooling, and warrants further research consideration. In any case, we do think that the magnitude of gender differences reported here should at least give the mathematics education research community pause when considering moving away from reporting and exploring differences between girls and boys, even in an environment where the notion of gender as a binary construct has become problematised (Hall & Norén, 2021).

## References


Australian Curriculum, Assessment and Reporting Authority (2015). *The Australian curriculum: Mathematics*. ACARA. http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10

Bailey, D. H., Littlefield, A., & Geary, D. C. (2012). The codevelopment of skill at and preference for use of retrieval-based processes for solving addition problems: Individual and sex differences from first to sixth grades. *Journal of Experimental Child Psychology, 113*(1), 78–92.

Baroody, A. J. (2006). Why children have difficulties mastering the basic number combinations and how to help them. *Teaching Children Mathematics*, *13*(1), 22–31. https://doi.org/10.5951/tcm.13.1.0022

Carr, M., & Davis, H. (2001). Gender differences in arithmetic strategy use: A function of skill and preference. *Contemporary Educational Psychology*, *26*(3), 330–347.

Carr, M., & Jessup, D. L. (1997). Gender differences in first-grade mathematics strategy use: Social and metacognitive influences. *Journal of Educational Psychology*, *89*(2), 318.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's mathematics: Cognitively guided instruction* (2nd ed.). Heinemann.

Geary, D. C., Bow-Thomas, C. C., Liu, F., & Siegler, R. S. (1996). Development of arithmetical competencies in Chinese and American children: Influence of age, language, and schooling. *Child Development*, *67*(5), 2022–2044.

Gervasoni, A., Guimelli, K., & McHugh, B. (2017). The development of addition and subtraction strategies for children in kindergarten to grade 6: Insights and implications. In A. Downton, S. Livy, & J. Hall (Eds.), 40 years on: We are still learning! *Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 269–276). Melbourne: MERGA.

Hall, J., & Norén, E. (2021). Innovations in "gender issues" research in mathematics education. Mathematics *Education Research Journal, 33*(4), 787–791. https://doi.org/10.1007/s13394-021-00404-8

Hopkins, S., & Bayliss, D. (2017). The prevalence and disadvantage of min-counting in seventh grade: Problems with confidence as well as accuracy? *Mathematical Thinking and Learning*, *19*(1), 19–32. https://doi.org/10.1080/10986065.2017.1258613

Hopkins, S., Russo, J., & Siegler, R. (2022). Is counting hindering learning? An investigation into children's proficiency with simple addition and their flexibility with mental computation strategies. Mathematical *Thinking and Learning*, *24*(1), 52–69. https://doi.org/10.1080/10986065.2020.1842968

Russo, J., & Hopkins, S. (2022). Not so Simple Addition: Comparing Student Performance and Teacher Perceptions of Retrieval. *International Journal of Science and Mathematics Education*, 1–23. https://doi.org/10.1007/s10763-022-10346-7.

Wheeler, L. R. (1939). A comparative study of the difficulty of the 100 addition combinations. *The Pedagogical Seminary and Journal of Genetic Psychology*, *54*(2), 295–312.

# A Framework for Designing Green Mathematics Tasks

Soma Salim
*The University of Queensland*
s.salim@uq.edu.au

Climate change is a complex issue that requires collective actions from various disciplines, including educating future generations about sustainability. Some countries such as Indonesia and Australia have included sustainability into their school curricula, which require teachers to design and implement tasks related to a sustainability issue. However, teachers may not have the capability and understanding to do so. The challenges are difficult for mathematics teachers because designing a mathematics task pertaining to sustainability is rarely explored. Therefore, this paper draws from mathematics education literature to provide and discuss a new framework for designing sustainability-related, mathematics tasks (green mathematic tasks).

Climate change as a global issue accelerated by human activities requires collective actions to address the issue from various disciplines, including education (Abtahi et al., 2017). Some students have not been well-informed regarding climate change, suggesting the importance to educate the future generations (Oliver & Adkins, 2020). Although most students know what climate change is, this does not guarantee that they understand what about the changing climate is problematic, or that they would want to take action to tackle the environmental challenge (Nugroho, 2020). Such actions are more likely to happen if people can comprehend and are aware of the environmental issue as well as the projected effect it has on the earth and living creatures, including humans (Endsley, 2017). Therefore, it is essential to make the future generations aware of the global issue and encourage them to project possible actions to tackle the problem (Barwell & Hauge, 2021).

In terms of educational policy, some countries have included environmental issues and sustainability in their school curricula. For example, Australian Curriculum has cross-curriculum priorities, one of which is sustainability (Dyment & Hill, 2015). The new curriculum of Indonesia called "*Merdeka Belajar*" (*Emancipated Learning*) has co-curricular, thematic projects; one of the themes is related to environmental issues, global change, and sustainability (Anggraena et al., 2022). Both curricula require teachers to integrate sustainability into their teaching plans or to create projects that involve issues around sustainability. However, teachers may currently not have capability and understanding of how to implement this part of curriculum at schools (Dyment & Hill, 2015).

Mathematics education plays a pivotal role to make students aware of real-world issues and contextual mathematics tasks may assist students to understand sustainability-related issues (Barwell, 2018). This is because mathematics can serve both as an interpreting and formatting tool to understand such sustainability challenges as climate change (Steffensen et al., 2021). Unfortunately, designing contextual, authentic mathematics tasks pertaining to sustainability is challenging for teachers (Paredes et al., 2020). Considering the ethical and policy needs to educate the future generations about sustainability as well as the challenge that mathematics teachers may encounter in designing a sustainability task, this paper provides a framework of how to design sustainability-related, mathematics tasks (green mathematics tasks).

## Mathematics Tasks Related to Sustainability Issues

A task plays a pivotal role in mathematics classrooms because learning takes place when students are involved in a mathematical activity as required in a task (Ineson & Povey, 2020). There are many different types of mathematics tasks, which have been categorised as abstract, pseudo-contextual, or authentic (Palm, 2007), routine or non-routine (Schoenfeld, 2016), reasoning and proofing (Stylianides, 2009), and problem solving (Schoenfeld, 2016). Therefore, which types of

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 451–458). Newcastle: MERGA.

mathematics tasks should teachers use when integrating sustainability issues in order to make students aware of such issues?

The most common mathematics tasks used by teachers in classrooms are abstract because mathematics itself is considered an abstract, deductive system that can be used to explain and solve real-world situations (Williamson, 2018). Providing students with more abstract mathematics tasks may give students less opportunity to understand the use of mathematics in real world (Hernandez-Martinez & Vos, 2018). As such, connecting real-world situation and mathematical ideas can assist students to understand the relevance of mathematics out of school; for example, Realistic Mathematics Education was introduced to promote the importance of teaching mathematics in meaningful and useful ways (Freudenthal, 1968).

However, some mathematics tasks designed to be contextual seem unrealistic for students (Palm, 2007). To illustrate, the following water tank task (Figure 1) is considered contextual in a textbook, but students may not interpret themselves as experiencing such a problem as described in the task. They may not see the purpose of figuring out the height that the water will reach. Information is missing or assumed, such as where the water comes from or whether the water tank was empty at the start or not. Despite the lack of purpose and missing information, this task may have sufficient complexity to develop students' reasoning, problem solving, and mathematical thinking (Schoenfeld, 2016).

> The water tank shape cuboid has a width of 2.3 m and a length of 4.6 m. How high will the water reach in the tank if the water flows into the tank at 19 litres of water per second for 52 minutes?
>
> *Source:* Adapted from HackMath (https://www.hackmath.net/en/math-problem/981)

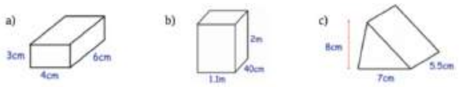
Figure 1. Water tank task.

Sustainability has a realistic purpose, hence the information and situation described in green mathematics tasks should be realistic for students as well. This is important so that students understand not only how mathematical ideas are relevant in real world, but also how to solve a problem in a real situation (Schmidt et al., 2022). Bushnell (2018) designed what he called the *melting ice sheet task* (see Table 1) as an attempt to integrate mathematics and environmental sustainability. He wanted his students to be able to understand the *utility* of mathematical concepts, but the students missed the *purpose* of working on the task (Ainley et al., 2006). Ainley et al. (2006) define *purpose* from students' perspectives of why and how they solve a problem meaningfully and *utility* as to how through the task students perceive the mathematical ideas as useful tools to construct meaning.

When implementing the task, most of Bushnell's students struggled with the concept of prism volume because he deliberately had not taught the topic yet. This made it difficult for his students to find out the volume of water needed to cause a 50 m rise of sea levels. In such situation, some teachers tend to provide their students with explicit, procedural instructions to pave the way to the intended answers (Bushnell, 2018). However, this may lead the students to develop instrumental understanding, instead of relational understanding (Skemp, 1976). Instrumental understanding (knowing how) is easier to demonstrate and assess, but it is also easier to forget while relational understanding (knowing how, when, and why) is more difficult to develop, but the reasoning required strengthens knowledge constructions (Skemp, 1976).

**Table 1**

*Melting Ice Sheet Task Adapted from Bushnell (2018)*

| Abstract/common tasks | Environment-related tasks |
|---|---|
| Convert the following lengths into kilometres: 6000 m, 450 m, 125 m, and 90000 m. | Suppose that the ice sheets have melted enough to cause a 50 m rise in sea levels. Convert 50m into kilometres. |
| Calculate the volume of each prism:<br><br> | Given that the global surface area is 361,132,000 $km^2$, and using your answer to the previous question, calculate the volume of water needed to cause a 50 m rise in sea levels. |
| Calculate the distance that a train travels in 20 minutes at 90 mph. | What is the formula of calculating density? Calculate the mass of water needed to cause a 50 m rise in sea levels (density of water is $1 \times 10^{12}$ $kg/km^3$). |
| Calculate the density of a rod of aluminium that has a mass of 575.4 g and a volume of 210 $cm^3$. Calculate the volume of a 770 g block made of brass which has a density of 8.67 $g/cm^3$. | Given the mass of water is the same as the mass of ice, calculate the volume of ice needed to cause a 50 m rise in sea levels (density of ice is $9 \times 10^{11}$ $kg/km^3$). |
| A cuboid container is used to store boxes. Each box is a cube with side length 1 m. How many boxes can be stored in the container. (The container size is 12 m length, 5 m width, and 2 m height) | The volume of an Olympic sized swimming pool 2.5 x $10^{-6}$ $km^3$. How many pools would be needed to contain all of the melted ice that causes a 50 m rise in sea levels? |

Analysis of the issues raised by these two tasks lead to the following argument. If a task consists of mathematical ideas and problem solving (e.g., water tank task in Figure 1), it may give opportunity for students to develop reasoning and problem solving, but not awareness of sustainability issues (Barwell, 2018). On the other hand, if a task has mathematical ideas and sustainability issues (e.g., melting ice sheet task in Table 1), it may allow students to notice sustainability issues, but not seeing themselves becoming critical and taking actions to solve the sustainability issues (Barwell, 2018; Bushnell, 2018). If a task only includes sustainability issues and problem solving, it cannot show the utility of mathematical ideas (Ainley et al., 2006). Meanwhile, mathematics plays a critical role to format the understanding of sustainability issues (Steffensen et al., 2021). Therefore, all three interdependent elements— sustainability issues, mathematical ideas, and problem solving—are needed in the design of green mathematics tasks.

## Designing Green Mathematics Tasks: A Framework

One way to indirectly support students to keep working on a challenging task is providing them with a *purpose* in a task (Ainley et al., 2006), so that they are willing to make mistakes and struggle, which is necessary for them to develop a strategy to solve such tasks (Boaler, 2022). Working on problematic tasks can encourage active engagement in students' construction of mathematical understanding (Schoenfeld, 2016). The following is an example of a challenging, problematic task (designed by the author) called *sea level task* (see Table 2), in which students may find purpose and utility.

**Table 2**

*Sea Level Task*

| Investigate these two different scenarios by considering the given facts | |
|---|---|
| *Scenario 1*: If carbon emissions keep increasing, scientists predict that by 2100 the sea level will raise up to 2 m above that in 2000.<br>*Fact 1:* The global surface area of the ocean is 361,900,000 km$^2$.<br>*Fact 2*: Antarctic Ice is melting at an average rate of about 150 billion tons per year, and Greenland Ice is losing about 280 billion tons per year. | *Scenario 2*: On the other hand, if carbon emissions are constantly reduced, the sea level will only raise up to 0.3 m by 2100 above that in 2000.<br>*Fact 3:* If 100% of Ice Sheet in Antarctica and Greenland melt, it can raise sea level by 57.9 m and 7.42 m respectively.<br>*Fact 4:* Earth's temperature has risen by 0.08° C per decade since 1880, but the rate since 1981 is more than twice that: 0.18° C per decade. |

*Note.* The facts provided in this task are gathered from official sources such as https://www.climate.gov, https://www.ngdc.noaa.gov, https://climate.nasa.gov, and https://www.antarcticglaciers.org.

For example, from Fact 1 students want to know how much water is needed to cause sea level rises of 2 m and 0.3 m, and eventually utilise mathematical ideas (area and volume) meaningfully to achieve the purpose.

In addition to the elements of sustainability issues, mathematical ideas and problem solving, the sea level task has three salient characteristics (*authenticity, complexity,* and *projection* in the frame of awareness), derived from the main elements (see Figure 2). The task includes real information (see the note in Table 2) and existing environmental issues, which make it authentic from this aspect (Ainley et al., 2006; Palm, 2007). The sea level task is complex and perplexing, which can encourage students to understand the problem and become a confident mathematical problem solver (Schoenfeld, 2016). Finally, the instruction of the sea level task (investigating the scenarios) might also encourage students to project what will happen in the future and what alternatives are available to address the issue, which eventually can awaken students' awareness of an environmental issue (Endsley, 2017). Next, I will explore each characteristic and how they may lead students to take sustainable actions.

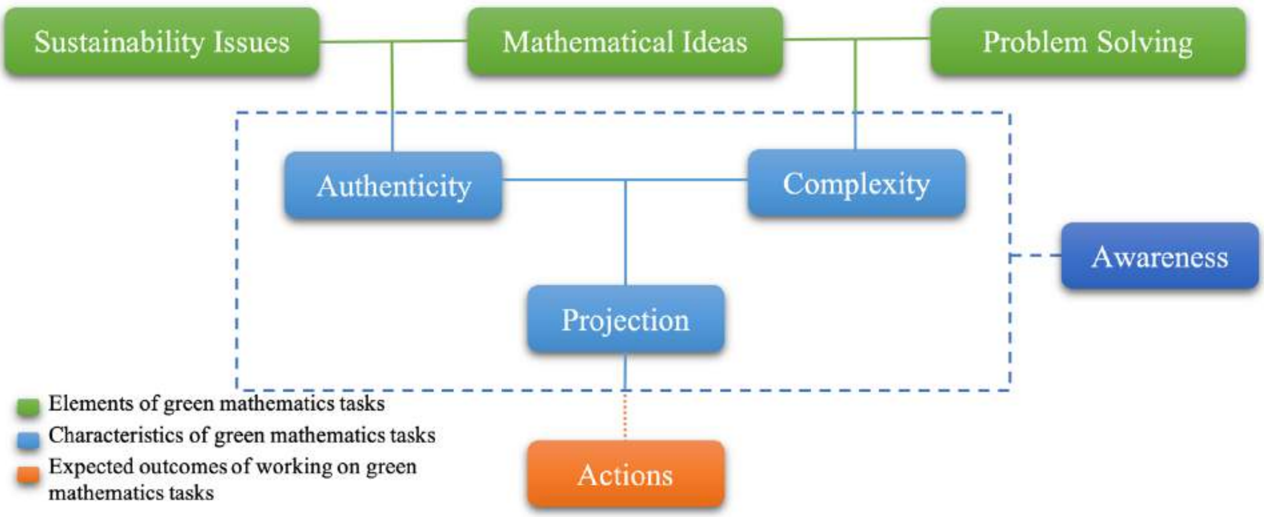



Figure 2. The framework of green mathematics tasks.

### *Authenticity in Green Mathematics Tasks*

The first characteristic of the framework for designing green mathematics tasks is authenticity. The authenticity of contextual tasks refers to real situations that have happened or are predicted to eventuate; authentic tasks are important not only to show the use of mathematics in real world, but also to engage students in solving such tasks (Palm, 2007). To create authentic tasks, the real-world issues are identified first then figuring out mathematical ideas that can be used to work on the tasks (Palm, 2007). However, some contextual mathematics tasks are designed by first identifying the mathematical concepts in the curriculum, then finding real-world issues or contexts where the concepts can be used (Paredes et al., 2020). These different methods of designing contextual tasks result in a 'planning paradox' because the former method may cause unfocused activities whereas the latter can make the tasks inauthentic (Ainley et al., 2006). The designer's intention is therefore to encourage students to see *purpose* and *utility* in green mathematics tasks (e.g., sea level task in Table 1) that can make the task more authentic and avoid such paradox (Ainley et al., 2006).

Palm (2007) proposed eight aspects of authentic tasks: event, question, purpose, information/data, presentation, solution strategies, circumstances, and solution requirements. However, it is challenging for teachers to design a task that meets all these aspects (Paredes et al., 2020). Therefore, de-authenticating some aspects is necessary for educational purposes (Vos, 2018). A question remains; which aspects are essential to be authentic and which ones can be de-authenticated? Vos (2018) suggests the authenticity of the questions and activities are more important than the context of the task. This is because these two aspects can help students to find the purpose and utility of mathematical ideas. However, he refers to mathematical modelling tasks where the context is related to professional work such as planning a city bus schedule, a problem context which would be very challenging for students. Thus, de-authenticating the context of such tasks for educational purposes is likely necessary.

On the other hand, sustainability-related issues as the context in green mathematics tasks should be as authentic as possible so that students can interpret the sustainability issue better (Tran & Dougherty, 2014; Wernet, 2017). Hence, beside the questions and activities, the authenticity of the event and information or data in green mathematics tasks is equally essential. For students to believe the authenticity of a task, these authentic aspects can be validated either by professionals or through official resources, including websites (Vos, 2018) and having the problem described as relevant to students' lives and experiences (Walkington & Hayata, 2017; Wernet, 2017). To conclude, the five salient aspects of Palm's (2007) work for the authenticity of green mathematics tasks in this framework are the event or context, purpose, data/information, questions, and activities.

### *Complexity in Green Mathematics Tasks*

The second characteristic of the framework for designing green mathematics tasks is complexity. The complexity of mathematics tasks refers to the extent to which the problems are unusual (non-routine) for students to solve (Schoenfeld, 2016). One method of exposing mathematical ideas to students is through explicit instruction followed by routine exercises or tasks; another method is through dialogic instruction with problematic, unfamiliar tasks (Clark et al., 2012; Munter et al., 2015). Despite the debates on explicit or dialogic instruction, the role of complexity in mathematics tasks is crucial because it can determine how students perceive mathematical ideas, whether they see the tasks as related to memorising or understanding (Hewitt, 1999). If students memorise the procedures of working on a mathematics task, they tend to become confused when attempting to solve another modified task even if both tasks require the same mathematical ideas (Lubienski, 2000; Salim, 2019). In addition, solving a mathematics problem does not only rely on the ability to memorise mathematical ideas (mathematical knowledge), but also understanding how, when, and whether to use the ideas suggesting mathematics tasks need to be problematic or complex (Schoenfeld, 2016).

Some mathematics tasks explicitly aim to show the use of mathematical ideas in a real-life situation (e.g., melting ice sheet task in Table 1) while other tasks may be designed to develop students' reasoning and problem-solving skills without considering whether they are in a contrived, problematic situation (e.g., water tank task in Figure 1) (Wernet, 2017). While the authenticity of a task helps students to find purpose, working on a complex task challenges students' problem solving as to how mathematical ideas are utilised in a particular context (Lubienski, 2000). As such, the complexity of green mathematics tasks is designed to enhance students' problem solving.

Lubienski (2000) identified difficulties that students may experience when solving a complex, contextual task. First, students can be unfamiliar with vocabulary in the contextualised task; for example, in green mathematics tasks 'carbon footprint' or 'carbon emissions' may be unfamiliar. Providing students with a glossary can assist them to understand the task. Another struggle that students may experience is dealing with uncertainty about what to do, particularly if students are accustomed to learning rules to solve a mathematics problem. To address this issue, sense making as part of problem solving plays a pivotal role to guide students exploring different approaches to comprehend a complex problem (Lubienski, 2000).

When solving a complex task, students are expected to struggle in order to solve the problem using mathematical ideas, and with dialogic, indirect guidance, students can succeed both in problem solving and learning (Kapur, 2016; Simon & Tzur, 2012). In addition, reflecting on their activities of solving a complex task is essential because students can examine their own thinking and connect the task to what they have learned or experienced (Carpenter & Lehrer, 1999). Reflecting on how mathematical ideas are utilised can help students to develop understanding and also realise that similar mathematical ideas can provide different interpretations of data and information pertaining to sustainability (Barwell & Hauge, 2021; Brendefur & Frykholm, 2000). All in all, the complexity of green mathematics tasks in this framework requires students to struggle, develop sense making, and reflect on their activities and mathematical ideas used.

### *Projection of Sustainability Issues*

The third characteristic of the framework for designing green mathematics tasks is projection. Projection refers to the extent to which students relate to and envisage the situation raised by a task. Awareness of a situation proceeds in three levels: Level 1 'perception of elements in a situation', Level 2 'comprehension of a situation', and Level 3 'projection of future status' (Endsley, 2017). The theory of situation awareness is usually illustrated through professional work such as pilots, nurses, and—in education—teachers (Sherin et al., 2011). However, this paper adapts the theory to raise students' awareness of a sustainable-related issue when they work on a green mathematics task.

The authenticity of green mathematics tasks, in relation to the first level of awareness, can help students in perceiving a real sustainability-related event or context, information and data, questions, as well as purpose (Ainley et al., 2006). These authentic aspects can be either provided or gathered by students. The complexity of green mathematics tasks can help students to grasp the given situation through sense making of and reflecting on mathematical ideas used when working on the task (Barwell & Hauge, 2021). These two salient characteristics of green mathematics tasks are essential for students before moving to the next part: projection.

Projection can guide students to envisage the future status of a situation and possibly propose solutions to address the relevant issue (Endsley, 2017). According to the theory of situation awareness, this projection affects students' decision as to what actions they want to take given their perceptions and comprehensions of the situation. Therefore, after working on an authentic and complex task related to sustainability issues (green mathematics tasks), students can project what will happen in the future, so that they become aware of sustainability issues. From this point, they may see themselves taking action to tackle sustainability issues of which they are aware (Endsley, 2017). Actions (as shown in Figure 2) refer to students' behaviours and beliefs regarding sustainability issues and mathematics.

## Conclusions

This paper draws from mathematics education literature to provide and discuss a new framework for designing green mathematics tasks. Designing green mathematics tasks that can make students aware of sustainability issues requires three interdependent elements: sustainability issues, mathematical ideas, and problem solving. In addition to these elements, the tasks have three characteristics that can raise students' awareness when working on such tasks. They are authenticity, complexity, and projection. The authenticity of the tasks relies on five important aspects: event or context, purpose, data or information, questions, and activities. The complexity of the tasks allows students to struggle, develop sense making, and reflect on their activities in utilising mathematical ideas. Finally, the projection of what might happen in the future, allows students to decide what sorts of actions to take to address a sustainability issue. Investigations (in progress) are required to justify how teachers design and implement a sustainability-related, mathematics task in classrooms in relation to the framework.

## Acknowledgements

This paper would not be possible without the support of my supervisors, Katie Makar and Jana Visnovska. Their guidance helped me to develop the framework presented in this paper.

## References

Abtahi, Y., Gøtze, P., Steffensen, L., Hauge, K. H., & Barwell, R. (2017). Teaching climate change in mathematics classrooms: An ethical responsibility. *Philosophy of Mathematics Education Journal*, *32*(November 2017), 1–18.

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*, *32*(1), 23–38. https://doi.org/10.1080/01411920500401971

Anggraena, Y., Felicia, N., Eprijum, D., Pratiwi, I., Utama, B., Alhapip, L., & Widiaswati, D. (2022). *Kajian akademik kurikulum untuk pemulihan pembelajaran*. Pusat Kurikulum dan Pembelajaran, Badan Standar, Kurikulum, dan Asesmen Pendidikan, Kementerian Pendidikan, Kebudayaan, Riset, dan Teknologi.

Barwell, R. (2018). Some thoughts on a mathematics education for environmental sustainability. *The philosophy of mathematics education today* (pp. 145–160). Springer.

Barwell, R., & Hauge, K. H. (2021). A critical mathematics education for climate change: A post-normal approach. *Applying critical mathematics education* (pp. 166–184). Brill.

Boaler, J. (2022). *Mathematical mindsets: Unleashing students' potential through creative mathematics, inspiring messages and innovative teaching*. John Wiley & Sons.

Brendefur, J., & Frykholm, J. (2000). Promoting mathematical communication in the classroom: Two preservice teachers' conceptions and practices. *Journal of Mathematics Teacher Education*, *3*(2), 125–153. https://doi.org/10.1023/A:1009947032694

Bushnell, K. (2018). Learning mathematics for an environmentally sustainable future. *Mathematics Teaching*(263), 35–39.

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. *Mathematics classrooms that promote understanding* (pp. 19–32). Routledge.

Clark, R., Kirschner, P. A., & Sweller, J. (2012). Putting students on the path to learning: The case for fully guided instruction. *American Educator*, *36*(1), 5–11.

Dyment, J. E., & Hill, A. (2015). You mean I have to teach sustainability too?: Initial teacher education students' perspectives on the sustainability cross-curriculum priority. *Australian Journal of Teacher Education*, *40*(3), 21–35.

Endsley, M. R. (2017). Toward a theory of situation awareness in dynamic systems. In *Situational awareness* (pp. 9–42). Routledge.

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics*, *1*, 3–8. https://doi.org/10.1007/BF00426224

Hernandez-Martinez, P., & Vos, P. (2018). "Why do I have to learn this?" A case study on students' experiences of the relevance of mathematical modelling activities. *ZDM*, *50*(1), 245–257. https://doi.org/10.1007/s11858-017-0904-2

Hewitt, D. (1999). Arbitrary and necessary part 1: A way of viewing the mathematics curriculum. *For the Learning of Mathematics*, *19*(3), 2–9.

Ineson, G., & Povey, H. (2020). *Debates in mathematics education*. Routledge.

Kapur, M. (2016). Examining productive failure, productive success, unproductive failure, and unproductive success in learning. *Educational Psychologist*, *51*(2), 289–299. https://doi.org/10.1080/00461520.2016.1155457
Lubienski, S. T. (2000). Problem solving as a means toward mathematics for all: An exploratory look through a class lens. *Journal for Research in Mathematics Education*, *31*(4), 454–482. https://doi.org/10.2307/749653
Munter, C., Stein, M. K., & Smith, M. S. (2015). Dialogic and direct instruction: Two distinct models of mathematics instruction and the debate (s) surrounding them. *Teachers College Record*, *117*(11), 1–32. https://doi.org/10.1177/016146811511701102
Nugroho, A. W. (2020). What students know about climate change? a case study of high school students in Samboja, Indonesia. *IOP Conf. Ser.: Earth Environ. Sci*, *487*(1), 12001. https://doi.org/10.1088/1755-1315/487/1/012001
Oliver, M. C., & Adkins, M. J. (2020). “Hot-headed” students? Scientific literacy, perceptions and awareness of climate change in 15-year olds across 54 countries. *Energy Research & Social Science*, *70*, 101641. https://doi.org/10.1016/j.erss.2020.101641
Palm, T. (2007). Features and impact of the authenticity of applied mathematical school tasks. *Modelling and applications in mathematics education* (Vol. 10, pp. 201–208). Springer Nature.
Paredes, S., Cáceres, M. J., Diego-Mantecón, J.-M., Blanco, T. F., & Chamoso, J. M. (2020). Creating realistic mathematics tasks involving authenticity, cognitive domains, and openness characteristics: A study with pre-service teachers. *Sustainability*, *12*(22), 1–17. https://doi.org/10.3390/su12229656
Salim, S. (2019). Identifying students’ understanding of missing angles in parallel lines: a case study of year 7 students in The United Kingdom. *Journal of Physics: Conference Series*, *1317*(1), 12126. https://doi.org/10.1088/1742-6596/1321/2/022102
Schmidt, W. H., Houang, R. T., Sullivan, W. F., & Cogan, L. S. (2022). When practice meets policy in mathematics education: A 19 country/jurisdiction case study. *OECD education working papers* (Vol. 268). OECD Publishing. https://doi.org/10.1787/07d0eb7d-en
Schoenfeld, A. H. (2016). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics (Reprint). *Journal of Education*, *196*(2), 1–38. https://doi.org/10.1177/002205741619600202
Sherin, M., Jacobs, V., & Philipp, R. (2011). Situation awareness in teaching: What educators can learn from video-based research in other fields. *Mathematics teacher noticing* (pp. 81–95). Routledge.
Simon, M. A., & Tzur, R. (2012). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Hypothetical learning trajectories* (pp. 91–104). Routledge.
Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*(1), 20–26.
Steffensen, L., Herheim, R., & Rangnes, T. E. (2021). The mathematical formatting of how climate change is perceived: Teachers’ reflection and practice. *Applying critical mathematics education*. Brill. https://doi.org/10.1163/9789004465800_009
Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning*, *11*(4), 258–288. https://doi.org/10.1080/10986060903253954
Tran, D., & Dougherty, B. J. (2014). Authenticity of mathematical modeling. *The Mathematics Teacher*, *107*(9), 672–678. https://doi.org/10.5951/mathteacher.107.9.0672
Vos, P. (2018). “How real people really need mathematics in the real world”—authenticity in mathematics education. *Education Sciences*, *8*(4), 195. https://doi.org/10.3390/educsci8040195
Walkington, C., & Hayata, C. A. (2017). Designing learning personalized to students’ interests: balancing rich experiences with mathematical goals. *ZDM*, *49*(4), 519–530. https://doi.org/10.1007/s11858-017-0842-z
Wernet, J. L. W. (2017). Classroom interactions around problem contexts and task authenticity in middle school mathematics. *Mathematical Thinking and Learning*, *19*(2), 69–94. https://doi.org/10.1080/10986065.2017.1295419
Williamson, T. (2018). Alternative logics and applied mathematics. *Philosophical Issues*, *28*(1), 399–424. https://doi.org/10.1111/phis.12131

# Evidencing Relational Trust Within Mathematics Leadership Activity

Matt Sexton
*Australian Catholic University*
matthew.sexton@acu.edu.au

Janeen Lamb
*Australian Catholic University*
janeen.lamb@acu.edu.au

We explore how relational trust was evidenced within the leadership activity of one mathematics leader using cultural-historical activity theory as a theoretical perspective. We use data from interview and observation sources to suggest that relational trust was a motive object of that mathematics leader's activity. Findings contribute knowledge about how relational trust dimensions, originally situated within middle leadership theory, were evidenced within that mathematics leader's activity whilst leading a professional learning experience in their school.

Recently, mathematics leadership has received attention by the mathematics education research community, however, it remains under-theorised (Sexton & Lamb, 2017). We used the cultural-historical activity theory (CHAT) concept of motive object (Kaptelinin, 2005; Leont'ev, 1978) and concepts about relational trust within middle leadership literature (e.g., Edwards-Groves et al., 206) to contribute to the theorisation of mathematics leadership. In our paper, we demonstrate the applicability of those concepts to mathematics leadership, showing how relational trust was evidenced through the activity of one primary mathematics leader. Although we only draw on one set of data from interview and observation sources, we contribute a nuanced understanding of how relational trust was evidenced as a conscious motive object of that mathematics leaders' activity.

## Literature Background

With its surfacing as an educational leadership construct in recent times (De Nobile, 2018), middle leadership has been positioned as a form of practice, enacted in spaces between the executive leadership team and teaching teams in schools (De Nobile, 2018; Grootenboer, 2018; Lipscombe et al., 2021). Several definitions have provided insights into middle leadership; however, they have caused some contention (Lipscombe et al., 2021). The middle leader has been defined as a staff member who enacts a formally recognised position within the school whilst also undertaking classroom teaching responsibilities (Edwards-Groves et al., 2016; Grootenboer, 2018). Lipscombe et al. (2021) stated that a middle leader held a formally recognised school leadership position; engaged responsibilities for which the role was accountable; enacted leadership between the executive leadership and teaching teams; and that leadership actions were undertaken to influence teacher and student learning in positive ways.

De Nobile (2018), when identifying school middle leadership positions, did not state the mathematics leader as a middle leadership one. We believe, however, that the mathematics leader is indeed a middle leadership position when we consider the definition that Lipscombe et al. (2021) offered. Our reasons are that the mathematics leader is a formal position within school leadership systems (Copping, 2022; Sexton & Lamb, 2017); they undertake responsibilities, requiring active membership of executive leadership and teaching teams (Copping, 2022; Driscoll & Cheeseman, 2022); and their responsibilities include school-based professional development (PD) leadership (Bolyard & Baker, 2021; Driscoll & Cheeseman, 2022; Sexton & Lamb, 2017). Our definition of mathematics leadership also perceives it as an enactment of middle leadership activity that influences the dispositions, practices, and knowledge about and for mathematics education within schools (Sexton & Lamb, 2017).

Theorising of middle leading has highlighted the importance of PD leadership, and how relational trust mediates the establishment of relationships that are required for effective school-based PD (Edwards-Groves et al., 2016). That theorisation, drawing primarily on practice architecture theory (Edwards-Groves & Grootenboer, 2021), extended upon beliefs held about

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 459–466). Newcastle: MERGA.

relational trust within school leadership generally, where it is understood as a crucial resource required for school improvement (Bryk & Schneider, 2003). Relational trust is perceived as necessary for enabling conditions that allow for enactment of developmental work through teachers' PD (Edwards-Groves et al., 2016). The reason for that is because relational trust has been perceived as collegial relationships, characterised by respect, competence, and regard, all of which are deemed necessary for meaningful PD (Bryk & Schneider, 2003).

To illustrate enactment of relational trust within middle leading practice of PD leadership, a new conceptualisation about the nature and function of relational trust has been offered by Groves et al. (2016) and Edwards-Groves and Grootenboer (2021). Those authors have positioned relational trust as a multidimensional construct, consisting of five distinct, yet interrelated dimensions. According to those authors, relational trust forms the "social resources needed for securing sustainable practice development in schools" (Edwards-Groves et al., 2016, p. 384). The relational trust dimensions are summarised in Table 1.

**Table 1**

*Summary of Relational Trust Dimensions*

| Relational trust dimension | Definition of relational trust dimension |
|---|---|
| Interpersonal trust | demonstration of practice that nurtures mutual respect, trustworthiness, confidence, and relationships between staff including enactment of approachability, genuine care and empathy for, interest in, and recognition of colleagues' expertise and differences |
| Interactional trust | establishment and maintenance of opportunities that facilitate interactive spaces for colleagues to engage in collaborative and cooperative dialogue, characterised by integrity, equity, and independent and interdependent pedagogical reasoning |
| Intersubjective trust | demonstration of practice that sees the middle leader working alongside and journeying with colleagues as they develop community about the developmental work, characterised by collaborative decision-making, co-construction of shared understanding and language, and co-investment in problem-solving and sensemaking |
| Intellectual trust | enactment of leadership where the middle leader presents themselves as knowledgeable about the developmental work focus through demonstrations of wisdom, knowledge, and practical expertise in response to needs and requirements of that work focus |
| Pragmatic trust | leadership of the developmental work that presents it as having relevancy and practicality to colleagues' teaching practice, and presented in coherent and achievable ways that acknowledges the reality and complexity of teachers' work |

Middle leaders are understood to enact those relational trust dimensions through their PD leadership in schools. Those dimensions are a part of practice and are not positioned as characteristics or traits of the middle leader (Edwards-Groves & Grootenboer, 2021). The relational trust dimensions are understood to be in a constant state of formation and transformation because each one influences and is influenced by the others (Edwards-Groves et al., 2016). Enactment of relational trust is perceived as dynamic, enabling conditions for middle leading that mediates professional learning (Edwards-Groves & Grootenboer, 2021).

Turning to the place of trust and mathematics leadership, we have reports of enactment of trust by mathematics leaders within some literature sources. Copping (2022) shared survey results about mathematics leaders' perceptions of their leadership, claiming that relationships were important to mathematics leaders, forming part of a larger theme that Copping named "culture" (p. 150). Bolyard and Baker (2021) and Driscoll and Cheeseman (2022) purported that the development of constructive relationships was crucial to the work of leading primary school mathematics. Their claim confirms that of Eden (2018) who reported that trust was vital when teachers needed to discuss tensions about teaching practice with peers. According to Eden (2018), those trusting relationships mediated interactions about practice that led to practice shifts, which in turn, nurtured further trust amongst teachers.

Although we have access to knowledge about how relational trust is realised within middle leadership (e.g., Edwards-Groves et al., 2016), and recognising that trust is positioned as crucial in mathematics leadership (Copping, 2022; Driscoll & Cheeseman, 2022; Eden, 2018), we are yet to know how the relational trust dimensions are evidenced within mathematics leadership. That gave rise to the problem we explored where we sought to evidence the relational trust dimensions within the leadership activity of a mathematics leader. Drawing on an example provided by that mathematics leader, we seek to respond to the following question: *How are the relational trust dimensions evidenced within the activity enacted by a School Mathematics Leader through their professional learning leadership?*

## Methodology

Recognising middle leadership as a form of practice (Grootenboer, 2018), a practice-based theory was required to perceive our problem. Practice can be understood as a form of activity (Nicolini, 2012), and with its unit of analysis on activity (Engeström, 2015), CHAT offered a way of responding to our question. With relational trust constituted as part of practice (Edwards-Groves et al., 2016), further reason for using CHAT was established.

### *Theoretical Perspective*

As a practice-based theory (Engeström, 2015; Nicolini, 2012), CHAT offers ways to study human activity using several concepts, most of which are related to the activity system (Engeström, 2015). CHAT privileges the notion that within all activity, there is always a subject who pursues a motive object (Kaptelinin, 2005). The concept of motive object is understood to be the entity at which activity is directed, the driving force of activity (Kaptelinin, 2005). It provides the motivation for activity (Engeström, 2015; Kaptelinin, 2005; Nuttall et al., 2015).

CHAT, however, perceives motivation beyond its prevalent understanding that sees it as an individual and internal force of will. Instead, CHAT understands motivation as directing psychological and practical activity, drawn forward in simultaneous and conscious ways, as the subject seeks to realise the motive object, resulting in desired outcomes (Engeström, 2015). The motive object can also be seen as undertakings enacted by the subject as they engage in activity (Nuttall et al., 2015). Leont'ev (1978) claimed that there is a hierarchical structure of activity, meaning that as the subject pursues the motive object of activity, they enact a series of actions. Those actions are undertaken to meet the goals associated with the motive object.

It is also possible for activity to be directed at multiple motive objects, meaning that human activity can be multi-motivational (Leont'ev, 1978; Nuttall et al., 2015). Within CHAT, labour as a form of human activity, is considered to fulfill two functions: it is enacted to achieve the motive object that directs the activity, and it is intended to influence other people who are participants within that activity (Leont'ev, 1978). Therefore, the motive object acts as an essential analytical tool in understanding the what and the why of activity (Kaptelinin, 2005).

### *Participant and Data Generation*

The data we use are from the lead author's doctoral study that explored how mathematics leaders contributed to project sustainability through leadership of school-based PD. We report on just one event within the data generation phase of that study. Those data were generated with Cindy, a School Mathematics Leader, who participated in the Contemporary Teaching and Learning of Mathematics (CTLM) project (Clarke et al., 2013) in 2011 and 2012. Cindy had maintained the mathematics leadership position in her school at the time of data generation (November 2017). Mathematics planning meetings were used by Cindy as PD opportunities, and as a way of sustaining project-initiated reforms in her school. Cindy also fulfilled the Learning and Teaching Leader and classroom teacher roles during the data generation period.

In November 2017, Cindy was interviewed prior to and after a whole-school planning meeting she led. Prior to that meeting, Cindy was interviewed (~15 minutes) about the intentions she had set for the planning meeting. Observation was used to generate data about enactment of Cindy's leadership of that 50-minute planning meeting. The generation of data focused on the *sayings* and *doings* (Grootenboer, 2018) of Cindy's leadership, with attention paid to what she focused her activity on during that meeting. After her planning meeting took place, Cindy was interviewed again (~40 minutes) responding to questions about observation data generated by the lead author. The planning meeting and the interviews were audio recorded. The planning meeting recording was used to cross-check the accuracy of observation data, and those data were transcribed to electronic Word™ files.

### *Data Analysis*

With its analytical potential, the CHAT concept of motive object (Kaptelinin, 2005) and the relational trust dimensions (e.g., Edwards-Groves et al., 2016) were used as sensitising concepts to analyse the dataset. A coding scheme (Saladaña, 2013) was developed that included pre-determined codes (e.g., motive object, interpersonal trust, pragmatic trust) along with definitions informed by CHAT (e.g., Kaptelinin, 2005; Nuttall et al., 2015) and relational trust literature sources (Edwards-Groves et al., 2016; Edwards-Groves & Grootenboer, 2021).

Data files were uploaded into an NVivo™ project and nodes were set up and named within that project using concepts from the coding scheme. The lead author used that scheme to deductively analyse data, looking for evidence of the motive object and relational trust dimensions within the dataset. Inductive analysis approaches were also used as a means of exemplifying how those concepts were enacted specifically by Cindy in that planning meeting. Analysis focused on Cindy's sayings and doings as evidenced in the dataset. The analysis approach was an iterative process of comparing examples with the definitions within the coding scheme, and cross-checking definitions with the examples in the dataset (Saladaña, 2013).

## Findings and Discussion

We present and discuss findings simultaneously in attempts to show how the relational trust dimensions (Edwards-Groves et al., 2016) were evidenced within Cindy's leadership of that planning meeting. Our attention was first drawn to the importance of relational trust and its dimensions through insights that Cindy provided in the pre-planning meeting interview.

In response to a question about what she intended to work on in the meeting, Cindy stated:

> I've decided to share the mathematics NAPLAN results from this year. We're looking at growth analysis, and we're going to have a look at the cohort of students from when they were in Grade 3 and then to Grade 5. I want the teachers to see that we're not accommodating the needs of the top-end students. I have to be careful though. I know teachers can get upset with how NAPLAN is presented. I have to play it out carefully so that they don't feel like they're being blamed. There's a lot of emotion with NAPLAN data that I have to think about as the maths leader.

Through our CHAT perspective, Cindy revealed to us that her leadership activity for that planning meeting was directed by, what we have called, a *developmental* motive object (Kaptelinin, 2005). Cindy wanted to influence her teachers' reading of NAPLAN data and to develop a shared interpretation of the results. She also wanted the teachers to realise that teaching practice at the school was not meeting the needs of higher-achieving students.

We were also sensitised to another motive object that extended beyond that developmental one. Cindy highlighted to us an awareness she had about her teachers' affective responses to NAPLAN data through her anticipation that the teachers may experience blame about the data results. We perceived that as evidence of a *relational* motive object within Cindy's leadership activity. Drawing on the relational trust dimensions, we further interpreted that attention to affect as Cindy's way of wanting to work on interpersonal trust (Edwards-Groves & Grootenboer, 2021). Cindy's comment led us to believe that there was a conscious intention (Engeström, 2015) to enact care and empathy in ways that drove her leadership activity (Kaptelinin, 2005) of that planning meeting with her teachers.

Our interpretation of that conscious relational motive object of Cindy's leadership activity was further supported by the following comment:

> I need to take my time today and go slowly. I know there will be a lot of emotions in there. I have to make sure that I focus on what we need to do with the NAPLAN data, but also make sure that the staff are okay. We have to talk about what we are going to do about the data, but I don't want staff feeling blamed or upset. There will be a lot going on for me in there today.

Cindy revealed that for her and the teachers at her school, NAPLAN results were highly imbued with affect. We have interpreted that NAPLAN data use in that PD setting was emotionally freighted for the teachers and for Cindy herself, surfacing that relational motive object of her activity. Our interpretation confirms findings by Thompson and Mockler (2016) who found that teachers can experience anxious responses to NAPLAN data. We have extended that knowledge to include how mathematics leaders may also be affected as they pay attention to teacher anxieties and their professional vulnerabilities in ways that Cindy shared. We have presented evidence to suggest that mathematics leaders may be consciously motivated to nurture interpersonal trust when working with NAPLAN results in PD settings because of their awareness of teachers' potential responses to NAPLAN data.

Recognising that within CHAT, actions are undertaken as a means of achieving motive objects (Leont'ev, 1978), we now turn to Cindy's leadership actions during that planning meeting. Due to constraints, we do not include all observation data. We use examples and include interpretation of Cindy's leadership actions to further show how relational trust was worked on by her through the developmental and relational motive objects of activity. We present that in Table 2, with the observation data presented in a way that acknowledges the temporality of Cindy's leadership actions during that mathematics planning meeting.

Data in Table 2 confirm that Cindy worked on the developmental motive object of activity. We interpreted that the developmental motive object was pursued by Cindy as she encouraged teachers to identify and discuss teaching practice that would meet the needs of higher-achieving students in the school. Cindy' leadership actions that sought to influence her teachers' knowledge and use of open-ended tasks and extending prompts (Sullivan et al., 2015) are further evidence of that developmental motive object. Her work on that developmental motive object also included how she engaged the teachers in a goalsetting about those open-ended tasks.

**Table 2**

*Observation Data Examples, Interpretation, and Relational Trust Dimension*

| Observation data example | Interpretation of leadership action(s) | Relational trust dimension(s) |
|---|---|---|
| Cindy mentions that it is alarming that high-achieving students made "low progress" from Year 3 in 2015 to Year 5 in 2017. She quickly highlights that several students who did make "high progress". Cindy reminds the teachers in the meeting that "it's only NAPLAN data." | Managing teachers' affect and emotional responses to NAPLAN data; preserving teacher self-esteem | Interpersonal trust |
| Cindy says to the teachers about "being in this together" and that the NAPLAN data are "everyone's responsibility." Cindy adds that it is important that the teachers decide as a team about "ways forward with using the data." | Reminding teachers of team approach; reiterating a sense of collaboration and the importance of shared responsibility for data results | Intersubjective trust |
| Cindy asks teachers to offer questions that clarify her interpretation of the NAPLAN data. She invites teachers to talk about ways of addressing the needs of high-achieving students, "What can we do in our maths teaching to make sure we are meeting the needs of the top kids?" | Opening spaces for dialogue by inviting clarifying questions about NAPLAN data; creating opportunities for shared decision-making about ways of using data to inform practice | Interactional trust<br>Intersubjective trust |
| Cindy references CTLM ideas, specifically open-ended tasks and differentiation prompts. She says the NAPLAN data as reason to continue with the practices learned in CTLM. Cindy invites input from teachers for agreement with her interpretation of NAPLAN data and the need to continue with CTLM practices that she said she knows "works". | Demonstrating knowledge of mathematical tasks (task selection and implementation); using data as influencing tool to create shared understanding and purpose for collective work about mathematics teaching practice | Intellectual trust<br>Interactional trust<br>Intersubjective trust |
| Cindy sits with a group of teachers and gives advice on how to choose an open-ended task and how changing the number range can extend its demands. Cindy shares a story of her use of open-ended tasks, giving an example from her own teaching and how she changed the task demand using extending prompts that increased the number range. | Demonstrating knowledge of mathematical tasks, including selection and implementation through differentiation prompts; using own stories of practice to highlight practicality and relevancy of teaching strategies | Intellectual trust<br>Pragmatic trust |
| Cindy asks teachers to plan the use of one open-ended task to be used in the following week. Cindy prompts teachers to identify a goal about using open-ended tasks with extending prompts. She asks teachers to email their goal so that she is aware: "I want to know what you want to get better at with extending the top kids with the open tasks. I can help you more then." | Building in goal setting and making the work of extending students' learning practical, relevant, and achievable; demonstrating interest in teachers' professional learning goals | Pragmatic trust<br>Interpersonal trust |

Drawing on their definitions (e.g., Edwards-Groves et al., 2016), we applied the relational trust dimensions as we interpreted Cindy's leadership actions. We interpreted that as Cindy sought to influence teachers' practice and knowledge (Sexton & Lamb, 2017), she worked simultaneously on relational trust and its dimensions. Cindy's leadership actions, as evidence of her work on that relational motive object, suggest to us that she directed her activity at development of the relational trust dimensions, as reported within literature about middle leading (Edwards-Groves & Grootenboer, 2021). Our interpretation of Cindy's leadership activity, however, shows that those relational trust dimensions hold nuance when observed through the perspective of mathematics leadership activity.

Taking just a few examples from Table 2, we can see that intersubjective trust was developed by Cindy through her attempts to nurture shared understanding and collective responsibility for the NAPLAN results. She also worked on creating space for teachers to contribute to collective responses concerning the use of those data to improve mathematics teaching practice. A different example of relational trust development was evidenced in how Cindy enacted intellectual trust when she, as School Mathematics Leader, shared her knowledge of task selection and implementation when she interacted with her teachers about the use of open-ended tasks and enabling prompts. Cindy's leadership activity highlighted how the relational trust dimensions surfaced in multi-dimensional ways through her leadership activity, confirming that the enactment of relational trust as dynamic (Edwards-Groves & Grootenboer, 2021).

During the post-meeting interview, Cindy explained her attention to relationship, noted by the first author during observation of that planning meeting. Cindy revealed the vitality of relationships within her leadership activity, evidencing further that relational motive object:

> Relationship plays a big part in mathematics leadership, even more so with maths. There's something about mathematics, trust, and relationships that allows me to know how my teachers really feel about maths and their teaching, especially the teachers with maths anxiety. I always make sure that people are okay, and we have trust before I push. I have to make sure we are confident in our relationships.

As the School Mathematics Leader and Learning and Teaching Leader, Cindy was afforded opportunities to compare leadership roles. We have interpreted that for Cindy relational trust held a unique space within her mathematics leadership activity when compared to her other roles. By investing in secure relationships (e.g., Driscoll & Cheeseman, 2022), Cindy created conditions for her to generate knowledge about teachers' dispositions (i.e., teacher anxieties) as well as insights into their mathematics teaching practice. We also interpreted Cindy's comment to mean that as the middle leader of mathematics, she understood how relational trust formed conditions for the teachers to engage in the professional learning agenda that she sets within her school site (Edwards-Groves & Grootenboer, 2021).

For Cindy, the work on nurturing relational trust as a motive object enabled conditions for teachers to engage in PD in her school, providing further evidence of its importance in enabling middle leadership that mediates teachers' professional learning (Edwards-Groves et al., 2016). We have interpreted that for Cindy, developing relational trust as a motive object of activity (Kaptelinin, 2005) played a crucial role in mediating her leadership of mathematics, confirming previous research (e.g., Bolyard & Baker, 2021; Driscoll & Cheeseman, 2022; Eden, 2018).

Drawing those interpretations together, we have reason to believe that Cindy's leadership activity was multi-motivational (Leont'ev, 1978), in the way that there existed developmental and relational motive objects that she worked on during that planning meeting. By this we mean that for Cindy, along with the intention of influencing teachers' understanding of NAPLAN results and the implications of that for teaching practice, fostering relational trust was also a driving force of activity (Kaptelinin, 2005), realised through her mathematics leadership.

## Conclusion

We asked how relational trust, as offered in middle leadership literature, was evidenced within Cindy's mathematics leadership activity. We shared our interpretation of her leadership of one planning meeting. For that reason, we are cognisant of limitations and mindful of not making grand claims. Through our CHAT perspective, however, we evidenced relational trust as a motive object of activity pursued by Cindy. We provided examples of our application of the relational trust dimensions to Cindy's leadership actions in that mathematics planning meeting to support our interpretation of her activity. CHAT allowed us to perceive relational trust as a conscious motive object of activity. This is a nuanced contribution to knowledge about mathematics leadership not yet considered, and is therefore, worthy of further study as a means of extending the theorisation of mathematics leadership in school settings.

## References

Bolyard, J. J. & Baker, C. K. (2021). Examining the practice of elementary mathematics specialists through narratives: Implications for professional learning and development. *Professional Development in Education.* https://doi.org/10.1080/19415257.2021.1876150

Bryk, A. S., & Schneider, B. (2003). Trust in schools: A core resource for school reform. *Educational Leadership, 60,* 40–45.

Clarke, D. M., Roche, A., Wilkie, K., Wright, V., Brown, J., Downton, A., Horne, M., Knight, R., McDonough, A., Sexton, M., & Worrall, C. (2013). Demonstration lessons in mathematics education: Teachers' observation foci and intended changes in practice. *Mathematics Education Research Journal, 25*, 207–230.

Copping, K. (2022). Perceptions of the role of primary mathematics leaders. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 146–153). Launceston: MERGA.

Driscoll, K., & Cheeseman, J. (2022). School mathematics leaders' support of primary teachers' professional learning in meetings. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 186–193). Launceston: MERGA.

Eden, R. (2018). Opening classroom practice to challenge: The role of trust in mathematics teachers' collaborative inquiry involving co-teaching. In J. Hunter, P. Perger, & L. Darragh, (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 282–289). MERGA.

Edwards-Groves, C., & Grootenboer, P. (2021). Conceptualising five dimensions of relational trust: implications for middle leadership. *School Leadership & Management, 41*(3), 260–283.

Edwards-Groves, C., Grootenboer, P., & Rönnerman, K. (2016). Facilitating a culture of relational trust in school-based action research: Recognising the role of middle leaders. *Educational Action Research, 24*(3), 369–386.

Engeström, Y. (2015). *Learning by expanding: An activity-theoretical approach to developmental research* (2nd ed.). Cambridge University Press.

Grootenboer. P. (2018). *The practices of school middle leadership: Leading professional learning.* Springer.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Prentice-Hall.

Lipscombe, K., Tindall-Ford, S., & Lamanna, J. (2021). School middle leadership: A systematic review. *Educational Management Administration & Leadership*, 1–19. https://doi.org/10.1177/ 1741143220983328

Nicolini, D. (2012). *Practice theory, work, and organisation: An introduction.* Oxford University Press.

Nuttall, J., Edwards, S., Mantilla, A., Grieshaber, S., & Wood, E. (2015). The role of motive objects in early childhood teacher development concerning children's digital play and play-based learning in early childhood curricula. *Professional Development in Education, 41*(2), 222–235.

Saladaña, J. (2013). *The coding manual for qualitative researchers* (2nd ed.). SAGE.

Sexton, M., & Lamb, J. (2017). Using activity theory to understand a mathematics leaders' motivations and use of mathematical knowledge for teaching. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 466–473). MERGA.

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A., Roche, A., & Walker, N. (2015). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education, 18,* 123–140.

Thompson, G., & Mockler, N. (2016). Principals of audit: testing, data and 'implicated advocacy'. *Journal of Educational Administration and History, 28*(1), 1–18.

# How Children and Their Teacher Use Different Ways of Talking During Whole Class Interactions in a New Zealand Primary Classroom

Shweta Sharma
*University of Canterbury*
shweta.sharma@canterbury.ac.nz

The interactional role of language use in the mathematics classroom is explored in the last few decades. This paper adds to the knowledge base by exploring different ways of talking that children and their teacher use in a New Zealand primary geometry English-medium classroom. Bakhtin's concept of speech genres is used for an analysing transcript of one audiovisually recorded whole-class interaction episode from a geometry lesson in a Year 5/6 classroom. The analysis suggests that the teacher and children use various genres to participate in classroom interactions. The use of several genres provides insights into what children and the teacher construct as mathematical in the real sense. The paper finished with a few implications for teaching and research.

In the last few decades, mathematics education research has widely acknowledged and emphasised the role of language in teaching and learning of mathematics. This paper takes a discursive approach to explore how Year 5/6 children (aged 9-11) and their teacher used different ways of talking to display their understanding of geometric shapes and their properties as they engage in classroom interactions. The discursive approach takes interactions as discursive practices where the focus is on the social actions performed by participants' utterances instead of just the content. Analysing the discursive nature of classroom discourse, Van Oers (2001) argued that genres act as a means for participants to evaluate their utterances as a legitimate part of the ongoing mathematical discussion. He argued that it is the genres used in mathematical discussions that allow the initiation of children into a mathematical culture by exposing them to a way of speaking, doing, and interacting that is considered mathematical. This paper aims to explore different genres that children and their teacher may use to convey their understanding of geometric shapes during classroom interactions.

I draw on Bakhtin's concept of speech genres (Bakhtin, 1986). Bakhtin (1986) talked about "speech genres" as "relatively stable types of utterances" (emphasis added in original, p. 60) and are specific to a particular sphere of life in which those utterances are used. Speech genres reflect a speaker's ideology in terms of the participation role that they assign to their addressee through their use of common expressions (Joyner, 2018). Additionally, for Bakhtin (1986), speech genres reflect specific conversational conditions at a particular moment in time as well as reveal the specific intention or action orientation of the speaker at that moment within the conversational space (Bakhtin, 1986; Sullivan, 2012). In other words, speakers use a specific kind of speech genre depending upon the social action they engage in. For example, in criminal court proceedings, the defendant makes use of justifications as one kind of speech genre when defending their actions. Sullivan (2012) argued that speech genres are also important for understanding how the speaker brings "intonation or emotional attitudes" into speech (p. 109). Thus, speech genres account for the overall composition of the utterance with content, style, and intention embedded into it (Bakhtin, 1986; Sullivan, 2012).

Bakhtin (1986) further suggested that the diversity of speech genres is inexhaustible as "each sphere of activity contains an entire repertoire of speech genres that differentiate and grow as the particular sphere develops and becomes more complex. Special emphasis should be placed on the extreme heterogeneity of speech genres (oral and written)" (Emphasis added in original, p. 60). Bakhtin (1986) further argued that there are primarily two significantly different kinds of speech genres: primary and secondary. Primary speech genres are the simple genres used in everyday conversations. However, when these primary speech genres are adopted and shaped according to a

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 467–474). Newcastle: MERGA.

context, and become an integral part of a specific context, these primary genres combine with others to create complex secondary speech genres. For example, the act of formulating an argument in everyday life (a primary genre) may become part of the mathematics classroom, creating argumentation as a secondary speech genre where students are expected to provide either logic or evidence to support their claim, often identified in research as using socio-mathematical norm or argumentation.

Mathematics education research specifically focused on genres has studied genres as discursive practices to achieve a specific social action. For example, Gerofsky (1999) in one of her earlier studies analysed the Initial Calculus lectures of four mathematics professors at Simon Fraser University to explore the generic features of the lectures. In her analysis, Gerofsky found that the lecturers used ‘we’ to indicate a relationship of power and dependency between the lecturer and their students and seek conformity from the listener. Lecturers may also use tag questions, such as (“Ok?” or “Right?”) to elicit consent from their students. She argued that there is a clear link between the language of persuasion and that of mathematics lectures. Rockwell (2000) used the Bakhtinian concept of speech genres to analyse the teaching of speech genres in a teacher’s utterances during a lesson observed in a Grade 6 science classroom in a Mexican rural school. She noted that the teacher used a variety of speech genres, including informal talk, explanation, folklore and anecdote, although her analysis does not provide an exhaustive list of those speech genres. She found that the teacher often used two kinds of genre: “plática (informal talk or chat) and explicación (explanation)” (Rockwell, 2000, p. 267). She categorised informal talk as speech genres because they reflected generic overtones, established a particular way of expressing knowledge, and included relatively open turns for student utterances. She argued that explanation was interactionally used as another kind of speech genre, displaying assumptions that the students knew about the topic in question and were expected to provide details about the elements of that topic. Second, she argued that these genres both underwent and caused transformations in the structure of the student’s participation each time they were used. She suggested that the teacher’s use of a certain speech genre might create discursive conditions for the use of a particular kind of speech genre by the student. Finally, she claimed that teaching itself is like a complex speech genre, comprising several other speech genres which are embedded with “thoughts, values, and sentiments that are re-voiced and reinterpreted in each new situation” (p. 273). This paper aims to add to the existing research on using genres as different ways of talking in a mathematics classroom. The following research question informs the analysis in this paper:

- What speech genres do the teacher and the children use as they interact during a geometry lesson in Year 5/6 New Zealand primary classroom?

## Research Design

### *Context of School and Participants*

The paper reports on one aspect of a larger study. The study took place in a Year 5/6 class at a New Zealand English-medium primary school. This school catered to the multilingual children population. Participants included fifteen students with their mathematics teacher. Nine of the fifteen students were multilingual (1 Somali, 2 Tongan, 4 Māori, 1 Chinese, and 1 Filipino). The teacher had seven years of teaching experience. Parents filled out a short five-question questionnaire to provide information about languages that the children used at home or school and how long they have been in New Zealand. Informed and voluntary consent to participate in the study was sought from the participants.

### *Data Gathering Tools*

Six lessons on geometric shapes and their properties were observed and audiovisually recorded. Each lesson lasted for 45 to 50 minutes. Relevant documents including the New Zealand Curriculum

(Ministry of Education, 2007) and resources, teacher's unit plan, and students' work samples were also gathered. Field notes were taken during observations. I noted as many details as possible with keywords to note what and how classroom interactions were unfolding which were later developed in fuller notes with descriptions of settings, events, analytic ideas, inferences, memos, personal feelings and reflections after each lesson observation. All six geometry lessons were also audiovisually recorded. Audiovisual recording enabled the recording of non-verbal cues like gestures, body movements, pointing, and facial expressions that were found to be of particular relevance in revealing mathematical thinking and meaning construction in the pilot study. Two directional cameras were used to audio and video record the whole class and group interactions. I also used eye gear with an inbuilt camera and voice recorder to closely record moments of interaction that captured my attention. In addition, four to five audio recorders were kept on table-tops to record talk-in-interaction in-group settings. Using different audio and video recorders enabled me to record interactions from different angles, which were later corroborated to produce thorough and detailed transcription of data for analysis. Teachers' unit plan, resources used by the teacher to develop the unit plan, children's worksheets, pictures, drawings, and other classroom artefacts were also gathered.

### *Data Analysis*

Audiovisual data from one episode of whole-class interaction is presented in this paper, and participants' utterances in the selected episode are considered the unit of analysis. The analysis process involved analysing utterances using Conversation Analysis (CA) techniques to explore how the utterances were constructed and placed in the flow of interactions. CA techniques allow the explication of participants' utterances in terms of linguistic (including words, syntax, morphological and other grammatical forms) and paralinguistic (including prosodic features of the pitch, the volume of voice, silence along with gestures, laughter, and aspirations) features along with gestures, facial expressions that were used in an utterance. The selected episode was transcribed using an adapted version of Jefferson transcript conventions (Jefferson, 2004). The transcript is a product of an iterative process to ensure intra-rater reliability of the transcription process.

Paralinguistic features of the utterances were interpreted using insights from sociolinguistics research. For example, Hellermann (2003) found that repeating others' utterances with the same tonal pitch may indicate approval or acceptance; conversely, the repetition with different pitch may indicate disapproval. Ward (2019) has shown that English speakers use low pitch to signal authority over a knowledge claim. Moreover, the use of high rising intonation at the end of an utterance is often interpreted as a sign for questioning (Ward, 2019). However, in the New Zealand context, a high rising terminal (HRT) intonation pattern may indicate the speaker's intention to check if the listener is following the speaker or as a way to develop communicational solidarity (Metge & Kinloch, 1978; Warren, 2016). On the role of silence in interactions, Stubbe and Holmes (2000) argued that explicit verbal feedback is a norm of Pākeha (Pākeha is a Te Reo Māori term for a New Zealander of European descent) conversations, suggesting silence may be considered awkward by English speakers. Māori speakers, however, do not consider silence during a conversation problematic and may, therefore, refrain from giving immediate feedback in order to facilitate communicational solidarity (Metge & Kinloch, 1978). Māori participants have also been shown to engage in co-operating by overlapping, where they expand and elaborate on each other's suggestions (Stubbe, 1998). Since the data included children from multilingual contexts including Māori and Pasifica children, research insights from languages other than English were also used.

Peer consultation was also sought to receive comments and feedback on the interpretation and analysis from the other CA practitioners and Māori colleagues to ensure the reliability of the findings. Rockwell (2012) argued that intonation patterns (cues in the way utterances are made) act as an indication for the listener to identify the action and intention embedded in the speaker's utterance. Therefore, it is the content, style and intention embedded in the overall composition structure of utterance including ( linguistic (words, sentences) and paralinguistic features (pitch, silence, volume)) that defines the speech genre (Bakhtin, 1986; Rockwell, 2012; Sullivan, 2012). Sociolinguistic research insights allowed interpretation of the intended social action embedded in the utterances through interactional features such as pitch, volume, pauses, which led to the identification of the “relatively stable types” (Bakhtin, 1986, p. 60) of utterances that accounted for the speech genres at the macro-level analysis.

## Analysis and Discussion

This episode is extracted from the audiovisual data from the second lesson. During this lesson, the teacher provided children with playdough to make shapes that they already knew. During this episode, Zara (a female 9-year-old Māori-English bilingual Māori child with greater proficiency in English) made a few shapes using playdough and claimed that one of the shapes she made was a “perfect square” (see the circled shape in Figure 1).

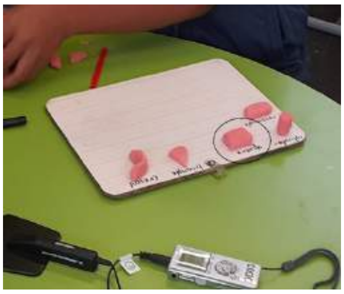


Figure 1. Playdough shape in a “perfect square” (circled).

The following transcript shows the conversation followed as Zara mentioned her shape.

| # | Speaker | Text |
|---|---|---|
| 189 | Zara | >look whaea[a] Jenny:< (1.0) whaea Jenny (.) a |
| 190 | | perfect square ((shows the shape by holding it in her hands)) |
| 191 | Teacher | is it perf (.) why is it a perfect square? Zara |
| 192 | Zara | I dun↓no |
| 193 | Teacher | what makes it a perfect square(2.0)>come on zara ↑I |
| 194 | | need< to ↑kno:w(0.5)because you said its perfect so |
| 195 | | what makes a perfect square a perfect square |
| 196 | | (1.0)= |
| 197 | Matiu | a ↑s[quare |
| 198 | Teacher | =[↑anyone ↑know why a perfect squa:re a perfect |
| 199 | | Square |
| 200 | Matiu | becoz its a square? |
| 201 | Garry | (h)(h) |
| 202 | Teacher | yeah because its a square doesnt tell me |
| 203 | | much(1.0)ELIE what do you think |
| 204 | Elie | becau::se um: [if you have to= (2.0) |
| 205 | Zara | [you put(on)((rolled her eyes)) |

| # | Speaker | Text |
|---|---|---|
| 206 | Elie | =um: because um::(1.0) if you have the right type |
| 207 | | of shape. or if you (1.0)if or if you have(.2) |
| 208 | | havin:g a right(0.5)type of equipment (.) °you can |
| 209 | have° | |
| 210 | Teacher | °okay° |
| 211 | Elie | so:: if you are trying to make square of that |
| 212 | | one(1.0)you can roll into a ball then you |
| 213 | | start pressing it down the other side >the |
| 214 | | other side and you can [get square< |
| 215 | Teacher | [oh thank thank you Elie |
| 216 | | (0.5)↑can any↑one ↑tell me why a perfect square |
| 217 | | might be (0.2)might be perfect square using geometry |
| 218 | | Language |
| 219 | Zara | [um ((looks at the roof trying to figure out how to say what she wants to |
| 220 | | say)) |
| 221 | Matiu | [um: °its got° |
| 222 | Teacher | Matiu |
| 223 | Matiu | because the face °no:(0.2)the si::des°(2.5) nah |
| 224 | | °I dun know° |
| 225 | Teacher | yeah you re on the right track. the si:des what |
| 226 | | (.)what would the sides be here |
| 227 | Matiu | perfectly:: aligned? with each other?= |
| 228 | Teacher | =aligned with each other? |
| 229 | Matiu | ah(1.0) perfectly the same? |
| 230 | Teacher | perfectly the sa:me the sides ↑are perfectly |
| 231 | | the same (1.0) |

[a] Whaea, meaning mother/aunt in Te Reo Māori, is used here as a term of respect to the teacher.

Zara tagged her teacher as the next speaker (lines 189 and 190) and claimed that the shape that she had made was a perfect square. She used a flat pitch with her claim of a perfect square. Ward (2019) suggests that English-speaker may use flat pitch to show their confidence in their knowledge claim. It seems that at this moment, Zara intended to declare her claim without providing any justification for the claim. This has been identified as one of the speech genres that children may use and is labelled as a *Declarative* speech genre.

To this claim, the teacher responded with a question to Zara, thus initiating the Initiation-Response-Evaluation/Feedback conversational pattern (McHoul, 1978) in the classroom. The Initiation-Response-Feedback (McHoul, 1978) In the classroom interaction, the sequence of the teacher's question in the first turn, the child's response in the second turn, and the teacher's feedback in the third turn were observed in the data. The teacher initially structured her question to ask if the shape was a perfect square (line 191); however, in the same turn, she rearticulated her question as "why is it a perfect square?", thereby leaving more space for Zara's explanations. Initiating a conversation with a question and providing cues to support children's thinking is identified and labelled as the *Pedagogical* speech genre in this study.

Zara, in the following turn (line 192), stated she did not know, by saying "I dunno". She initially used a flat pitch and then lowered her pitch (indicated by ↓). Ward (2019) has shown that this type of construction, using a flat and low pitch, is often made to signal to the listener that the speaker has given in and is not able to provide any further explanation. Utterances signalling the action of giving up in a discussion were identified and labelled as the *Giving-up* speech genre.

Noticing that Zara is not able to provide reasoning for her claim, the teacher rephrased the question and emphasised "what" (line 193, indicated by <u>underline</u>) to encourage Zara to think about the shape's properties. The teacher used longer pauses of two seconds (line 193, indicated by (2.0)) and one second (line 196) in the same utterance to allow Zara to bring some explanation of her claim. This again signals the use of the Pedagogical speech genre.

In the next utterance, Matiu (male 11-year-old Māori-English bilingual child) self-selected and stated that being a square makes it a perfect square (line 197). In the following utterance (line 198), the teacher appeared to ignore Matiu's utterance. This may be because the teacher required children to raise their hands before speaking (Fieldnotes 1 to 6). It seems that the teacher ignored Matiu's response, probably because he had not followed this classroom norm. The teacher used high pitch (indicated by ↑) at the beginning of "anyone" and "know" to open the floor for all children to respond (line 198). Ward (2019) showed that high onset is often used for initiating a new topic. This time, the teacher looked at Matiu and provided him with her consent to speak. Matiu responded that a perfect square is perfect because it is a square, as he stressed the word "square" (line 200). The use of the HRT (indicated by ?) in English spoken in New Zealand often implies the speaker's intention to check if the listener follows what the speaker is trying to say (Warren, 2016). Thus, Matiu's use of HRT at the end of his utterance may be interpreted as his way of checking with the teacher whether she agrees with his response, which again may be interpreted as his way of declaring his knowledge claim, therefore using Declarative speech genre.

In the next utterance, Garry (a male 11-year-old Filipino-English bilingual Philippines child) might not have understood the use of this specific intonation pattern in New Zealand English as he laughed at Matiu's response (line 201). Jefferson (1984) showed that laughter within talk-in-interaction sometimes signals trouble, as the recipient engages in laughter to embarrass the speaker. In this case, it seems that Garry might have evaluated Matiu's response as wrong and redundant. It seems that Garry used his laugh as a way to show his assessment of Matiu's response. Utterances that suggest assessment of others' responses are labelled as *Assessment* speech genre.

It is interesting that the teacher often used the Assessment speech genre where she provided an implicit assessment of the children's responses as either incomplete or incorrect and selected another speaker as evident in the following utterance in the interaction. It appears that the teacher did not accept Matiu's response as she said that "Being a square doesn't tell me much" (line 202). Matiu (line 200) used HRT to seek approval from the teacher. She selected Elie as the next speaker to answer, "why a perfect square is a perfect square". In lines 204, 206-209, Elie used "um", stretches and pause of one second to construct her utterance. These features are often a mark of a non-response (Sacks, 1987). Thus, it may be that she was not sure of what the teacher wanted her to comment on about the square. In the following utterance, the teacher again used the Assessment speech genre as she provided implicit rejection of Elie's response as the teacher thanked Elie for her response, however, did not provide specific feedback on her response to support her thinking.

The teacher at this moment rephrased her question (line 216) and stressed the words "geometry language" to direct the children's attention to the geometry-specific features of the shape that made it a perfect square. It seems that the social action embedded in this utterance was to provide feedback to children to provide their responses in a specific manner using geometry language. Thus, this could be interpreted as the use of the Pedagogical speech genre. Following this cue, Zara and Matiu self-selected. However, Zara used "um" as a filler and started looking at the ceiling of the classroom in an attempt to recall the shape (line 219).

Matiu (line 221) used "um" to hold the floor, and then he used his low tone (whispering, indicated by °) to state his utterance. The teacher continued to use this speech genre in utterances following Matiu's response as well. The teacher selected Matiu as the next speaker (line 222). He attempted to answer (line 223) by emphasising the word "face", but then he changed the term "face" to "sides". He used his whispering tone for his utterance. Ward (2019) suggests that speakers often use whispering at the end of their utterances to signal low confidence. Thus, Matiu's use of a whispering tone and pauses of 2.5 seconds (line 223) may be interpreted as doubt and uncertainty. He realised that he might be wrong, and therefore he stated that he did not know.

In the following turn (line 225), the teacher provided positive feedback and again stressed the word "side" as she stretched it and used a slightly high volume (for emphasis) to signify to the children that the answer she was looking for was related to the properties of a square in terms of equal sides. In doing so, she clearly showed her intent for children to use geometry-specific language by explicitly asking about the property of sides in the square. After receiving positive feedback from the teacher, Matiu responded that the sides needed to be perfectly aligned with each other (line 227). However, this time as compared to his previous utterance (line 223), Matiu used HRT to check if the teacher agreed with him. It seems that the teacher acknowledged that Matiu might have been looking for agreement as he used HRT; thus, in the following utterance (line 228), she responded with a question to Matiu to let him reconsider his response. She used HRT at the end of her utterance, probably to signal the partial correctness of Matiu's response. Matiu (line 229) realised that his answer was partially correct, but that he needed to restructure his response to meet the teacher's expectation. Thus, he used a filler and paused for one second to hold the floor while looking for the right word (line 229). He again used HRT with his utterance "perfectly the same" (line 229) to check with the teacher. This time, the teacher stretched the word "same" to emphasise its use (line 230). She used a slightly high pitch along with stretching the first syllable. Moreover, she reiterated the phrase "sides are perfectly the same" three times in her following utterance (lines 230-231).

## Discussion and Conclusion

Speech genres are those preferred utterances that speakers use to accomplish a certain social action, and are always embedded with the speaker's intentions, values and sentiments (Bakhtin, 1986; Rockwell, 2012; Sullivan, 2012). The analysis identified several speech genres such as pedagogical, assessment, declarative, and giving-up speech genres. The use of speech genres suggests that there are implicit messages in the way utterances are constructed while engaging in classroom interactions. The finding is an extension to the finding presented by Rockwell (2000). Rockwell (2000) analysed primary speech genres in teaching episodes in a Mexican rural school and found that the teacher used speech genres from different aspects of life, including informal talk, explanation, folklore, and anecdotes. Moreover, Rockwell (2000) had suggested that the use of speech genres in a teacher's utterances may create specific discursive conditions for children's participation. This is also evident in this paper. For example, the use of the Pedagogic speech genre by the teacher often called for an explanation or justification by the child by responding to a question provided as feedback, whereas, the use of the Assessment speech genre may only highlight a child's incorrect response and provide no further feedback to the child, which may hinder further participation of the child whose response is assessed. Thus, it can be argued that the teacher's practices in the use of these two speech genres (Pedagogic and Assessment speech genres) may work as implicit messages to indicate to children the kind of participation expected from the children.

The paper does not provide an exhaustive list of speech genres that can be identified in a classroom, rather draws our attention to the different ways of talking that may have implications for engaging students in mathematics learning. The paper suggests that teachers' use of the Pedagogic speech genre may be helpful in eliciting children's knowledge. The use of the Assessment speech genre, however, may deter children from participating in classroom interactions, even though negative evaluations are not provided explicitly. Being aware of speech genres may support teachers to consciously use the Pedagogic speech genre in preference to the Assessment speech genre. On the research front, the analysis calls for more research to explore what different ways children and teacher may use in a mathematics classroom. The analysis shows that the children may also construct their utterances in a way to show their declining interest in continuing the conversation as evident here in the use of the Giving up genre by the students, which needs to be explored more in research. It is possible that children construct their utterances in a variety of ways to indicate that they are not interested in continuing the discussion which may further lead to a loss of learning opportunities.

## References


Bakhtin, M. M. (1986). *Speech genres and other late essays* (V. W. McGee, Trans.). University of Texas Press.

Gerofsky, S. (1999). Genre analysis as a way of understanding pedagogy in mathematics education. *For the Learning of Mathematics, 19*(3), 36–46.

Hellermann, J. (2003). The interactive work of prosody in the IRF exchange: Teacher repetition in feedback moves. *Language in Society, 32*(1), 79–104.

Jefferson, G. (2004). Glossary of transcript symbols with an introduction. In G. H. Lerner (Ed.), *Conversation analysis: Studies from the first generation* (pp. 13–31). John Benjamins Publishing Company.

Joyner, K. (2018). Rethinking code-switching: Using Bakhtinian speech genres to revitalize classroom language. *Philosophical Studies in Education, 49*, 24–33.

McHoul, A. (1978). The organization of turns at formal talk in the classroom. *Language in Society, 7*(2), 183–213. https://doi.org/10.1017/S0047404500005522

Metge, J., & Kinloch, P. J. (1978). *Talking past each other: Problems of cross-cultural communication*. Victoria University Press.

Ministry of Education. (2007). *The New Zealand curriculum*. Wellington: Learning Media. https://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum

Rockwell, E. (2000). Teaching genres: A Bakhtinian approach. *Anthropology & Education Quarterly, 31*(3), 260–282. https://doi.org/10.1525/aeq.2000.31.3.260

Rockwell, E. (2012). Preambles, questions, and commentaries: Teaching genres and the oral mediation of literacy. In J. Kalman, & B. Street (Eds.), *Literacy and numeracy in Latin America: Local perspectives and beyond* (pp. 184–199). Routledge. https://doi.org/10.4324/9780203807798

Stubbe, M. (1998). Are you listening? Cultural influences on the use of supportive verbal feedback in conversation. *Journal of Pragmatics, 29*(3), 257–289. https://doi.org/10.1016/S0378-2166(97)00042-8

Stubbe, M., & Holmes, J. (2000). Talking māori or pakeha in English: Signalling identity in discourse. In A. Bell, & K. Kuiper (Eds.), *New Zealand English* (pp. 249-278). John Benjamins Publishing Company.

Sullivan, P. (2012). *Qualitative data analysis using a dialogical approach*. Sage Publications.

Van Oers, B. (2001). Educational forms of initiation in mathematical culture. *Educational Studies in Mathematics, 46*, 59–85.

Ward, N. G. (2019). *Prosodic patterns in English conversation*. Cambridge University Press. https://doi.org/10.1017/9781316848265

Warren, P. (2016). *Uptalk: The phenomenon of rising intonation*. Cambridge University Press.

# A Curriculum Comparison of Years 9–10 Measurement and Geometry in Australia and Singapore

Ellen Sugianto
*University of South Australia*
ellen.sugianto@mymail.unisa.edu.au

South Australia's PISA performance has been in constant decline since 2003 with the proportion of PISA participants meeting the Australian national proficient standard dropping from 73% (in 2003) to 50% (in 2018). In contrast, Singapore is a consistently strong performer. To better understand student readiness in answering PISA questions, this paper reports a curriculum comparison between Australia and Singapore for Years 9 and 10 in Measurement and Geometry. The findings highlight the similarities and differences in the topics covered in both countries' curricula and raise questions about potential implications for student outcomes in PISA.

The Programme for International Student Assessment (PISA) is a triennial assessment conducted by the Organisation for Economic Co-Operation and Development (OECD). PISA is targeted at 15 year olds as students at this age are reaching the end of compulsory education in most of the participating countries (OECD, 2013a). PISA attempts to assess how well students are able to apply what they have learnt from school in unfamiliar, real-world contexts. The PISA mathematical literacy assessment assesses four content categories: Change and Relationships, Space and Shape, Quantity, Uncertainty and Data. The outcomes are presented as mean scores, distributions of scores, and percentages of participants who attain defined proficiency levels (Thomson et al., 2013).

In the Measurement Framework for Schooling in Australia 2020, the national proficient standard for 15-year-old students participating is Level 3 on the PISA scales (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2020). Students who attain national proficient standard demonstrate that they have acquired more than the elementary skills expected at that year level. As such, the proportion of participating students achieving at or above the national proficient standard is a key performance measure.

South Australia (SA)'s PISA performance in mathematics has been reported as in constant decline since 2003. The 2018 PISA data indicated that SA's mean score was in the bottom three states in Australia (Thomson et al., 2019). Additionally, of SA students who participated in PISA, the proportion meeting the national proficient standard was only 50% in 2018 in contrast to 73% in 2003. This is concerning as SA data suggests a substantial increase in the proportion of students not meeting the national proficient standard. Across the four content categories in 2012, SA students recorded the lowest scores in Space and Shape (Thomson et al., 2013). Amongst the top three performers in 2012 in Space and Shape, Singapore attained a mean score of 580 points (Thomson et al., 2013). In comparison, SA's was 481 points.

This paper comprises six sections. First, it outlines what constitutes mathematical proficiency in PISA. Next, a case is made for the concept on *intended curriculum*. The third section reviews the curriculum structure of Australia and Singapore. This is followed by the methodology used in this paper. The comparative findings are then presented to give a highlight the differences and similarities of coverage of Measurement and Geometry in both curricula. Finally, the paper concludes with the potential implications for student readiness in answering PISA assessment items in Space and Shape content category.

## Mathematical Proficiency

The PISA 2012 framework identifies three mathematical processes and seven capabilities. Mathematical processes describe "what individuals do to connect the context of a problem with

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 475–482). Newcastle: MERGA.

mathematics and thus solve the problem” (OECD, 2013a, p. 28). The three processes are:(1) formulating situations mathematically, (2) employing mathematical concepts, facts, procedures, and reasoning, and (3) interpreting, applying and evaluating mathematical outcomes (OECD, 2013a). Each of the mathematical processes is underpinned by mathematical capabilities. Mathematical capabilities are learnable and complement mathematical content to be engaged with to solve PISA assessment items. The seven capabilities are (1) communication, (2) mathematising, (3) representation, (4) reasoning and argument, (5) devising strategies for solving problems, (6) using symbolic, formal and technical language and operations, and (7) using mathematical tools (OECD, 2013a).

Drawing on the mathematical processes and capabilities, the OECD formulated six increasing levels of mathematical proficiency. The proficiencies are intended to describe a series of mathematical capabilities required to solve PISA assessment items from each level. The difference in the activation and/or complexity level of mathematical capabilities across the mathematical processes is the key to describing the proficiency level (Stacey & Turner, 2015). For example, the description for Level 1 (see Figure 1) describes students as being able to carry out routine mathematical procedures with explicit direction in Level 1, while Level 6 describes students as being able to apply and use the aspects of all seven mathematical capabilities.

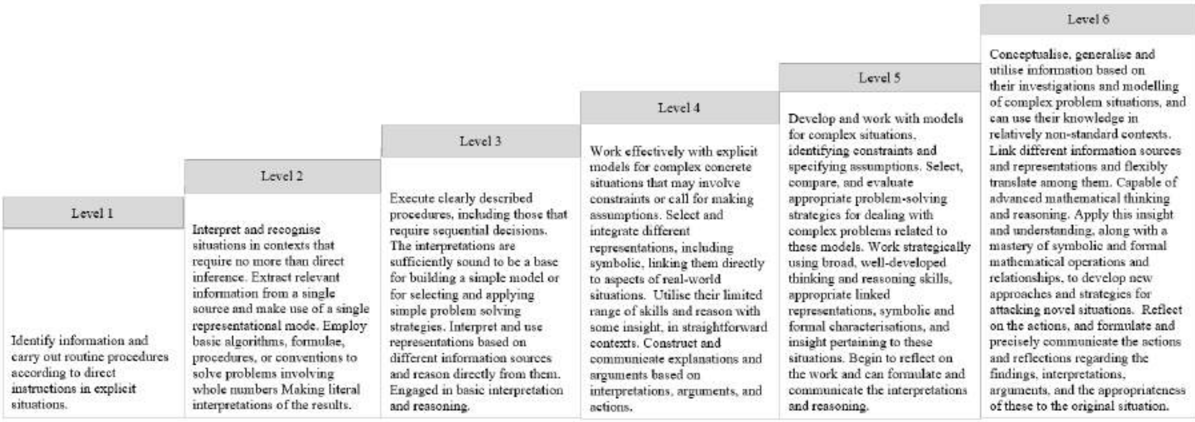



Figure 1. PISA 2012 mathematics proficiency levels (OECD, 2013b, p. 61).

The mathematical capabilities also require different levels of complexity to be activated. For example, in the communication capability, the lowest complexity level is in assessment items that simply require a numeric answer. Items requiring a justification or explanation have a greater level of communication complexity (OECD, 2013a). Figure 1 provides an overview of the difference in the proficiency level from Level 1 (the lowest) to Level 6 (the highest) (OECD, 2013b, p. 61) —the description of each proficiency level is intended to characterise the knowledge and skills of students at the level (Thomson et al., 2013; Turner, 2014).

## Intended Curriculum

Schmidt et al. (2004) argue that the differences in achievement among countries are related to what is taught, suggesting “the curriculum itself makes a huge difference" (pp.2–3). For example, students in a country where the education system has a greater emphasis on Geometry would perform better in the assessment items in the Space and Shape category of PISA (OECD, 2013b). Hypotheses like this have led researchers to compare curricula of specific countries against those from top-performing countries.

Studies such as Safrudiannur and Rott (2019), and Acar and Serçe (2021) have focused on curriculum comparisons, with curriculum defined to be the intended curriculum. Others have taken a broader approach to curriculum. Valverde et al. (2002) modify a tripartite model curriculum by adding the potentially implemented curriculum (mediating role of textbooks) into the intended curriculum (the educational system aims and goals), the implemented curriculum (the enactment of these goals in teaching and learning) and the attained curriculum (what students attained from the teaching and learning). In the comparison of mathematics achievement across countries, Valverde et al. (2002) created "a powerful link between the intended and the implemented curricula in their creation of the potentially implemented curriculum, affected primarily by the textbook" (O'Keeffe, 2013, p.3). Further, Valverde et al. (2002) explained that a key function of mathematics textbooks is to turn abstract curriculum policy into more concrete instructions for teachers and students; textbooks transform the intention of curricular policy into instructions in the classrooms. Therefore, mathematics textbooks can be seen as representative of curriculum (Remillard, 2005).

The larger study from which this paper stems analyses both the intended and potentially implemented curriculum layers related to the Year 9 and 10 Measurement and Geometry strands for South Australia and Singapore. Singapore is chosen for curriculum comparison based on their continual high performance in PISA. By conducting a curriculum comparison of the Years 9 and 10 Measurement and Geometry strands, followed by a content analysis of the mathematics textbooks, this larger study aims to inform the discussion of student readiness in answering PISA questions in the Space and Shape content category and to provide a clearer picture of the proficiency needed to solve tasks presented in the mathematics textbooks.

This paper presents the first phase of the study, which is a comparison of the Australian Curriculum: Mathematics (AC:M) Years 9 and 10 Measurement and Geometry strand and the Singaporean Express course for both O-Level Mathematics and O-Level Additional Mathematics (SC:M).

## Curriculum Structure in Australia and Singapore

The AC:M is a national curriculum for mathematics for Foundation to Year 10 and is adopted without modification in South Australia. In addition to the Year 10 curriculum is a Year 10A curriculum which caters for students who are seeking an extension in Year 10. The focus of this study is on Years 9–10A Measurement and Geometry strand in AC:M version 8.3 as the students who participated in the latest PISA in 2018 would have used mathematics textbooks written for this version of the curriculum. The AC:M is structured around the interconnection between three strands and four proficiency strands. The content strands, which describe what to be taught and learnt, are Number and Algebra, Measurement and Geometry, and Statistics and Probability. The proficiency strands, which describe how the content is explored, include understanding, fluency, problem solving, and reasoning. In Years 9–10A, the Measurement and Geometry strand is divided into three sub-strands: using units of measurement, geometric reasoning, and Pythagoras and trigonometry (ACARA, 2016).

The Singaporean mathematics curriculum (SC:M) consists of a set of connected syllabuses to cater to students' interests and strengths. There are five mathematics syllabuses in the secondary mathematics curriculum which are tied to the three core courses that are designed to match students' academic progress and interest such as Express course, Normal (Academic) course and Normal Technical course (Ministry of Education, 2019). The highest percentage of secondary school student enrolment in 2018 was those students undertaking the Express stream, which was about 63% (Ministry of Education, 2019). The Express course offers O-level Mathematics and O-level Additional Mathematics which assumes knowledge of O-level Mathematics content in addition to more in-depth coverage of topics (Ministry of Education, 2012a, 2012b). SC:M combines Secondary 3 and Secondary 4 O-Level Mathematics content strands into one syllabus (Ministry of Education, 2012b) and similarly for Secondary 3 and Secondary 4 O-Level Additional Mathematics (Ministry of Education, 2012a). Hence, the Express course pertaining to Measurement and Geometry in O-

Level Mathematics syllabus, and Geometry and Trigonometry in O-Level Additional Mathematics syllabus in Secondary 3 and 4 were selected for this paper.

## Research Design

Mayring's qualitative content analysis (Mayring, 2014) was used to inform the research design. Notably, Singapore was also selected by ACARA (2018) in their recent comparative curriculum study, which compared the Australian Curriculum against the Singapore Curriculum as part of a regular study of international comparison of Australia with high-performing countries. Of relevance to this study is the mathematics curriculum comparison for Years 9 and 10, which was based on the same version of the curricula analysed in this paper. The ACARA comparison comprised three layers of analysis: breadth, depth, and rigour of the content descriptions and elaborations in the AC:M against the content descriptions and learning experiences in the comparable Singaporean curriculum. For breadth analysis, ACARA counted the total number of content descriptions and elaborations for the AC:M, and content descriptions and learning experiences for the Singaporean curriculum, in addition to noting down content not present in the AC:M.

This study employs a similar approach to the analysis undertaken by ACARA (2018) however a point of difference is the "unit" that is counted. The ACARA (2018) analysis used the content description as the "unit", with each description counted as one "topic". In this study the detail provided in each content description is refined by using the elaborations to identify the number of single "topics" across the strand of measurement and geometry. In other words, one description could result in more than one topic being identified. For example, consider the content description ACMMG221 "Solve problems using ratio and scale factors in similar figures". The content description alone does not specify if the ratio and scale factors are used for corresponding sides or for areas of similar figures, however the elaboration "establishing the relationship between areas of similar figures and the ratio of corresponding sides (scale factor)" provides additional detail. The outcomes were two topics: a topic named "ratio and scale factors of the corresponding sides in similar figures" and a topic named "ratio of areas of similar figures".

All content descriptions from the AC:M Years 9–10A strand of Measurement and Geometry were examined to identify topics for inclusion in this study. This list of topics was then used as the basis for comparison, meaning that topics from the AC:M were used as a reference and SC:M topics mapped onto this. Finally, topics in SC:M Sec 3/4 Measurement and Geometry that are not in the AC:M were also identified.

## Findings

This section presents the findings of the curriculum comparison for the AC:M and SC:M Years 9-10A Measurement and Geometry strand. Once the topics included in the AC:M Measurement & Geometry strand for Years 9–10A and SC:M Secondary 3/4 were identified, the next step was to identify the topics that were common (or not) to both curricula. This mapping resulted in 52 topics, which can be categorised as shown in Figure 2.

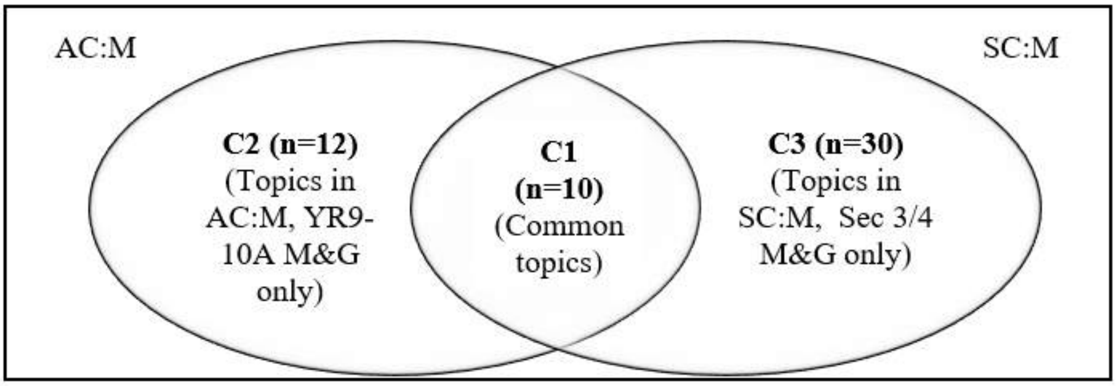



Figure 2. Classification used for topics in AC:M and SC:M Measurement & Geometry.

Table 1 lists the ten topics in C1, accounting for 19% of the topics identified in this analysis. Table 2 lists the twelve topics in C2. Eleven of these topics appear in AC:M Year 9, 10 or 10A Measurement and Geometry but appear in earlier year levels in the SC:M. These topics account for 21% of the total. However, it is worth noting that these eleven topics account for 50% of all topics identified in AC:M Years 9-10A Measurement and Geometry. The twelfth topic, "very small and very large time scales and intervals", is located in the comparable year level in SC:M but in the Number and Algebra strand.

**Table 1**

*C1: Topics in AC:M Years 9–10A Measurement and Geometry and SC:M Secondary 3/4 Measurement and Geometry*

| In the sub-strand: Geometric reasoning | In the sub-strand: Pythagoras and trigonometry |
|---|---|
| Ratio of areas of similar figures | Solve simple trigonometric equations |
| Formulate proofs involving congruent triangles and angle properties | Solve right-angled triangle problems including those involving direction and angles of elevation and depression |
| Use congruence and similarity to proof and numerical exercises involving plane shapes | Sine, cosine and area rules for any triangle and solve related problems |
| Angle and chord properties of circles | Apply Pythagoras' Theorem and trigonometry to solve three-dimensional problems |
| Use of properties of geometric figures | Unit circle to define trigonometric functions and graph them with and without the use of digital technologies (sine & cosine functions) |

**Table 2**

*C2: Topics in AC:M Years 9–10A Measurement and Geometry, but not in SC:M Secondary 3/4 Measurement and Geometry*

| In the sub-strand: Using units of measurement | In the sub-strand: Geometric reasoning | In the sub-strand: Pythagoras and trigonometry |
|---|---|---|
| Area of composite shapes (rectangles and triangles) | Enlargement and condition of similar triangle | Pythagoras' Theorem involving right-angled triangles |
| Surface area and volume of cylinders | Scale diagram | Use similarity to investigate the constancy of sine, cosine and tangent ratio in right-angles triangles |
| Surface area and volume of right prisms | Ratio and scale factors of the corresponding sides in similar figures | Apply trigonometry to solve right-angle triangle problems |
| Surface area and volume of prisms, cylinders and composite solids | | |
| Surface area and volume of right pyramids, right cones, spheres and related composite solids | | |

Table 3 lists 7 (of 30) topics in C3, which are from SC:M 3/4 Measurement and Geometry and in a different strand of the AC:M Years 9–10A curricula (in Number and Algebra). Table 4 lists the remaining 23 topics in C3, which are from SC:M Secondary 3/4 Measurement and Geometry but do not appear in the AC:M Years 9–10A curricula, presumably because they are covered in the senior years. These topics account for 44% of the total.

**Table 3**

*C3: Topics in SC:M Secondary 3/4 Measurement and Geometry, and in a Different Strand in AC:M Years 9–10A*

| In the strand: Number and Algebra | |
|---|---|
| Gradient of a line segment on the Cartesian plane | Solve problems involving parallel and perpendicular lines |
| Distance between two points on the Cartesian plane | Midpoint of a line segment on the Cartesian plane |
| Sketch linear graphs using the coordinates of two points | Describe, interpret and sketch circles |
| Solve linear equations from graphs | |

**Table 4**

*C3: Topics in SC:M Secondary 3/4 Measurement and Geometry, but not in AC:M Years 9–10A*

| Topics not in AC:M Years 9–10A | | |
|---|---|---|
| Arc length, sector area and area of segment of a circle | Unit circle to define trigonometric functions and graphs of y= tan(x) | Vectors in two dimensions (use of notation) |
| Use of radian measure of angle (including conversion between radians and degrees) | Other trigonometric functions for angles of any magnitude (secant, cosecant and cotangent) | Representing a vector as a directed line segment |
| Perpendicular and angle bisector | Principal values of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1}x$ | Translation by a vector |
| Ratio of volumes of similar solids | Unit of angles of trigonometric functions in radians | Position vectors |
| Mid-point theorem | Use of trigonometric identities, for example cosA/sinA = cot A; expansions of sin (A±B), cos (A±B), tan (A±B), formulae of sin2A, cos 2A and tan2A; expression of acosθ + bsinθ in the form Rcos (θ±α) or Rsin (θ±α) | Magnitude of a vector |
| Graphs of parabolas with equations in the form $y^2 =kx$ | Simplification of trigonometric expressions | Use of sum and difference of two vectors to express given vectors in terms of two coplanar vectors |
| Area of rectilinear figure | Proofs of simple trigonometric identities | Multiplication of a vector by a scalar |
| Transformation of given relationships, including $y=ax^n$ and $y=kb^x$, to linear form to determine the unknown constants from a straight-line graph | | Geometric problems involving the use of vectors |

## Summary and Conclusion

The findings show that there is a clear difference between the Measurement and Geometry strand of the intended mathematics curriculum for AC:M Years 9–10 AC:M and SC:M Secondary 3/4. There are more topics ‘not present’ (44%) in AC:M Years 9–10A Measurement and Geometry compared to topics that are ‘common’ in both curricula (19%). Topics that appear prior to Secondary 3/4 in Singapore account for 21% of the total number of topics, and the remaining proportion relate to topics that appear in a different strand in the comparable years in AC:M and SC:M (in Number and Algebra strand). Therefore, these findings imply that students in Years 9 and 10 in South Australia have less coverage of Measurement and Geometry content compared with students in Singapore, which may mean that students in Singapore are better prepared for solving PISA assessment items. The curriculum comparison by ACARA (2018) reported a similar observation.

The intended mathematics curriculum is one factor, of many, that contribute to students’ preparedness to achieve the national proficient standard in PISA mathematics. Other factors include previous outcomes and experiences, depth of knowledge in the topics being assessed, and confidence and capacity in problem solving. Nonetheless, the intended curriculum of each country, as a mandatory curriculum, is a common factor within that country. Hence, the intended curriculum is a useful unit of measure to provide insight into the differences in the content coverage in the Australian and Singaporean curricula in Measurement and Geometry.

## Acknowledgements

The author would like to acknowledge the assistance of her supervisors, Lisa O'Keeffe, Amie Albrecht and Hannah Soong, in preparing the manuscript.

## References

Acar, F., & Serçe, F. (2021). A comparative study of secondary mathematics curricula of Turkey, Estonia, Canada, and Singapore. *Journal of Pedagogical Research*, *5*(1), 216–242. https://doi.org/10.33902/JPR.2021167798

Australian Curriculum, Assessment and Reporting Authority. (2016). *Foundation to year 10 curriculum: Mathematics.* ACARA. https://webarchive.nla.gov.au/awa/20170421050854/http://www.australiancurriculum.edu.au/download/f10

Australian Curriculum, Assessment and Reporting Authority. (2018). *International comparative study: The Australian curriculum and the Singapore curriculum.* ACARA. https://www.australiancurriculum.edu.au/media/3924/ac-sc-international-comparative-study-final.pdf

Australian Curriculum, Assessment and Reporting Authority. (2020). *Measurement framework for schooling in Australia 2020.* ACARA. https://www.acara.edu.au/docs/default-source/default-document-library/measurement-framework-2020-amended-may-12-2022.pdf?sfvrsn=4ddc4c07_0

Mayring, P. (2014). Qualitative content analysis: Theoretical background and procedures. In Bikner-Ahsbahs, A., Knipping, C., Presmeg, N. (Eds.), *Approaches to qualitative research in mathematics education. Advances in mathematics education* (pp. 365–380). Dordrecht: Springer Netherlands. https://doi.org/10.1007/978-94-017-9181-6_13

Ministry of Education. (2012a). *Additional mathematics syllabuses: Secondary three to four: Express course & normal (academic) course*. Singapore: Author.

Ministry of Education. (2012b). *Mathematics syllabuses: Secondary one to four: Express course & normal (academic) course*. Singapore: Author.

Ministry of Education. (2019). *Education statistics digest*. Singapore: Author

OECD. (2013a). *PISA 2012 assessment and analytical framework: Mathematics, reading, science, problem solving and financial literacy.* OECD Publishing. https://doi.org/doi:https://doi.org/10.1787/9789264190511-en

OECD. (2013b). *PISA 2012 results: What students know and can do (Volume I)*. OECD Publishing. https://doi.org/doi:https://doi.org/10.1787/9789264201118-en

O'Keeffe, L. (2013). A framework for textbook analysis. *International Review of Contemporary Learning Research*, *2*(1), 1–13.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*(2), 211–246.

Safrudiannur, & Rott, B. (2019). The different mathematics performances in PISA 2012 and a curricula comparison: Enriching the comparison by an analysis of the role of problem solving in intended learning processes. *Mathematics Education Research Journal*, *31*(2), 175–195. https://doi.org/10.1007/s13394-018-0248-4

Schmidt, W., Houang, R., & Cogan, L. (2004). A coherent curriculum: The case of mathematics. *Journal of Direct Instruction*, *4*(1), 13–28.

Stacey, K., & Turner, R. (2015). The evolution and key concepts of the PISA mathematics frameworks. In Stacey, K., & Turner, R. (Eds.), *Assessing mathematical literacy* (pp. 5–33). Springer.

Thomson, S., De Bortoli, L., & Buckley, S. (2013). *PISA 2012: How Australia measures up: The PISA 2012 assessment of students' mathematical, scientific and reading literacy*. Melbourne: Australian Council for Educational Research (ACER).

Thomson, S., De Bortoli, L., Underwood, C., & Schmid, M. (2019). *PISA 2018: Reporting Australia's results. Volume I: Student performance.* Melbourne: Australian Council for Educational Research (ACER). https://research.acer.edu.au/ozpisa/35

Turner, Ross. (2014). *Described proficiency scales and learning metrics. Assessment GEMs no.4*. Melbourne: Australian Council for Educational Research (ACER).

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Springer Science & Business Media.

# A Further Investigation to Introducing the Equal Sign in China

Jiqing Sun
*Deakin University*
jiqing.sun@deakin.edu.au

Yifan Gu
*Lauriston Girls' School*
guec@lauriston.vic.edu.au

Fostering students' bidirectional conception of the equal sign (viewing the equal sign as indicating an equivalence of two sides rather than a 'show result' symbol) is challenging, and students' misconception of the equal sign is persistent. Some studies mention that in China, the pedagogical approach to introducing the equal sign supports students' development of bidirectional sense toward the equal sign. Built up on this body of literature, this paper further investigates the Chinese pedagogy from the researcher, students and the teacher's perspectives.

Forty years ago, Kieran (1981) foregrounded a prevalent students' misconception of the equal sign, as many students possessed a narrow conception that is treating the equal sign as a 'show result' symbol instead of a relational understanding of the equal sign (the equal sign indicates the equivalence of both sides). Students without relational understanding of the equal sign have difficulties in understanding the number sentences such as '3 = 3', and '3 + 2 = 4 + 1' (Kieran, 2004). Nowadays, this students' misconception of the equal sign is still widely documented (Blanton et al., 2018; Ralston & Li, 2022).

Literature has well argued that the relational understanding of the equal sign is a fundamental concept in students' transition from arithmetic to algebra (e.g. Carpenter et al., 2003). Without it, students will encounter difficulties learning algebra, such as equation solving. For instance, Blanton et al. (2018) showed that students with a narrow conception of the equal sign were not able to make sense of solving equations with unknowns on both sides (e.g., $3x + 2 = 2x + 1$), since they consider the equal sign should always be followed by a result carried out from the operation. There has been a great deal of research exploring pedagogical approaches to foster students' relational understanding of the equal sign (see below). In previous work (Sun, 2019), the author has illustrated how the equal sign has been introduced to Chinese students. Built on Sun (2019), this paper conducted a more thorough and detailed investigation of the Chinese pedagogical approach to the concept of the equal sign.

## Literature Review

In a seminal research, Carpenter et al. (2003) showed a number sentences comparison activity that could support students' development of conception on the equal sign. By evaluating a number of pairs number sentences as true or false (e.g., 3 + 5 = 8, 8 = 3 + 5, 8 = 8, 3 + 5 = 5 + 3), Carpenter et al. (2003) demonstrated that students' conception that considering the equal sign as showing results could be suspended, as this task could draw students' attention to the number sentences on both sides of the equal sign. Therefore, it can provide students with an opportunity to have a bidirectional view on the equal sign. This pedagogical approach is evidenced to be effective by other algebra researchers (e.g. Knuth et al., 2016).

Alternatively, others researchers such as McNeil et al. (2015), proposed a pedagogy that is modifying conventional arithmetic operation practice. Traditionally, arithmetic operation questions are generally presented in a form such as '1 + 2 = __' , and this traditional form of practice might lead to students think the right side of the equal sign always displayed an answer carried out from the calculation on the left side, thus activating or reinforcing students' 'show result' conception of the equal sign (McNeil, 2008). In this sense, McNeil et al. (2015) recommended in everyday mathematics classroom, students should be exposed more to non-conventional forms of arithmetic operation practice, such as '__ = 1 + 2' and '3 = 1 + __'. By doing this, students' conception of the equal sign will not be impeded by a one-directional calculation pattern, rather, students' bidirectional

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 483–491). Newcastle: MERGA.

view of the equal sign is possible to emerge. McNeil et al. (2015) showed this pedagogy took effect to improve students' relational understanding of the equal sign.

This study considers that while the starting points are different, both approaches mentioned above are essentially similar. Both pedagogies are to suspend students' one-directional conception of the equal sign by exposing them to a wide range of non-conventional forms of equation, so as to provide students with an opportunity to attend to the equivalence of both sides of the equal sign. In this sense, this study considers that diverting students from focusing on traditional forms of arithmetic operation could be a foundational way to develop relational understanding of the equal sign at an early stage.

While students' narrow conception of the equal sign is widely spread, this difficulty is not universal. The literature has shown the primary students in China commonly possess a robust relational understanding of the equal sign (Li et al., 2008). Li et al. (2008) concluded a few factors that contribute to Chinese students' success in understanding the equal sign, such as introducing the concept of the equal sign before students' exposure to arithmetic operation. Students understand the concept the equivalence by describing the quantities relationship (e.g. 'more than', 'the same', 'less than') in a concrete context before they are formally introduced to the equal sign. Following Li et al. (2008), Sun (2019) further explored the Chinese approach by analysing an official lesson plan used in the classroom through a theoretical lens of early algebra education. Echoing Li et al. (2008), Sun (2019) discovered that students in China learn the concept of the equal sign at the Kindergarten level, which is prior to that students learn arithmetic operation, so the interference of the 'show results' conception possibly brought by traditional forms of arithmetic operations can be reduced. Sun (2019) also identified other features of the Chinese approach to introduce the equal sign. First, the teaching sequence starts from students' own experience and informal solutions and continues to formal mathematical symbols. For instance, students learn the concept of the equality by comparing the quantities of two piles of concrete objects, and then students use own symbols to represent the equivalence before being exposed to the formal equal sign. Furthermore, the pedagogy emphasises both "left side is the same as the right side" and "right side is the same as the left side", which stresses a bidirectional conception of the equal sign. Finally, the lesson plan highlights the way of drawing an equal sign (i.e. two short horizontal lines with the same length). However, Sun (2019) is not an empirical study, so the feature exhibited in the lesson plan needs to be further investigated with empirical evidence. This study will examine how the pedagogy based on the lesson plan supports Chinese students' understanding of the equal sign.

## Methodology

### *Methodological Approach*

This research aims to explore how students learn the concept of the equal sign. Therefore, this study employs a qualitative case study, which has the affordance to provide fine-grained details of students' learning (Hamilton & Corbett-Whittier, 2012). With these details, research can have an in-depth understanding of what happened in the students' thinking in learning (Hamilton & Corbett-Whittier, 2012). Furthermore, according to Clarke (1998), to better understand students' learning, data sources that reflect different perspectives should be triangulated against each other. In this sense, three data sources are used for data collection and analysis in this study. The first one is the student interview, that reflects their own thinking and reasoning. The second data source is the teacher interview that reveals their opinions on students' learning and this pedagogy after the implementation. The third data source is students' work samples (the short quiz after the lesson), whereby this research could interpret their learning outcome.

### *Research Procedure*

One kindergarten class with twenty-nine pre-school students (about 5.5 - 6 years old) and one teacher participated in this study. The kindergarten is located in Wuxi, JiangSu province. A lesson to introduce the equal sign as per an above-mentioned official lesson plan in Sun (2019) (see Figure 1) had been conducted to students (this study focuses on the equivalence part, so the inequality part will not be investigated). After the lesson, students' work samples were collected. Ten students and one teacher were interviewed. Interviewee students are selected as per ethical approvals (ten students and their carers consented the interview).

*Activity 7: Are they equivalent? (Mathematics, Symbols)*

*Objective:*

1. Learn to use '=' or '≠' to indicate the quantity relationship between two sets of numbers.
2. Be able to think about the problems proactively, be able to use the appropriate language to present the results of the activity.

*Preparation:*

1. 3-4 cards with concrete objects, one '=' card and one '≠' card
2. Children's graphic book

*Activity:*

1. Recognising the equality, understanding how to represent the equivalent relationship between two sets of numbers.
   a) Teacher shows students two cards with the same number of fruits, for instance, six apples and six pears. Let students count the numbers of apples and pears, and let them decide whether they have the same number of fruit.
   b) Teacher asks students what symbols they can put between two cards so other people can clearly see that they have the same number. After that, you can introduce the equal sign by emphasising two short horizontal lines must be at the same length.
2. Recognising the inequality, understand the inequality and how to represent it.
   a) Teacher shows students cards with different numbers of fruits, asks students whether two cards have the same number now. If two sides have different numbers, asks students what symbols they can put to indicate the two sides have different numbers. After that, showing students the sign '≠'.
   b) Teacher shows students several pairs of cards with a mixture of equivalent and non-equivalent relationships, and then asks students what sign they should put in-between and why.
3. Practicing " the left side and the right side is equivalent" in the children's graphic book.

Figure 1. The lesson plan for introduction of the equal sign.

Student interview started from asking students to describe the meaning of the equal sign. Then they were encouraged to explain their thinking on the work samples. The prompt question is "can you explain to me why you did xxx?" Finally, they were queried about the most impressive part of the lesson. The prompt question is "Which part(s) of this lesson help you learn and why?" Some follow up questions might be asked to further probe students' thinking. In teacher interview, they were asked about their understanding and/or opinion on students' learning and how the lesson supports students' conception of the equal sign. It is worth mentioning that the literature has shown the kindergartners are capable of verbally expressing their mathematical thinking in interview (Lenz, 2022).

## Results and Discussion

Due to space limit, two students (Yang and Ming) and one teacher's (MS Q) data is reported here. Two students were chosen because their work samples and interview responses are typical as well as reflecting the diversity of data. The data will be presented in the following way. A snapshot of all twenty-nine students' work samples is shown first, to provide an overview of students' learning outcome, followed by an analysis of two students and the teacher interview.

### *Students Work Samples*

The Figure 2 shows twenty-nine students work samples and Figure 3 showed Yang and Ming's work samples.

Figure 2. All students' work samples.

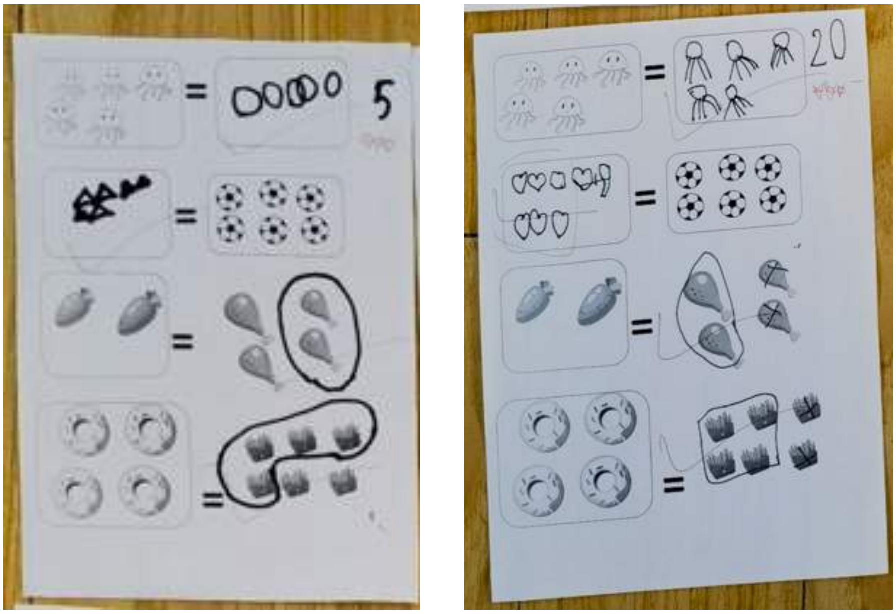


Figure 3. Yang (left) and Ming's (right) work samples.

The short quiz comprises four tasks. The first two are to match and draw the object, and the last two are to make two sides equivalent by circling the same amount of objects from the side with redundant items. It could be noted the missing parts of the first two tasks are shown at either right or left side of the equal sign. This tests students' bidirectional sense about the equal sign, referring to whether they understand the equal sign indicates the equivalence of both sides rather than from left to right. Almost all students completed four tasks correctly, except for a couple of students who filled the incorrect number of objects initially and made corrections afterward (e.g. Yang is one of these students, and his thinking will be elaborated below). This result tends to suggest that after the lesson most of students were able to develop a bidirectional view towards the equal sign, concurring with Sun (2019) in which a teacher stated 90% of his/her students could do a similar short quiz correctly. It is also worth mentioning that almost all students draw the missing objects which are not the same as the objects to be matched. This data could be interpreted as students focused on the quantity of objects on both sides of the equal sign instead of the objects per se. Therefore, it could be inferred that for students, four tasks are an informal way to represent a) 'five equals what number', b) 'what number equals six', c) 'two equals four takes away how many' and d), 'four equals six takes away how many'. These resemble the formal arithmetic operations a) 5 = __, b) __= 6, c) 2 = 4 - __ and d) 4 = 6 - __. In this sense, while students had not been exposed to formal arithmetic operations, their responses could be considered as being in line with the 'flexible operational level' of knowledge of the equal sign (students can accept the equation forms such as 'a = a' and 'a = b + c') suggested by Matthews et al. (2012). The 'flexible operational level' is a crucial step towards full relational understanding of the equal sign. Taken together, students' work samples evidence that students' conception of viewing the equal sign as an indication of the equivalence of the quantity on both sides has emerged.

*Students' Interview*

When Yang and Ming were asked to describe the meaning of the equal sign, Yang stated, "we use this to tell other people two sides as the same". Similarly, Ming said, "it shows left side and right side have the same number of things." These words "two sides" and "the same" appear to show that both students understand the equal sign as indicating the bidirectional equivalence. Ming explicitly pointed out the equivalence of the quantity. While Yang did not clearly refer to the quantity, his work sample shown above suggested he focused on the quantity, as discussed early.

Then two students' work samples are presented to them, and they were prompted to further explain their work. Yang's response is shown below,

Interviewer: For the question 1 and 2, do you think any difference when the missing part appears on the right side of the equal sign and when it is at left side?

Yang: No, we are just filling the same number of things.

Interviewer: How do you know you need to fill the same number?

Yang: Because of this sign [point to the equal sign]

Interviewer: Ok, you have explained this sign to me just now, let see this, these are octopuses and footballs, why you just draw the circles and triangles for the missing part?

Yang: It is easy to draw [laugh].

Interviewer: It's ok two sides are not the same thing?

Yang: Yes, I think is ok because the number are the same.

Interviewer: I see. Let see question 3 and 4, you just circle the same number of things and keep the redundant things out. Do you have the other way to make two sides as the same, if you don't use circle.

Yang: I can cross the more things.

Interviewer: Ok, that's right. What about if we do something from the left side, to make left side equal the right side?

Yang: um…oh… I can add more on left side.

The excerpt above confirmed Yang's bidirectional sense of the equal sign as he expressed left side and right side did not make difference. Yang also revealed that he did not have to match the shape of objects since only quantity mattered. This result corroborated this research's interpretation to students' work samples above, which is that students focused on quantities instead of objects per se. This research probed Yang's thinking further by prompting him to seek an alternative approach to make two sides equivalence for task 3 and 4. The data suggested that Yang did not understand the question asked, but following the prompt provided, he was able to find out that he can add more objects on left side instead of taking away things on right side. This tends to indicate that Yang accepts both forms 'a + b = c' and 'a = c - b' as legitimate, which showing Yang has achieved the flexible operational view towards the equal sign suggested by Matthews et al. (2012). It is worth mentioning that in the interview, Yang was encouraged to use numbers and symbols to represent task 1, and he was able to write '5 = 5', providing further evidence to his flexible operational thinking to the equal sign. Since students had not learnt the arithmetic operation yet, they were not asked to mathematically represent task 3 and 4 which addition and subtraction are involved. Therefore, this study does not seek the evidence to whether students achieve 'comparative relational level' understanding of the equal sign suggested by Matthews et al. (2012). Comparative relational level understanding of the equal sign refers to that students can use relational strategy to evaluate number sentences such as '67 + 86 = 68 + 85' without full calculation (Matthews et al., 2012).

In Ming's interview, he was first asked why he filled the wrong quantity of objects initially and corrected it. Ming responded,

I made a mistake on counting these hearts (the object drawn by Ming), and then the teacher showed me the number was not correct, so I counted again and change it.

Ming's words appear to suggest that his mistake is a result of miscounting rather than misunderstanding of the equal sign (his teacher, Ms Q, confirmed this interpretation, as will be seen later). Then Ming was also asked for task 1 and 2, whether there are any differences if the missing part is shown on left side or right side. Ming had a similar response as Yang, stating there were no difference and he explained,

As long as two sides has the same quantity, the equal sign works, it does not matter right side or left side.

Similar to Yang, Ming's explanation indicated that he had a clear bidirectional conception toward the equal sign. Next, the interviewer asked Ming why he drew the same octopuses for task 1 and draw the different things for task 2. The excerpt is shown below,

Ming: Because the octopuses are easy to draw but the footballs (objects in task 2) are more difficult to draw, so it is quicker to draw hearts.

Interviewer: Ok, so do you worry about one side is football and another side is heart they are different things.

Ming: No. I can draw the different things, as long as they have the same many of things.

Ming's words showed that, like Yang, he recognised that the equal sign is about showing the equivalence of the amount rather than the objects per se. Finally, Ming was also encouraged to find the alternative approach to make two sides in task 3 and task 4 equivalent. Ming provided a similar response as Yang: suggesting it can be done by drawing more objects at left sides. Finally, Ming's thinking was also further probed by requiring him to write a number sentence to represent the task 1, the excerpt is shown below,

Interviewer: Could you use mathematics to describe task 1, please?

Ming: Use mathematics to show? What do you mean?

> Interviewer: Like using numbers to represent the graph.
>
> Ming: Like five is the same as five?
>
> Interviewer: Can you write it down?
>
> (Ming wrote '5 is the same as 5')
>
> Interviewer: Can you use a symbol to replace the words here?
>
> Ming: oh, I can use the equal sign (then Ming wrote '5 = 5')

Unlike Yang who wrote '5 = 5' straightway, Ming needed more prompt to reach the answer. It might attribute to that he did not quite understand what he was required to do. Nevertheless, Ming was able to eventually write '5 = 5'. In this sense, together with the above mentioned alternative approach to task 3 and 4 provided by Ming, it could be argued, like Yang, Ming's understanding of the equal sign reached the flexible operational view suggested by Matthews et al. (2012).

Two students were asked which part of the activity helped them learn most. Yang revealed that the most impressive part was that Ms Q let them draw their own symbol to indicate the same quantity of two sides, and he explained that Ms Q let them do this first then introduce the formal the equal sign, so he had a strong impression that this is a symbol to show the equivalence of both sides. Ming provided a different response by suggesting that he thought the teacher's emphasis on that two short horizontal lines must be the same length when drawing the equal sign helped his understanding. Ming explained that the way of drawing the equal sign pressed a bidirectional sense of the equal sign to him. Ming and Yang's response made the connection between their conception of the equal sign and the learning activity, illustrating the important parts of the activity that help their learning. It is worth noting the way of drawing the equal sign and students' own construction of symbol are two well-mentioned parts of the activity in student interview. This finding tends to suggest these two parts took effect in supporting students' bidirectional conception of the equal sign. As will be seen next, in the teacher interview, the way of drawing equal sign is also mentioned.

*Teacher Interview*

In the interview with Ms Q, she first confirmed Yang and Ming's learning progress, as she stated "I believe both Yang and Ming understand the equal sign is to show two sides are equivalent quantitatively", and she followed, "Not only Yang and Ming, I think all other students also have the same understanding of the equal sign." These words echoed this research's interpretation to students' work samples and interviews, that all students were developing bidirectional conception to equal sign. Ms Q also mentioned Ming's mistake made in task 1, as she said, " some students like Ming had a mistake on counting, but they understand the concept of the equal sign." Ms Q also revealed her opinion about this pedagogical approach, as shown below,

> I think it is good to start with comparing the quantity of the real objects on cards such as apples and pears, it can give students an impression that on this lesson we focus on comparing the quantities of two sides. Then we let students put their own symbols to show this equivalence, I believe this step is important since it stars making connection between students' words such as 'the same number' and an abstract symbol. While not every student can draw '=' sign at this stage, they understand this symbol is to indicate the sameness. Lastly, I think emphasising the way to draw the equal sign, like two short horizontal lines with the same length, is most impressive to me, because it is making the equal sign that is an abstract mathematical symbol more sense to students, as they would think like "oh, these two short lines are the same, so that's why this one is an equal sign".

The except above tends to show that Ms Q highlighted the step in which students construct their own sign as she considers it progressively step students towards the abstract symbol. This finding echoes Sun (2019): in the teaching reflective journal, a teacher wrote that this step pressed to students a conception that the sign they were drawing represented the equivalence of amount on both sides. Furthermore, it appears that Ms Q values the way of drawing the equal sign the most. She considers this step is crucial for the students' sense making of the representation of formal symbol of the equal sign since students can visualise the meaning. This result also concurs with Sun (2019), in which the way of drawing the equal sign is reportedly to make students consider "it was very appropriate that the equivalent relationship was represented by two short horizontal lines with equal length" (p.55). It is worth mentioning that the effect of the way of drawing the equal sign is highlighted by both students and Ms Q, and this step appears has not been underscored by previous literature. Therefore, questions could be asked: what the value of the way of drawing the equal sign brings to students' learning and how this step supports students' understanding in detail. This might be worthy of further investigation in future research.

## Conclusion

By gathering the data from different perspectives (students, the teacher and the researcher), this paper showcases how Chinese kindergarten students develop the bidirectional conception of the equal sign. The data suggests the pedagogical approach mentioned in Sun (2019) helps students grapple the bidirectional sense towards the equal sign and achieve the 'flexible operational level' of the equal sign. While students need to continue developing 'comparative relational level' understanding of the equal sign when they start doing arithmetic, this research argues the pedagogy introduced here provides a solid foundation for students' comprehensive relational understanding of the equal sign.

Furthermore, this study discovers that the emphasis on the way of drawing (two horizontal lines with the same length) of the equal sign is widely mentioned by students and the teacher as supporting students' understanding of the equal sign. As mentioned above, this step seemingly has not been noticed by previous literature, therefore how it contributes to students' understanding and so might warrant further investigation.

Finally, this study recognises that this is a small-scale study, so the effectiveness of pedagogical approach proposed in this research could be further examined in possible large-scale quantitative research.

## References


Blanton, M., Otálora, Y., Brizuela, B. M., Gardiner, A. M., Sawrey, K. B., Gibbins, A., & Kim, Y. (2018). Exploring kindergarten students' early understandings of the equal sign. *Mathematical Thinking and Learning*, *20*(3), 167–201.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Heinemann.

Clarke, D. (1998). Studying the classroom negotiation of meaning: Complementary accounts methodology. In A. Teppo (Ed.), *Qualitative research methods in mathematics education*. Monograph number 9 of the *Journal for Research in Mathematics Education* (pp. 98-111+164-177). Reston, VA: National Council of Teachers of Mathematics. https://doi.org/10.2307/749949

Hamilton, L., & Corbett-Whittier, C. (2012). *Using case study in education research*. Sage.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*(3), 317–326.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it. *The Mathematics Educator*, *8*(1), 139–151.

Knuth, E., Stephens, A., Blanton, M., & Gardiner, A. (2016). Build an early foundation for algebra success. *Phi Delta Kappan*, *97*(6), 65–68.

Lenz, D. (2022). The role of variables in relational thinking: an interview study with kindergarten and primary school children. *ZDM–Mathematics Education*, *54*(6), 1181–1197.

Li, X., Ding, M., Capraro, M. M., & Capraro, R. M. (2008). Sources of differences in children's understandings of mathematical equality: Comparative analysis of teacher guides and student texts in China and the United States. *Cognition and Instruction*, *26*(2), 195–217.

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education*, *43*(3), 316–350.

McNeil, N. M. (2008). Limitations to teaching children 2+ 2= 4: Typical arithmetic problems can hinder learning of mathematical equivalence. *Child Development*, *79*(5), 1524–1537.

McNeil, N. M., Fyfe, E. R., & Dunwiddie, A. E. (2015). Arithmetic practice can be modified to promote understanding of mathematical equivalence. *Journal of Educational Psychology*, *107*(2), 423.

Ralston, N., & Li, M. (2022). Student conceptions of the equal sign: Knowledge trajectories across the elementary grades. *The Elementary School Journal*, *122*(3), 411–432.

Sun, J. (2019). Beyond the arithmetic operation: How the equal sign is introduced in Chinese classrooms. *Mathematical Association of Victoria annual conference proceedings*. Melbourne: Australia.

# Primary Pre-Service Teachers' Beliefs About Challenging Mathematical Tasks

Tammy Tran
*John The Baptist Primary School*
tammy.tran001@syd.catholic.edu.au

Janette Bobis
*The University of Sydney*
janette.bobis@sydney.edu.au

We explored primary pre-service teachers' beliefs about challenging mathematical tasks and the role they perceived their initial teacher education played in influencing those aspects. Fifty-seven pre-service teachers completed an online questionnaire, and four participants were individually interviewed. Results showed that most participants recognised the importance of teaching with challenging mathematical tasks even prior to exposure to such content in their teacher education program. The teacher education program was perceived to have positively impacted final year pre-service teachers' perspectives about challenging tasks. Implications for teacher education regarding challenging tasks are discussed.

Challenging mathematical tasks are usually thought-provoking problems that are cognitively demanding and initially difficult to solve (Sullivan et al., 2016). Emerging research indicates that challenging mathematical tasks are important for all students in the primary mathematics classroom because they promote conceptual understanding, learner autonomy and mathematical reasoning (Bobis et al., 2021; Russo et al., 2020; Sullivan et al., 2016). Thus, there is an important need to encourage and support teachers' use of challenging mathematical tasks. Previous research has shown that many primary teachers are reluctant to implement challenging mathematical tasks because of their own beliefs surrounding such tasks including a fear that if students struggle doing mathematical tasks, it will lead to poor engagement (Cheeseman et al., 2013; Ingram et al., 2020). Understanding what beliefs primary pre-service teachers' (PSTs) hold is essential because it is these beliefs that may influence their instructional practices in their future classrooms. Previous research has explored the beliefs of prospective teachers and practicing teachers towards mathematics and teaching mathematics more generally (Grootenboer & Marshman, 2016; Maasepp & Bobis, 2014). However, little research exists specifically on primary PSTs' beliefs on challenging mathematical tasks. Identifying such information is vital for PSTs to reflect and change their beliefs to create classrooms where all students can engage in challenging mathematical tasks. There is a significant need to study what prospective teachers think or believe about challenging mathematical tasks and whether initial teacher education (ITE) courses have an impact on reshaping or changing their beliefs.

The aim of the study reported here was to explore (1) primary PSTs' beliefs about challenging mathematical tasks, and (2) their perceptions of how their ITE courses impacted their beliefs and knowledge about such tasks.

## Literature Review

Two areas of research provide the background to this study: teachers' (including PSTs') beliefs about the teaching and learning of mathematics, and research about challenging mathematical tasks.

According to Maasepp & Bobis (2014), beliefs refer to an individual's opinion on a specific issue or practice that is considered true. Regarding PSTs' beliefs about mathematics, research indicates that they are quite narrow and rigid and can have a negative impact on their future teaching and students' learning (Liljedahl, 2009). Maasepp and Bobis's (2014) research showed that PSTs' beliefs affected the way they felt, acted, and thought about mathematics and could impact their knowledge development and future experiences with mathematics. The study also confirmed that experiences provided within ITE programs have the potential to reshape PSTs' beliefs about key aspects of teaching and learning mathematics.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 493–500). Newcastle: MERGA.

Challenging tasks provide students with the opportunity for prolonged thinking and reason promoting rich student-centred learning involving productive struggle, persistence and risk-taking (Russo et al., 2019; Clarke et al., 2014). Risk-taking resonates with Growth Mindset theory, where taking risks and making mistakes is part of learning by persisting and making the effort to gain deeper understanding (Dweck, 2008). Challenging tasks generally involve open-ended designed tasks or tasks that have more than one solution or solution strategy for all students to access tasks starting from a "low-floor" to their "high-ceiling" potential and take time to solve with limited teacher instruction (Bobis et al., 2021; Ingram et al., 2020).

The very nature of challenging mathematical tasks could be the reason why teachers are reluctant to pose challenging tasks to their students. For example, open-ended tasks are not always readily available to teachers. They may not know how or be confident to design such tasks themselves. Teachers may not understand the benefits of productive student struggle or have limited knowledge of appropriate instructional strategies to deliver and support students as they work on challenging tasks. Research surrounding teachers' beliefs about challenging mathematical tasks indicates that they generally reserve such tasks for highly mathematically capable students (Cheeseman et al., 2013; Russo et al., 2020).

Acquiring new knowledge or experiences may assist a teacher's beliefs to change. Changing beliefs relating to challenge in mathematics can help reshape teacher practices to include those that encourage students to productively struggle and persist working on challenging mathematical tasks (Clarke et al., 2014; Maasepp & Bobis, 2014). Notably, literature surrounding teacher beliefs about mathematics, challenge and challenging tasks reveals the effect that such beliefs can have on their instructional practices and ultimately on their students learning of mathematics. To date, there is little research that explores primary PSTs' beliefs relating to challenging mathematical tasks.

### *Research Questions and Conceptual Framework*

This study was guided by the research questions:

- What beliefs do primary PSTs hold towards challenging mathematical tasks?
- How do primary PSTs perceive that their initial teacher education program impacted their beliefs and knowledge about challenging mathematical tasks?

The study was designed from a constructivist theoretical perspective (Cobb, 1994). Namely, the construction of primary PSTs' beliefs about challenging tasks was viewed as being influenced by a range of factors and experiences. In accordance with this view, we used Maasepp and Bobis' (2014) adapted framework of primary PSTs' mathematical beliefs to structure the survey items and interview questions. The framework represents how various elements can impact PST's beliefs, including their prior schooling, current ITE courses, their knowledge and confidence of mathematics, and experiences teaching with challenging mathematical tasks.

## Methodology

All students enrolled in The University of Sydney's Bachelor of Education (Primary) ITE program across years one to four of the degree were invited to participate in an online survey via an email sent from the university's online Learning Management System. Respondents to the questionnaire consisted of 57 adults (49 females, 7 males and 1 other), representing approximately 13% of the total number of PSTs enrolled in the program. Of the 57 participants, 53 were aged between 18-24, one was aged between 25-34, two were aged between 35-44 and one was aged between 45-54. Across the four year groups, 18 from year one, nine from year two, 13 from year three and 17 from year 4 students responded to the questionnaire. All participation occurred on a voluntary basis and participants were able to opt out at any time. Participants who indicated their willingness on the questionnaire to participate in phase two were contacted and invited to an interview.

*Questionnaire.* The questionnaire was conducted via the platform Qualtrics and took participants 15 to 20 minutes to complete. Part A asked for biographical information of participants such as their age range, mathematics level in high school and year level of the course. Part B contained 14 closed response items adapted from a previous survey designed to assess inservice teachers' beliefs about challenging tasks (Russo et al., 2020). Using a 5-point Likert scale (1 Strongly Disagree to 5 Strongly Agree), PSTs were asked to respond to measures of beliefs about mathematics (e.g., *I believe if I work hard, I will be able to perform better in mathematics),* about challenging tasks *(e.g., Challenging mathematical tasks enable me to think more creatively),* their learning about challenging tasks *(e.g., My university units have helped me to understand challenging mathematical tasks),* and of teaching students with challenging tasks *(e.g., It is important for my students to struggle in mathematics before I intervene).* Part C required an open-ended response asking PSTs to elaborate on why they had or had not changed their views about challenging tasks in the past few years.

*Interview.* Participants from each year level were invited to a semi-structured interview that was guided by open-ended questions and discussion prompts to further understand participants' beliefs towards mathematical tasks. Semi-structured interviews have in-built flexibility during data collection to adapt to respondents and situations to elicit the PSTs' beliefs among different year levels of the degree (Punch & Oancea, 2014).

Pseudonyms are used to report the interview data. No student in the second year of the program accepted the invitation to be interviewed. The interviews were conducted via zoom, which allowed for scheduling flexibility and availability of participants. All interviews were audio-recorded with participants' consent then transcribed by the first author.

*Analysis*. Descriptive statistics were used to summarise the biographical information collected from Part A in the questionnaire. Part B also used descriptive statistics such as percentages, mean scores and measures of frequency to analyse items requiring quantitative responses. The interview recordings were re-visited multiple times to ensure accuracy of transcripts and obtain a better understanding of participants' thoughts and beliefs. Participant responses were coded using a deductive coding process to account for information relevant to each element in the conceptual framework and corresponding interview questions (Braun & Clarke, 2013). The analysis was approached with pre-empted responses and anticipated themes relevant to the beliefs of PSTs. An inductive analysis followed to apprehend unexpected codes that emerged in the data and connecting it to prior research (Fereday & Muir-Cochrane, 2006).

## Results

*Questionnaire*. Table 1 summarises participants' responses to item 6 (prior learning relating to challenging mathematical tasks) in Part A of the questionnaire. Other information gleaned from Part A was included in the description of participants.

Table 1 data reveals that nearly 30% of PSTs had no prior experiences learning about challenging tasks. Of the 71% who had prior experience, over half of them had some form of exposure as part of their ITE course. Approximately 35% of respondents have either read or had experienced PL about challenging tasks when on their professional placements in schools. These findings correspond with the fact that over 31% of respondents were in their first year of their ITE and had not yet undertaken any mathematics education courses.

**Table 1**

*Questionnaire Participants' Biographical Data (N=57)*

| Item | Questionnaire Item | Responses | No. (%) |
|---|---|---|---|
| 6 | Prior learning relating to challenging mathematical tasks | No prior experiences | 25 (29.4) |
| | | PST education Unit of Study | 28 (32.9) |
| | | Professional readings | 17 (20.0) |
| | | Mathematics teacher conferences | 2 (2.4) |
| | | School-based PL | 13 (15.29) |

Of the 71% of the respondents who had prior experiences with challenging tasks, sixteen elaborated on their reasons in the open response section of the questionnaire as to why their beliefs about them had changed in the past few years. Four PSTs specifically referred to a mathematics education unit of study or the readings associated with a unit as having helped develop more positive beliefs towards challenging tasks. These changes were associated with a deeper knowledge of what challenging tasks are, how "they can be beneficial to students' learning" of mathematics and feeling "more equipped to teach" and "confident" with challenging tasks. Three respondents described how their beliefs about challenging tasks had changed because their view "of maths has changed". For instance, one respondent claimed she had "started to see mathematics as a creative subject that allows for open exploration rather than having a single answer". Several PSTs reported changed beliefs because they had changed their beliefs about how students learn mathematics and referred to "student participation", "engagement" and the "importance" of challenge when learning mathematics for conceptual understanding. Overall, survey respondents who reported changes to their beliefs about challenging mathematical tasks in the last few years, indicated that these changes were mostly because of new knowledge acquired either from their ITE course, professional readings or other PL that helped them recognise the benefits of challenging tasks hold for students' conceptual understanding of mathematics.

Table 2 summarises participants' beliefs about mathematics, challenging tasks, learning, and teaching challenging tasks from Part B of the questionnaire. The data shows that participants generally possess a growth mindset in terms of their own learning of mathematics as indicated by the high and low mean scores of items 16 and 8 (3.68 and 1.74 respectively). Responses to the three items (9, 13, 17) intended to measure participants' beliefs about doing challenging mathematical tasks indicated that most held moderate to moderately high positive beliefs. Similarly, in terms of PSTs' learning about challenging tasks during their ITE course, responses were moderate to moderately high. This is a positive result given that almost 30% of respondents were yet to experience any mathematics methods courses. It seems that ITE units have assisted PSTs' understandings of what challenging tasks are and helped them appreciate their affordances (e.g., item 14, mean 3.42).

PSTs' beliefs about personally teaching challenging tasks also reflect a growth mindset as shown in responses to items 15 and 20, which acknowledge the importance of challenging tasks and students need to expend effort to perform better in mathematics. Most respondents disagreed with item 11, indicating that they believed challenging tasks were not just for gifted and talented students. However, most respondents agreed with item 21, which is the opposite belief to what would be expected of someone comfortable using challenging tasks to teach mathematics to all students. This ambiguity in the findings could be explored in future research.

**Table 2**

*Participants' Mean Responses to Part B of the Questionnaire*

| Item | Questionnaire Item | Mean |
|---|---|---|
| Beliefs about mathematics | | |
| 8 | There will always be some people who will never 'get' math concepts no matter how hard they try. * | 1.74 |
| 12 | I consider myself to be a 'maths' person. | 2.12 |
| 16 | If I believe I work hard, I will be able to perform better in mathematics. | 3.68 |
| Beliefs about challenging mathematical tasks | | |
| 9 | I like solving mathematics problems that can be solved in many different ways. | 3.33 |
| 13 | I like tasks where I do not know the answer straightaway and need to spend time to solve. | 2.67 |
| 17 | Challenging mathematical tasks enable me to think more creatively. | 3.23 |
| Beliefs about the learning of challenging mathematical tasks | | |
| 10 | My university units have helped me to understand challenging maths tasks. | 2.89 |
| 14 | My university units have helped me see the benefits of teaching with challenging mathematical tasks. | 3.42 |
| 18 | My personal experiences of learning with challenging mathematical tasks have shown me that they are too difficult and unnecessary in learning mathematics. * | 1.44 |
| Beliefs about personally teaching challenging mathematical tasks | | |
| 11 | Challenging maths tasks should be reserved for gifted and talented students. * | 0.70 |
| 15 | It is important for my students to struggle in mathematics before I intervene. | 3.37 |
| 19 | I think implementing challenging tasks will be difficult as students will become disengaged and struggle. * | 2.47 |
| 20 | It is important to teach primary school students with challenging maths tasks. | 3.96 |
| 21 | I think students should master basic mathematical facts before they tackle challenging tasks. * | 4.14 |

*Reverse-scored item.

*Interviews.* Interview data were first categorised under three domains relevant to the conceptual framework: beliefs, ITE, and implementing challenging tasks in the classroom. Table 3 summarises the codes and themes identified during analysis and provides illustrative quotes. Due to page limitations, interview data are interpreted in the Discussion section.

**Table 3**

*Codes and Themes Identified During Analysis of Interviews*

| Domain | Theme | Code | Sample quotes |
|---|---|---|---|
| Beliefs | Growth mindset vs. Fixed mindset | Struggle | "productive struggle came about by the challenge question—students had to figure out a method" (Casey year 3) |
| | | Challenge | "There are certain people who are predetermined to enjoy a challenge more than others … there are others who may or may not enjoy the challenge and would prefer it to be something easier" (Anthony year 1) |
| | Past experiences in teaching / learning mathematics | Gifted and talented students, enrichment—for everyone | "I think all of them. There's no reason why not everybody could be involved in something" (Anthony year 1) |
| | Beliefs about teaching practice | Teaching for understanding | "I think there may be some problems in comprehension if you do it, especially with early stage 1 or maybe stage 1… especially like big-worded questions" (Anthony year 1) |
| | | | "When it comes to the elements in mathematical concepts, the key underlying concept has to be clear… the other elements can be challenging" (Lexi year 4) |
| Impact of initial teacher education (ITE) | Exposure to challenging mathematical tasks | Primary Education Degree | "I haven't heard of challenging tasks, but it sounds interesting" (Anthony year 1) |
| | | | "I have heard of challenging tasks during the course of my degree in Primary Education… over the years, I realise the fact that its open-ended nature actually helps every student" (Susan year 4) |
| | Mathematical knowledge for teaching | Using representations, confidence | "if you use a wide range of fraction models, like the area model or the discrete model, it helps the knowledge be more solid and its consolidated" (Lexi year 4) |
| Implementing challenging tasks | Students' disengagement | Not enjoyable, too challenging/ difficult | "Challenging tasks can possibly negatively impact the kids who are struggling just maybe going I can't, I give up, this is way too hard" (Casey year 3) |
| | Teacher guidance and differentiation | Teaching strategies (enabling/extend prompts) | "you can use a wide range of fraction models, like the area model or the discrete model" (Lexi year 4) |

## Discussion and Conclusion

Regarding RQ1 and PSTs' beliefs about challenging mathematical tasks, questionnaire results showed participants generally had confidence in themselves to succeed and expend effort to perform better in mathematics revealing a growth mindset towards their own learning (Dweck, 2008). However, interview data revealed that a fixed mindset was present in some PSTs in terms of student learning with challenging tasks. Anthony (year 1) believed that some students may be "predetermined to enjoy a challenge more than others…". Similarly, Casey (year 3) expressed a belief reminiscent of some practicing teachers that less able students might become disengaged by challenging tasks (Ingram et al., 2020). Nevertheless, questionnaire data revealed most PSTs recognised the importance of teaching with challenging mathematical tasks. This belief is also reflected in the final year PST interviewees' responses.

In terms of RQ2, and if PSTs perceived their ITE had impacted their beliefs and knowledge about challenging tasks, questionnaire data highlighted that overall, participants held positive beliefs towards challenging mathematical tasks even though nearly 30% had not had any prior learning experiences with them. Meanwhile, 32.9% of participants have been exposed to challenging mathematical tasks in their ITE unit/s of study contributing to their belief about the teaching and learning of challenging tasks. These findings resonate with interview data, where the first year PST reported that he had "not heard of" or had limited exposure to challenging tasks. In contrast, PSTs in the final year of the program indicated familiarity with them and could relate implementation strategies to differentiate learning involving challenging tasks (e.g., using representations, and extending and enabling prompts). Through the ITE, PSTs like Susan recognised the benefits of such tasks for "every student". Corresponding with what Maasepp and Bobis (2014) and Lijedahl (2009) stated, ITE can impact PSTs' beliefs.

One interpretation of the variation between first and final year PSTs' beliefs and knowledge relating to challenging tasks is that exposure to the ITE methods courses and information about challenging tasks had positively impacted the perspectives of final year PSTs. The revised perspectives of challenging tasks were more notable in final year PSTs' responses. For instance, Susan noted that "over the years, I realise the fact that its (challenging tasks) open-ended nature actually helps every student". These findings emphasise the potential impact ITE can have in reshaping PSTs' beliefs and reveals the importance of explicitly addressing the use of challenging mathematical tasks in ITE.

Limitations of this study include the fact that the sample size was quite small and non-representative of the majority of primary PSTs in Australia. Namely, PSTs were from only one university. Nonetheless, PSTs from all year levels of the program were represented in the questionnaire. Future studies with a greater number of questionnaire respondents would increase the reliability of results. Similarly, the interviews were conducted with only four participants, one from each year group except for a second year primary PST. Whilst not a broad cross section of students in the primary ITE course, the number was sufficient to reveal PSTs' beliefs about challenging tasks and how their ITE has the potential to impact their beliefs and knowledge.

The aims of this study were to explore primary PSTs' beliefs about challenging mathematical tasks and their perceptions of how their ITE courses have impacted their beliefs and knowledge about such tasks. The findings indicate that primary PSTs in this study generally held positive views towards the teaching and learning of challenging mathematical tasks. These are important findings because new curricula around the globe focus on the skills of problem solving and reasoning—processes inherent when working with challenging tasks. ITE can change or reshape beliefs by raising awareness of the importance and affordances of including challenging mathematics instruction.

## References


Bobis, J., Russo, J., Downton, A., Feng, M., Livy, S., McCormick, M., & Sullivan, P. (2021). Instructional moves that increase chances of engaging all students in learning mathematics. *Mathematics, 9*(6), 582. https://www.mdpi.com/2227-7390/9/6/582

Braun, V., & Clarke, V. (2013). *Successful qualitative research: A practical guide for beginners.* London, UK: Sage Publications.

Cheeseman, J., Clarke, D., Roche, A., & Wilson, K. (2013). Teachers' views of the challenging elements of a task. In V. Steinle (Ed.), *Proceedings of MERGA 36 annual conference of the Mathematics Education Research Group of Australasia.* Melbourne: MERGA.

Clarke, D., Roche, A., Cheeseman, J., & Schans, S. (2014). Teaching strategies for building student persistence on challenging tasks: Insights emerging from two approaches to teacher professional learning. *Mathematics Teacher Education & Development*, *16*(2), 46–70.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23(*7), 13–20. https://doi.org/10.3102/0013189X023007013

Dweck, C. (2008). *Mindsets and math/science achievement*. New York: Carnegie Corporation of New York, Commission on Mathematics and Science Education.

Fereday, J., & Muir-Cochrane, E. (2006). Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *International Journal of Qualitative Methods, 5*(1), 80–92. https://doi.org/10.1177/160940690600500107

Grootenboer, P., & Marshman, M. (2016). *Mathematics, affect and learning middle school students' beliefs and attitudes about mathematics education*. Singapore: Springer.

Ingram, N., Holmes, M., Linsell, C., Livy, S., McCormick, M., & Sullivan, P. (2020). Exploring an innovative approach to teaching mathematics through the use of challenging tasks: A New Zealand perspective. *Mathematics Education Research Journal*, *32*(3), 497–522. https://doi.org/10.1007/s13394-019-00266-1

Liljedahl, P. (2009). Components of mathematics teacher training. In D. L. Ball, & R. Even (Eds.), *The professional education and development of teachers of mathematics* (pp. 25–31). New York: Springer.

Maasepp, B., & Bobis, J. (2014). Prospective primary teachers' beliefs about mathematics. *Mathematics Teacher Education and Development*, *16*(2), 89–107.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Sage Publications.

Punch, K., & Oancea, A. (2014). *Introduction to research methods in education.* Sage Publications.

Russo, J., Bobis, J., Downton, A., Hughes, S., Livy, S., McCormick, M., & Sullivan, P. (2019). Teaching with challenging tasks in the first years of school: What are the obstacles and how can teachers overcome them? *Australian Primary Mathematics Classroom*, *24*(1), 11–18.

Russo., Bobis., J., Downton, A., Hughes, S., Livy, S., McCormick, M., & Sullivan, P. (2020). Elementary teachers' beliefs on the role of struggle in the mathematics classroom. *Journal of Mathematical Behavior*, *58*, 100774. https://doi.org/10.1016/j.jmathb.2020.100774

Sullivan, P., Borcek, C., Walker, N., & Rennie, M. (2016). Exploring a structure for mathematics lessons that initiate learning by activating cognition on challenging tasks. *The Journal of Mathematical Behavior, 41*, 159–170.

# Teachers' Experiences of Developing Ethnomathematical Ideas for Classroom Teaching: A Case Study in the Solomon Islands

Raynier Tutuo
*University of the South Pacific*
rayniertuto34@gmail.com

Hem Dayal
*University of the South Pacific*
hem.dayal@usp.ac.nz

Robin Averill
*Victoria University of Wellington*
robin.averill@vuw.ac.nz

Kay Owens
*Charles Sturt University*
kowens@csu.edu

We report on how professional development on ethnomathematics can support teaching and learning in mathematics. The perspectives of six teachers are analysed, highlighting three key themes: designing ethnomathematics activities, ethnomathematics activities, and teachers' perceptions on ethnomathematics activities. Firstly, the findings on teachers' motivation for designing ethnomathematics activities are presented, followed the different types of ethnomathematics activities teachers developed. Finally, teachers' perceptions of ethnomathematics activities are explored. The findings offer potential for implementing teaching approaches that can support teaching and learning of mathematics.

Solomon Islands (SI) consists of diverse groups of people from Melanesian, Polynesian, Micronesian origins, and other ethnic groups. SI students bring rich cultural values to the school. By understanding and recognising these important cultural ideas in mathematics, teachers can help students connect important concepts in mathematics to their everyday experiences (Bills & Hunter, 2015).

Ethnomathematics is the study of the relationships between culture and mathematics (D'Ambrosio, 2001). From an ethnomathematical point of view, students will likely understand their mathematical learning if mathematics is taught in a way that is familiar to them. Ethnomathematics fits well within the constructivist and sociocultural theory of learning where prior knowledge and interaction is of paramount importance. Given the importance of ethnomathematics and the fact that SI is largely a society that values its cultures, this study focuses on guiding secondary mathematics teachers to develop activities from SI cultural context with a view to helping students in their learning. As part of a larger study, a professional development (PD) workshop on ethnomathematics was carried out by the first researcher with a small sample of SI secondary mathematics teachers. After the PD, teachers made their own mathematics examples from SI cultural contexts. In this paper, we explore the research question:

- When given professional learning support, how do a small sample of SI mathematics teachers derive their own ethnomathematical teaching ideas?

## Literature Review

In recent years, there has been a growing interest in ethnomathematics, with researchers from around the world working to better understand the ways in which mathematics is embedded in cultural practices (D'Ambrosio, 2001; Owens, 2012; Rosa et al., 2016). Ethnomathematics recognises the importance of cultural diversity and the value of diverse ways of knowing and learning mathematics. It also aims to challenge the dominant Eurocentric perspective of mathematics education and promote more inclusive and culturally responsive approaches to teaching (D'Ambrosio, 2001).

Studies have highlighted the importance of ethnomathematics for enriching mathematics education. Ethnomathematics can help promote student engagement and cultural identity (Suherman & Vidakovich, 2022), cultural diversity and inclusivity (Brant & Chernoff, 2015; Matang & Owens,

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 501–508). Newcastle: MERGA.

2014), and help students see the relevance of mathematics in their cultural traditions (Weldeana, 2016).

These findings apply to the Pacific Island cultures. Walls (2006) argued that incorporating Indigenous knowledge into the mathematics curriculum can help promote cultural identity and pride among students. Hunter and Anthony (2011) and Averill (2012) emphasized the significance of recognizing the value of Maori and Pacific knowledge systems in mathematics education, stating that their mathematical knowledge is deeply embedded in their cultural practices, while Furuto (2014) supported this idea and argued that understanding the mathematical practices of Pacific cultures can enrich mathematics education for Pacific Islander students and can improve student engagement and achievement. Also Matang and Owens (2014) argued that understanding the relationship between Papua New Guinea and ethnomathematics is crucial for creating culturally responsive and relevant mathematics education in the country.

While many of these studies highlight the importance of incorporating cultural practices and knowledge into mathematics education to promote cultural diversity, inclusivity, and student engagement, they do not tell us much on how ethnomathematical ideas could be taken up by teachers. This study aims to uncover some ethnomathematical ideas from the SI by allowing teachers to think about and come up with their own ethnomathematics ideas after engaging them in a short PD. PD is a way of enhancing teachers' knowledge and practices (Timperley, Wilson, Barrar, & Fung, 2007). The current study engaged a small sample of SI mathematics teachers in a PD activity to introduce them to the idea of ethnomathematics. The next section gives further details on this.

## Research Methods

This study focused on sociocultural perspectives and drew on the interpretative framework. Interpretive case study approach was used as it provided an in-depth exploration of a small sample of SI secondary mathematics teachers' understandings of ethnomathematics. Case studies are useful because they emphasise understanding the meaning and context of social phenomena from the perspective of the participants involved (Cohen, Manion, & Morrison, 2018). Another reason we chose a case study approach is that we wanted to see how we could use online PD approaches as we were affected by Covid-19 lockdown during the time of the data collection. There were two case study secondary schools in the SI. These schools were purposively chosen because they had good internet facilities to enable the principal researcher to maintain regular virtual contact with the six teachers who volunteered to be part of this study. Out of the six, four were males and two females. All of the teachers were experienced mathematics teachers, with average teaching experience of approximately 10 years. The study reported in this paper is focused on Phase 3 of the larger study. In this phase, six teachers were asked to develop their own ethnomathematical activities from SI cultural context.

All these teachers had undergone a PD with the principal researcher who is from the SI and speaks the same language, Solomon Pijin. Using Pijin facilitated a constructive relationship between the researcher and the participants. This enabled the participants to express their thoughts and ideas openly and honestly, fostering dialogue. As the small-scale investigation occurred during the mandated lockdown in Fiji and SI, the professional PD and semi-structured interviews were carried out virtually. The PD included topics such as the definition of ethnomathematics, examples of ethnomathematics from around the world and SI context. Prior to the actual PD, a pilot study was done to trial the actual PD. The pilot teachers were not part of the study reported in this paper. At the end of this phase of the larger study, the participants were able to come up with 15 ethnomathematics activities with lesson plans. Later the principal researcher interviewed them regarding their views on developing ethnomathematics. Each interview lasted approximately 20 minutes. The researcher translated Solomon Pijin interviews into English and analysed them thematically. This research was covered by USP ethics clearance and we maintained strict confidentiality and anonymity in reporting the findings. As such, we use pseudonyms T1–T6 for the participants.

## Findings and Discussion

The findings of this investigation are organised around teachers' motivations for designing ethnomathematics activities, ethnomathematics activities developed by the teachers, and teachers' perceptions of ethnomathematics activities. Extracts from interviews are provided to show teachers' perspectives within these themes.

### *Motivations for Designing Ethnomathematics Activities*

The data provided shed light on why teachers' chose the particular ethnomathematical activities. The findings suggest that teachers created these materials for a variety of reasons, including addressing teaching challenges, improving student understanding, and developing skills. All the participants mentioned designing activities that address the challenges their students face and improve their understanding of the subject matter. Additionally, one participant designed activities to help students develop skills they can use in the future, while another focused on incorporating student interests into their lessons. All of these approaches demonstrate a commitment to personalising instruction to meet the diverse needs of students.

The fact that four out of six participants mentioned that they design activities because their students struggle with certain mathematical topics highlights the importance of addressing teaching challenges to help students learn effectively:

> T1: What makes me to come up with the activities and questions is seeing that algebra concept is difficult for the students in my classes.
>
> T2: I see that my students are struggling with understanding measurements and shapes.
>
> T5: I come up with the idea for making this lesson, I find that many students in this country find it difficult to understand Pythagoras trigonometry.

Hattie (2009) argued that designing activities and questions that target specific learning difficulties can be an effective way to improve student learning. In our study, the teachers created activities and questions to improve students understanding and assist the school in other areas. Participants mentioned that they designed these learning experiences to help their students understand mathematics better, as well as to help the school manage food consumption in the dining hall or the school farm:

> T6: Since I find it hard to work with the school farmers to find out how many plots we need to work on for the semester, I came up with this activity.

T6, who is a mathematics teacher but also assists in managing the school farm, was able to design an ethnomathematical activity on potato farming. This finding highlights the importance of designing activities and questions that have real-world applications and can help students develop problem-solving and critical thinking skills (National-Research-Council, 2012). Skill development is also highlighted. One participant mentioned that they design activities and questions to help students gain skills that they can use in the future:

> T5: The reason I came up with this teaching house profile is because Pythagoras is there, and they will learn a skill that will be useful to make their own house, local kitchen, or small shed in the future.

T5 prepared an ethnomathematics activity on building profile since students struggle with the concept of Pythagoras and at the same time help students gain skills for future use. This finding highlights the importance of designing activities and questions that go beyond teaching specific mathematical concepts and focus on developing transferable skills that can be used in various contexts (National-Council-of-Teachers-of-Mathematics, 2014). Lastly, some teachers created activities and questions based on their personal interests. One participant mentioned that they design these materials to learn more about their students' interests, such as their favourite food or fruit. Students were tasked to collect data on other students' favourite fruits and then relate it to statistics:

> T4: I want to know student's food interest. What food they liked so much so it is good to find out which teacher they like the most etc.

While this motivation may not directly relate to mathematics content, it highlights the importance of building relationships with students and understanding their individual needs and interests (Pekrun, Elliot, & Maier, 2009). Overall, the data provided above suggested that teachers have various motivations behind designing activities and questions for their students. These motivations can be driven by teaching challenges, improving student understanding, developing transferable skills, and personal interest. Designing effective activities and questions that meet these motivations can contribute to improved student learning and engagement.

### *Ethnomathematics Activities*

After the PD session, the teachers were tasked to develop their own mathematics activities. They developed a range of ethnomathematics activities that integrate everyday activities, local cultural practices and traditions with mathematical concepts. These activities include collecting and analysing data on favourite fruits (T4), measuring cassava garden plots (T3), measuring potato garden plots (T6) and watermelon garden plots (T3), making traditional cassava pudding (T3), finding the perimeter of shapes around the school buildings (T4), building a house profile (T5), using traditional forms of money (T1 & T2), using fruit to calculate circumference or perimeter (T2), and using traditional ornaments (T1). These activities promote an understanding of the importance of local fruits, traditional food practices, cassava as a staple crop, shapes and structures around them, traditional housing practices, and traditional forms of exchange.

The paragraph above highlights the importance and benefits of incorporating ethnomathematics activities into mathematics education in the SI (D'Ambrosio, 2001). Firstly, by incorporating these cultural practices into mathematics education, students develop essential mathematical skills while also gaining an appreciation and understanding of their local culture and traditions (D'Ambrosio, 2001). These ethnomathematics activities offer a unique and culturally relevant approach to teaching mathematics in the SI, bridging the gap between local cultural practices and mathematical concepts. Secondly, these activities are designed to connect local cultural practices and traditions with mathematical concepts, making learning more meaningful and relevant for students. By integrating traditional practices such as measuring cassava garden plots, making cassava pudding, or using traditional forms of money into mathematical activities, students can see the practical applications of mathematics in their daily lives. These activities not only develop essential mathematical skills, but also promote an appreciation of local culture and traditions, helping to preserve and celebrate them (Gonzalez, Moll, & Amanti, 2005). Furthermore, ethnomathematics activities can help to create a more inclusive and culturally responsive learning environment (Rosa & Orey, 2015). By connecting mathematical concepts with local cultural practices and traditions, these activities make learning more meaningful and relevant, promote an appreciation of local culture and traditions, and create a more inclusive and culturally responsive learning environment (Rosa & Orey, 2015).

### *Teachers' Perceptions on Ethnomathematics Activities*

At the end of the three weeks of developing ethnomathematics activities, teachers were asked to reflect on their task. All teachers mentioned challenges and benefits while developing ethnomathematics activities.

Designing lesson plans and activities that are grounded in cultural contexts can present both challenges and benefits for educators. In terms of challenges, T1 and T2 identified their limited understanding of their local traditional historical culture as a major challenge due to growing up in the capital city rather than their original villages. For example, T1 explicitly stated 'I do not understand my culture'. This shows that teachers who are not well grounded in their cultural activities will have some difficulties in designing ethnomathematics activities. This finding aligns with the view that ethnomathematics should be grounded in the cultural practices and traditions of the local community (D'Ambrosio, 2002).

Another challenge was that of a lack of time. Two participants (T1 & T2) highlighted that it takes time to think and prepare the ethnomathematics activity. For example, T1 and T2 mentioned that it takes time to think, prepare and design the activities. Moschkovich (2002) argues that the preparation and implementation of ethnomathematics activities require time and effort, but can lead to a deeper understanding of both mathematics and culture. This shows that ethnomathematics education requires careful planning and consideration of cultural context. Educators must be willing to invest time and effort into researching, preparing, and implementing ethnomathematics activities in order to create meaningful learning experiences for their students. In addition, T1 highlighted other challenges such as the need to incorporate cultural artefacts into numerical contexts and the challenge of using pictures instead of real-life artefacts. Cultural artefacts can influence how we measure and quantify things, and it is important to ensure that examples used are relevant and meaningful to students from diverse cultural backgrounds. While pictures can be useful for conveying information, they may not provide the same level of detail and information as real-life artefacts and this can impact the accuracy and effectiveness of the activity. These challenges highlight the importance of careful planning and consideration when deriving ethnomathematics examples. D'Ambrosio (2001) emphasizes the importance of careful planning and consideration when selecting examples, noting that the cultural context of the examples must be taken into account to ensure that they are meaningful and relevant to learners. He argues that careful consideration of cultural context can also help to avoid perpetuating stereotypes and biases in mathematical education.

The use of modern tools versus traditional tools is another challenge mentioned by T5. For example, T5 mentioned that "I face a challenge when deriving examples from traditional buildings since they are constructed differently from modern ones." While modern tools may offer greater precision and accuracy, they can also present challenges when comparing measurements to those taken with traditional tools. This highlights the importance of understanding the tools being used and how they impact the accuracy of measurements (Eyeoglu, Ozdemir, & Eyeoglu, 2021).

Access to resources mentioned by T3 can impact the effectiveness of a measurement activity. For example, T3 highlighted that "students do not have access to measuring tools in the classroom such as rulers." Without access to measuring tools, students may struggle to engage with the activity, and the accuracy of their measurements may be compromised. This highlights the importance of ensuring that students have access to the necessary resources to engage with the activity effectively (Eyeoglu et al., 2021).

The importance of designing an activity that is fitting for the students mentioned by T4 is also a challenge. For example, T4 mentioned "One challenge I face is to design an activity that is fitting for the students." Activities that are engaging and relevant to the students are more likely to be effective in promoting understanding of measurement concepts (Boaler & Staples, 2008). Designing an activity that is both relevant and engaging can present challenges, but it is a critical aspect of ensuring that the activity is effective.

Finally, T6 mentioned that the thought of implementing the ethnomathematics activity can be a challenge. For example, T6 strongly emphasised that "the thought of successfully implementing the ethnomathematics activity into practice is a challenge." While the lesson may be prepared for only a period, the actual activity may extend beyond that initial timeframe, presenting challenges in terms of planning and execution. This highlights the importance of considering the logistics of implementing the activity and ensuring that it is feasible within the available timeframe (Ayuwanti, Marsigit, & Siswoyo, 2021).

Despite the challenges mentioned above, there are also benefits gained when developing ethnomathematics activities. Three participants (T1, T3, and T4) noted that the lesson plans used for the ethnomathematics activities helped them with self-assessment and self-learning, while also improving their teaching practices. For example, T1 mentioned "the benefits of preparing ethnomathematics activities with lesson plans are really helpful." Lesson plans have been found to help teachers in organizing and planning their teaching effectively while also providing a means for self-reflection and self-assessment (Lewis, Perry, & Hurd, 2009).

Three participants (T2, T3 and T5) remarked that the ethnomathematics activities broadened their thinking and helped them to think outside the box, leading to greater creativity in lesson planning. For example, T2 explicitly stated "it broadens my mind when relating cultural activities to mathematics." Ethnomathematics activities have been shown to be effective in broadening teachers' perspectives and helping them to think more creatively in designing and implementing their teaching strategies (D'Ambrosio, 2001).

Additionally, two other participants (T1 and T4) found ethnomathematics activities to be motivating and exciting, providing opportunities for culturally-relevant examples and resources that could be integrated into their teaching. For example, T4 highlighted "developing ethnomathematics activities helped me to understand that mathematics can be an exciting subject." Ethnomathematics has been found to be motivating for students, particularly when it provides culturally relevant examples and resources that students can relate to and understand (Rosa & Orey, 2011).

Another participant (T5), an untrained teacher, expressed gratitude for the training sessions, which helped him to learn about lesson planning and activities that could be used in their classroom. For example, T5 argued that "Even untrained teachers can benefit from this process, and I have learned a lot during my time spent with this program." PD programs provide teachers with opportunities to acquire new skills, knowledge, and strategies for teaching, which they can then apply in their classrooms (Wei, Darling-Hammond, Andree, Richardson, & Orphanos, 2009).

Finally, one participant (T6) highlighted the potential benefits of hands-on activities, such as a potato planting project, which could help students in rural areas better understand the subject matter and provide valuable information to the school about the sustainability of the farm. For example, T6 affirmed that "a significant benefit is that it helped both the students and the school as a whole." The use of hands-on activities in ethnomathematics has been shown to be effective in promoting student learning and understanding, particularly in rural areas where students may have limited access to resources and may benefit from experiential learning (Sobel, 2004).

Overall, these perceptions highlights the importance of considering multiple factors when deriving ethnomathematics activities, including time, the use of modern tools versus traditional tools, access to resources, activity design, and the thought of implementation challenges. By considering these factors, it is possible to create ethnomathematics activities that are engaging, effective, and relevant to the students. On the other hand, the benefits of developing ethnomathematics activities with lesson plans as discussed by the participants in the study are consistent with the existing research on the use of ethnomathematics in education. These findings highlight the potential of ethnomathematics to enhance teaching and learning in mathematics, particularly in promoting cultural relevance and creativity in teaching practices.

## Conclusion

In this study, a group of mathematics teachers from the SI participated in PD training on ethnomathematics and were then observed as they developed their own ethnomathematics activities. The aim was to help teachers develop their own ethnomathematics activities after participating in

PD training on ethnomathematics. The research question was "When given professional learning support, how does a small sample of SI mathematics teachers derive their own ethnomathematical teaching ideas?" Three main themes were highlighted in the findings. In terms of our first theme, we concluded that developing ethnomathematics align with teachers' motivation and knowledge of their culture. The findings related to the second theme indicated that the teachers designed ethnomathematical activities to promote cultural appreciation and create a more inclusive learning environment. With respect to the final theme on teachers' perception on ethnomathematics activities, our findings suggest that despite challenges encountered during the development of ethnomathematics activities, the teachers gained important knowledge and skills in creating their own ethnomathematics activities. The above findings are from a small sample of teachers only, which is the main limitation of our study. Future research could include a larger sample size of mathematics teachers from different regions of the SI to investigate whether the findings are consistent across different contexts. A longitudinal study also could be conducted to investigate the long-term impact of the PD training on teachers' practice and students' learning outcomes.

## References


Averill, R. (2012). Reflecting heritage cultures in mathematics learning: The views of teachers and students. *Journal of Urban Mathematics Education, 5*(2), 1578–1181. doi:http://education.gsu.edu/JUME

Ayuwanti, I., Marsigit, & Siswoyo, D. (2021). Teacher-student interaction in mathematics learning. *International Journal of Evaluation and Research in Education (IJERE), 10*(2), 660–667. doi:https://files.eric.ed.gov/fulltext/EJ1299252.pdf

Bills, T., & Hunter, R. (2015). The role of cultural capital in creating equity for Pasifika learners in mathematics. In M. Marshman, A. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 109–116): MERGA.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record, 110*(3), 608–645. doi:https://www.youcubed.org/wp-content/uploads/2017/09/Creating-Mathematical-Futures.pdf

Brant, A., & Chernoff, E. J. (2015). The importance of ethnomathematics in the math class. *Ohio Journal of School Mathematics, 71*, 31–36. doi:https://kb.osu.edu/bitstream/handle/1811/78917/OJSM_71_Spring2015_31.pdf

Cohen, L., Manion, L., & Morrison, K. (2018). *Research methods in education*: Routledge.

D'Ambrosio, U. (2001). What is ethnomathematics and how can it help children in schools? *Teaching Children Mathematics, 7*(6), 308–311.

D'Ambrosio, U. (2002). Ethnomathematics and its place in the history and pedagogy of mathematics *Revista Latinoamericana de Investigacion en Matematica Educativa, 5*(1), 21–39. doi:https://www.jstor.org/stable/40247876

Eyeoglu, O. K., Ozdemir, B. G., & Eyeoglu, U. (2021). Conceptual understanding of 8th grade students based on big ideas: The case of measurement. *Journal of Pedagogical Research, 5*(4), 19–42. doi:https://doi.org/10.33902/JPR.2021472786

Furuto, L. H. (2014). Pacific ethnomathematics: Pedagogy and practices in mathematics education. *Teaching Mathematics and its Applications: An International Journal of the IMA, 33*(2), 110–121. doi:https://scholarspace.manoa.hawaii.edu/server/api/core/bitstreams/cf5cc535-95ee-4c93-8211-e29e66406bd8/content

Gonzalez, N., Moll, L. C., & Amanti, C. (2005). *Funds of knowledge: Theorising practices in households, communities, and classrooms*: Routledge.

Hattie, J. (2009). *Visible learning: A synthesis of over 800 meta-analyses relating to achievement.* Routledge.

Hunter, R., & Anthony, G. (2011). Forging mathematical relationships in inquiry-based classrooms with Pasifika students. *Journal of Urban Mathematics Education, 4*(1), 98–119. doi:http://education.gsu.edu/JUME

Lewis, C. C., Perry, R. R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*, 285–304. doi:https://doi.org/10.1007/s10857-009-9102-7

Matang, R., & Owens, K. (2014). The role of indigenous traditional counting systems in children's development of numerical cognition: Results from a study in Papua New Guinea. *Mathematics Education Research Journal, 26*, 531–553. doi:10.1007/s13394-013-0115-2

Moschkovich, J. N. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning, 4*(2–3), 189–212. doi:https://www.tandfonline.com/doi/abs/10.1207/S15327833MTL04023_5

National-Council-of-Teachers-of-Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. National Council of Teachers of Mathematics.

National Research Council. (2012). *Education for life and work: Developing transferable knowledge and skills in the 21st century*: National Academies Press.

Owens, K. (2012). Papua New Guinea Indigenous knowledges about mathematical concepts. *Journal of Mathematics & Culture,6*, 15–50. doi:https://www.researchgate.net/publication/277019441_Papua_New_Guinea_Indigenous_knowledges_about_mathematical_concepts

Pekrun, R., Elliot, A. J., & Maier, M. A. (2009). Achievement goals and achievement emotions: Testing a model of their joint relations with academic performance. *Journal of Educational Psychology, 101*(1), 115–135. doi:http://dx.doi.org/10.1037/a0013383

Rosa, M., D'Ambrosio, U., Orey, D. C., Shirley, L., Alangui, W. V., Palhares, P., & Gavarrete, M. E. (2016). *Current and future perspectives of ethnomathematics as a program*: Springer Open.

Rosa, M., & Orey, D. C. (2011). Ethnomathematics: The cultural aspects of mathematics. (Etnomatemática: os aspectos culturais da matemática). *Revista Latinoamericana de Ethnomatemática, 4*(2), 32–54. doi:https://dialnet.unirioja.es/descarga/articulo/3738356.pdf

Rosa, M., & Orey, D. C. (2015). Ethnomathematics: Connecting culturally aspects of mathematics through culturally relevant pedagogy. In S. Mukhopadhyay, & B. Greer (Eds.), *Proceedings of the eighth International Mathematics Education and Society Conference* (Vol. 3, pp. 898–911). MES8.

Sobel, D. (2004). Place-based education: Connecting classrooms and communities. *Nature and Listening, 4*(1), 1–7. doi:https://kohalacenter.org/teachertraining/pdf/pbexcerpt.pdf

Suherman, S., & Vidakovich, T. (2022). Tapis pattern in the context of ethnomathematics to assess students' creative thinking in mathematics: A rasch measurement. *Mathematics Teaching Research Journal, 14*(4), 56–79. doi:https://files.eric.ed.gov/fulltext/EJ1361683.pdf

Timperley, H., Wilson, A., Barrar, H., & Fung, I. (2007). *Teacher professional learning and development: Best evidence synthesis iteration*: Wellington: Ministry of Education.

Walls, F. (2006). Discursive practices as cultural cargo. *For the Learning of Mathematics, 26*(3), 2–8. doi:https://www.jstor.org/stable/40248540

Wei, R. C., Darling-Hammond, L., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional learning in the learning profession: A status report on teacher development in the United States and abroad*. National Staff Development Council.

Weldeana, H. N. (2016). Ethnomathematics in Ethiopia: Futile or fertile for mathematics education? *Momona Ethiopian Journal of Science (MEJS), 8*(2), 146–167. doi:https://www.ajol.info/index.php/mejs/article/view/148967/138469

# Conversations About Place Value: A Survey of Literature Across Three International Research Communities

Pamela Vale
*Rhodes University*
pamela.vale@ru.ac.za

Lise Westaway
*Rhodes University*
l.westaway@ru.ac.za

Place value is a foundational competency for primary school mathematics and for this reason we have sought to investigate what the recent and current academic conversations are around this important concept. In this paper we present a survey of literature presented in the Australasian, European and Southern African contexts through a review of purposively selected conference proceedings and journals to establish what the conversations have been about the teaching and learning of place value in these research communities from 2013 to 2022.

An understanding of place value is a foundational competency for primary school mathematics. Lambert and Moeller (2019) maintain that understanding place value is a predictor of success in primary school and for later number competencies. Recognising this importance, this paper asks: what are the conversations about place value teaching and learning in mathematics education research communities? Specifically, we examine research originating from three international communities: Australasia, Europe and Southern Africa. There is clear potential for conversations to happen across these research communities and value to be had in doing so.

Place value understanding develops over a period of time, therefore, it is important for learners to be familiarised with the base-10 decimal system in the early years. The base-10 system refers to the value of each digit as determined by the position within a number. According to Dehaene and Cohen (1999), place value understanding is based on a triple code model, meaning that there are three representations of number: understanding quantities, number words and number symbols. The focus in schooling is often on procedures, such as, the standard algorithm taught by rote (e.g., Graven et al., 2013). The importance of developing a conceptual understanding of place value requires children to be able to construct the algorithm as opposed to being reliant on a taught procedure (e.g., Benton, et al., 2018). Important in constructing algorithm is the need for teaching the place value concept through resources (manipulatives, iconic representations, and digital tools) (e.g., Larkin et al., 2019).

## Methodology

This literature survey was done with five purposively selected conference proceedings and journals that were deemed representative of the three academic research communities: Europe, Australasia and Southern Africa. The content of these publications was conceptualised as representative of the conversations about place value teaching and learning in the respective mathematics research communities. For the European context, we included the proceedings of the bi-annual Congress of the European Society for Research in Mathematics Education [CERME]. For the Australasian context we included the proceedings of the annual conference of the Mathematics Education Research Group of Australasia [MERGA] and papers published in the associated *Mathematics Education Research Journal* [MERJ]. Similarly, for the Southern African context, we included the proceedings of the annual conference of the Southern African Association for Research in Mathematics, Science and Technology Education [SAARMSTE] and papers published in the associated *African Journal for Research in Mathematics, Science and Technology Education* [AJRMSTE]. We searched for the term 'place value' in all the above-mentioned sources from 2013-2022 to identify the papers for inclusion. Only full research papers were included. In total, the corpus of papers totalled 158. We further classified the papers into 3 categories: (1) papers that focus explicitly on the teaching and learning of place value; (2) papers in which the focus is not on place

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 509–516). Newcastle: MERGA.

value, but it is an important concept in the paper and the work does contribute to conversations about the teaching and learning of place value either through its findings, implications or other commentary; (3) papers in which place value is mentioned, but the mention is inconsequential to the findings and it does not contribute to conversations about the teaching and learning of place value. Those classified as (1) or (2) were read in full to determine the contribution made by each paper to conversations about the teaching and learning of place value.

**Table 1**

*Number of Papers Reviewed and Classification of the Papers*

| Context: Proceedings / Journal | Total | Category (1) | Category (2) | Category (3) |
|---|---|---|---|---|
| Australasia: MERGA | 41 | 12 | 6 | 23 |
| MERJ | 32 | 3 | 4 | 25 |
| European: CERME | 65 | 11 | 11 | 43 |
| Southern African: SAARMSTE | 12 | 1 | 2 | 9 |
| AJRMSTE | 8 | 3 | 1 | 4 |

## Findings

In this section, we present an analysis of the papers making substantive reference to place value across the three international mathematics education research communities. We draw out the themes evident across the papers and summarise their main findings. We make reference to category (1) papers only but provide a full list of papers at: http://bit.ly/3JDiIuk.

In total, there were 15 category 1 papers from Australasia between 2013 and 2022. The themes emerging from these papers include assessment of children's place value understanding (5 papers); the use of resources (4); the role of place value understanding in learning algorithms (4) and teachers' pedagogical practices (2). Twelve are focused on children's understanding, 2 on pre-service teachers [PSTs] and one on teachers' pedagogical practices.

Hurst's (2014) study examines how a diagnostic assessment of children's place value understanding can assist pre-service teachers [PSTs] in making appropriate decisions about learning tasks and resources. Through using the assessment, the PSTs identified that children have difficulty reading and writing numbers, specifically when zero is a place holder and also struggled to interpret numbers and their values. The results show that the diagnostic assessment enabled the PSTs to think about place value in a conceptual way. Bicknell and Young-Loveridge's (2015) research focuses on two assessment tasks given to Year 1-3 children. The tasks focused on the placement of numbers on a 0-10 and 0-20 empty number lines. The results show that learners were more accurate in placing numbers on the 0-20 number line and that those with a better place value understanding were better with placing the numbers on the number lines. They suggest that teachers should make explicit the connections between different representations of two digit numbers. The paper by Gervasoni and Peter-Koop (2015) also focuses on assessment of learner knowledge, in a comparative study of the counting and whole number understanding of Grade 2 children in Germany and Australia. Their findings indicate that there was a significant difference in place value understanding at the end of Grade 1, but that the results of the two cohorts were the same after Grade 2. The authors surmise that the curriculum expectations may have some influence on the differences and suggest that more research is required on this as there may be different pedagogical approaches in the two countries. Young-Loveridge and Bicknell (2016) developed a framework to assess 5-7-year olds' place value development. They show that young learners are able to perform early place value tasks. The authors present important implications for teachers and curriculum showing that: (1) learning of the facts 5+5 and 120+10 are important for place value; and that (2) learning how to combine single digit numbers to multiples of 10; and (3) making explicit links between the digits in two-digit numbers and groups of tens and ones are important. Hurst and Hurrell (2016a) sought to develop a tool, the Multiplicate Thinking Quiz, to assist teachers in assessing mathematical reasoning. They show that students are taught procedures rather than having the opportunity to develop a conceptual understanding. Of relevance to this review is the finding that the notion of 'times bigger' is not understood by most children.

There is also a focus in the Australasian research on the role of place value understanding in the application of algorithms. Hurst and Huntley (2017) explore whether children make the connection between place value partitioning of a number and the distributive property of multiplication. Their analysis shows that children who only demonstrate a partial understanding of place value partitioning do not consistently apply the distributive property in their calculations. They offer several clear teaching implications, including that teaching should focus on establishing a link between the distributive property and place value partitioning. Downton et al. (2020) also focus on multiplication and contrast the use of a place value partitioning method with a truncated strategy of ignoring the zeros and then re-placing them in the final solution. They argue that this truncation strategy is frequently demonstrated in classrooms and reinforced in textbooks, and they show that the children who thought in this way did not understand place value partitioning and were manipulating the zeros without understanding. Jazby and Pearn's (2015) focus is similarly on multiplication algorithms and compare algorithms based on how they work with place value. They present algorithms that "suspend place value" (p. 311) but explain that this requires cognitive work to reinstate place value at the end of the calculation. Their argument is that different algorithms work with place value differently and as a result require different aspects of cognitive work and thus individual children may prefer different algorithms for different reasons related to the mental work involved. Jacobson and Simpson (2019) turn their attention to PST's conceptions of multi-digit numbers in a replication study. The PSTs were tasked with explaining two worked examples making use of the vertical addition and subtraction algorithms. Relevant to this place value review is their finding that PSTs "with less sophisticated conceptions tend to rely on a calculational or algorithmic approach to multidigit addition and subtraction problems and often speak in terms of position rather than value" (p. 86). They offer the implication that place value understanding evident in addition contexts is not necessarily generalised to subtraction contexts and suggest that addition and subtraction should be intermixed in teacher education.

In the research on the use of resources in teaching and learning of place value, Hurst and Hurrell (2016b) used the 'Marvelous Multiplier', a "sliding strip ….to assists students to understand that when numbers are multiplied or divided by a power of ten, all the digits move one place to the left (for multiplication) or one place to the right (for division) for each power of ten" (p.330). Understanding this idea is seen as an indicator of conceptual understanding, while 'adding a zero' is deemed to be an indicator of procedural understanding. The findings indicate that the use of the manipulative helped children's conceptual understanding of multiplicative relations. The research of Gorman and Way (2018) also focuses on the use of a resource to assist learners' mathematical understanding. In their case, the resource is virtual zoomable number line to develop Year 4 learners' understanding of decimal fractions. They argue that the virtual number line provides more opportunities for children to develop an understanding of decimal density than a static number line. Rogers (2021) examined the use of a computer-based Place Value Assessment Tool and the online version with Year 3-6 children. Rogers (2021) noted that while both tools saved time for teachers, they lacked transparency as the "teachers' judgement and involvement in the process was removed" (p.334). The suggestion made is that teachers be supported through professional development to develop their assessment literacy skills to assist them in interpreting the data. Litster et al. (2019) also explore the use of a digital resource, specifically an iPad app allowing virtual manipulation of the Montessori Number Base-10 blocks. The activities focus on grouping to form tens and hundreds.

Their focus is on the affordances that are offered by the app and they compare this with the affordances offered by the corresponding physical manipulative. Findings showed that the prior achievement of the children influenced which affordances they were able to access. They conclude that in selecting and designing virtual manipulative apps, consideration needs to be made of the prior achievement of the children who will be interacting with the app.

The paper by Choy et al. (2022), "contributes to conversations around making a teacher's thinking visible and enhancing a teacher's pedagogical reasoning by exploring the use of pedagogical documentation" over a series of lessons on division. Pedagogical documentation in this research included a single teacher's Padlet (a digital notice board) entries. The findings indicate that the pedagogical documentation made the unseen practices of the division lesson visible (e.g., the teacher's preference for the formal division algorithm and the use of a mnemonic device to teach the algorithm, rather than making a connection between the chunking strategy and the formal algorithm) and enables teachers to learn from their practices and uncover the invisible aspects of their teaching. Nutchey et al. (2016) report on the use of a Reality, Abstraction, Mathematics and Reflection framework to describe children's mathematical reasoning by observing lessons and semi-structured interview with the teachers. Regarding place value teaching, there were limited opportunities for learners to engage with manipulatives or iconic experience (e.g., a place value chart) in grouping and ungrouping tasks involving standard and non-standard partitioning. They argue that resources (manipulatives and iconic) are critical in creating effective learning for students in secondary schools.

In the European research 11 papers were classified as category 1. Ten of these papers focused on the place value understanding of children, and 1 focused on teacher knowledge. Themes emerging across these papers included the role of language (2); the use of resources (6); numeration units in place value understanding (3); and the role of place value understanding in learning algorithms (1). Ten papers are focused on children's understanding of place value, and one is focused on teacher knowledge.

Houdement and Chambris (2013) focus on the teaching and learning of multi-digit numbers and they present a design study in which they aimed to construct a relation between written numbers, numbers units and quantities (the triple code, Dehaene & Cohen, 1999). They include discussion of the different representations of these numbers in the written number (26), number name (twenty-six) and 'numbers-units-number' (2 tens 6 ones) and compare the number names in French and English. They note the 'irregular' number names in French, and the challenges that these number names pose to children learning about place value. Nguyen and Gregoire (2013) also focus on the French language and conducted a study investigating Vietnamese and Belgian (French-speaking) children's performance on place value tasks. They indicate that Vietnamese has a more "transparent name-number system" (p. 1926) and their findings show that the Vietnamese children performed better when the task was related to the number name. Chambris and Tempier (2017) build on this work in relation to large numbers. They argue that a base-1000 approach is useful in teaching large numbers and might contribute to the development of a sense of quantity. The authors make reference to base 10 'numeration units' and base 1000 'numeration units' and explain that an understanding of the relations between units is important. This work is taken further by Coulange and Train (2019) who write of the usefulness of the "discursive register of numeration units in conceptualising decimal numbers" (p. 403). Their findings of an analysis of three classroom episodes include that children found unit-conversions difficult as a deep conceptual understanding is required of the relationship between units, tenths and hundredths etc. There is a continuity evident in the cross-referencing in these papers, showing a coherent strand of conversation on this topic.

Several papers focused on resources. Tsiapou and Nikolantonakis (2013) examined the use of the Chinese abacus with a group of 12-year-olds, showing that the participants did achieve an understanding of place value concepts when using the tool, but struggled to transfer this understanding to their work in calculations. Jeannotte and Corriveau (2019) explored Grade 3 children's use of base ten blocks and a "homemade abacus" (p. 443), which comprised a colour-coded place value chart with small objects to represent the numbers in each position on the chart, when solving an arithmetic task. They noted that the children had some difficulty in using these manipulatives to solve the task and comment that the teacher's role is important in helping children to operate with manipulatives and not only rely on them to count and represent numbers. In the paper by Morais and Serrazina (2017), several models representing decimal numbers are explored in a teaching experiment. One in particular, the Decimat, is recognised as offering an important part-whole model which can "promote the understanding of partitioning by powers of ten connected with decimal place value" (p. 393).

Three papers focus on virtual manipulatives and representations of place value. Two make use of the Place Value Chart app (Kortenkamp & Ladel, 2013). Behrens (2015) provides a compelling theoretically driven explanation of the potential for this app to foster substantial understanding of the decimal place value system, and this work is referenced in Behrens and Bikner-Ahsbahs (2017) who report on findings from a research project implementing use of this app. In this research, their findings indicate that the actions and gestures of 'dragging' required when using the app "can accumulate more and more aspects of bundling and de-bundling" (p. 2728) which they argue are important place value concepts leading to the development of the concept of decimal fractions. Schulz and Walter (2019) present the *Stellenwerte üben* app and report on the use of this app by primary school children. They show that "the existence of the described mathematical didactic features of virtual representations does not automatically lead to an intended use" (p. 2947) but report that they did nevertheless see evidence of children using the linked representations of place value.

One paper focused on the role of place value understanding in the performance of standard algorithms for the four operations. Zembat et al. (2022) investigated the nature of teachers' knowledge in relation to their articulation of the role of place value in understanding arithmetic operations. They show that "the teachers rely mostly on common content knowledge that has little or no connections to a solid place value understanding" (p. 3735). There is need for further conversations about teachers' understanding of place value.

There are very few place value papers emerging from the Southern African context. There were 4 category 1 papers. The key themes emerging from the Southern African papers were the role of language in developing an understanding of place value (2) and the assessment of learners' place value understanding (2). Two papers focused on children's understanding of place value and one focused on teacher educators.

The research by Hertzog et al. (2017) and Graven et al. (2015) both explore learners' place value competence using different frameworks. Graven et al. (2015) draw on the Learning Framework in Number [LFIN] of Wright et al. (2006), while Hertzog et al. (2017) focus on the Conceptual Understanding of Place Value which they initially developed for learners in Germany. Their findings indicate that most of the Grade 2–4 South African learners could trade tens and ones and work with non-canonical representations provided that they had visual support. Graven (in Graven et al. 2015) used the Conceptual Place Value aspect of the LFIN framework, to assess two learners' understanding of place value as part of a broader study that explored learners' numeracy proficiency and progression. Both papers suggest frameworks that provide a hypothetical trajectory for place value understanding.

The role of language in developing an understanding of the base-10 decimal number system featured in two research studies. Mostert (2019) examined the linguistic features of isiXhosa and English and the affordances that the spoken and written number words offer in learning place value. In contrast to English, isiXhosa is a transparent language. This means that the spoken numbers correspond with the written numbers. Mostert (2019) argues that teachers need to capitalize on the transparency of isiXhosa to develop their children's knowledge of place value. The second study that focused on language paid attention to teacher educators (Longwe et al., 2022). The research investigated teacher educators' word use when teaching PSTs to develop early year children's knowledge of place value. The findings reveal that 66% of the naming was mathematical and 34% was non-mathematical (everyday terms). In addition, almost two-thirds of the non-mathematical terms included ambiguous pronouns (using 'this', 'that', 'those' to refer to a mathematical object). The research indicated that teacher educators need to pay more attention to the mathematical terms used that relate to place value.

## Discussion and Concluding Remarks

The context contributing the most papers to this review was the Australasian context followed closely by the European context and then the Southern African context. Most papers across all contexts are focused on children's understanding of place value, with a secondary focus on teachers and PSTs. There are many 'best practices' proposed across the papers, which points to the need for a systematic review of place value literature to synthesise these ideas.

In the Australasian papers there were a large proportion of papers focused on the assessment of place value understanding (e.g., Hurst, 2014). In addition, there were papers that examined place value understanding in the context of learning and applying algorithms, which is necessary due to the focus in schooling on algorithms (Graven et al., 2013). There is also attention given to the role of resources in the teaching and learning of place value, particularly virtual resources (e.g., Litster et al., 2019). Notable though is the absence of a focus on language in this context, whereas this is present in the European and Southern African work. In the European context, there is a focus evident on the 'triple code' (Dehaene & Cohen, 1999) through the work of Houdement and Chambris (2013), Chambris and Tempier (2017) and Coulange and Train (2019) in which there is exploration of the numeration units as well as exploration of the role of language in developing place value understanding. As with the Australasian research, there is also a focus evident on the role of resources, including virtual resources (e.g., Behrens, 2015) and there is work that addresses the role of place value understanding in learning to understand and apply algorithms (Zembat et al., 2022). The Southern African context offers the smallest number of papers to this review, pointing to a need for growth in this area. Language is included in the conversations in this context, as seen in the work of Mostert (2019). Assessment also appears as a theme in the Southern African work (e.g., Hertzog et al., 2017). The Southern African research does offer one unique contribution, however, in the work of Longwe et al. (2022) which focuses on teacher educators' work with place value. This is a population that is not researched in the other contexts.

As is evident in this review, each research community has overlapping areas of interest but also has absences in their conversations. Language is not explored in the Australasian work reviewed, virtual manipulatives are not explored in the Southern African literature and neither the Australasian nor the European conversations include consideration of teacher educators. In all contexts, the dominant focus is on children's understanding of place value, with proportionally far fewer studies examining teacher, PST and teacher educator knowledge and practices. One of the implications of this research is that there is clear potential for conversations to happen across these research communities and value to be had in doing so. We propose that collaborations involving more than two research communities hold great potential for moving the field forward.

# References


Behrens, D. (2015). How a digital place value chart could foster substantial understanding of the decimal place value system. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the 9th CERME* (pp. 2467–2472). CERME.

Behrens, D., & Bikner-Ahsbahs, A. (2017). The perspective of indexicality: How tool-based actions and gestures contribute to concept-building. In T. Dooley, & G. Gueudet (Eds.), *Proceedings of the 10th CERME* (pp. 2721–2729). CERME.

Benton, L., Saunders, P., Kalas, I., Hoyles, C., & Noss, R. (2018). Designing for learning mathematics through programming: A case study of pupils engaging with place value. *International Journal of Child-Computer Interaction, 16*, 68–76.

Bicknell, B., & Young-Loveridge, J. (2015). Young children's number-line placements and place-value understanding. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 101–108). Sunshine Coast: MERGA.

Böttcher, M., Huethorst, L., Walter, D., Gutscher, A., Selter, C., Bergmann, A., Dobbrunz, T., & Harrer, A. (2022). Design and research of digital case-based learning platform for primary pre-service teachers. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the 12th CERME* (pp. 2685–2692). CERME.

Chambris, C., & Tempier, F. (2017). Dealing with large numbers: What is important for students and teachers to know? In T. Dooley, & G. Gueudet (Eds.), *Proceedings of the 10th CERME* (pp. 322–329). CERME.

Choy, B. H., Dindyal, J., & Yeo, J. B. W. (2022). Making visible a teacher's pedagogical reasoning and actions through the use of pedagogical documentation. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of Mathematics Education Research Group of Australasia* (pp. 138–145). Launceston: MERGA.

Coulange, L., & Train, G. (2019). Teaching and learning decimal numbers: The role of numeration units. In U. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th CERME* (pp. 403–410). CERME.

Dehaene S., & Cohen L. (1997). Cerebral Pathways for Calculation: Double Dissociation between Rote Verbal and Quantitative Knowledge of Arithmetic. *Cortex, 33*(2), 219–250.

Downton, A., Russo, J., & Hopkins, S. (2020). Students' understanding of the associative property and its applications: Noticing, doubling and halving, and place value. *Mathematics Education Research Journal, 34*, 437–456.

Gervasoni, A., & Peter-Koop, A. (2015). Comparing the development of Australian and German 7-year-old and 8-year-old's counting and whole number learning. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th annual conference of Mathematics Education Research Group of Australasia* (pp. 261–268). Sunshine Coast: MERGA.

Gorman, A., & Way, J. (2018). Zooming-in on decimals. In J. Hunter, P. Perger, & L. Darragh (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 337–344). Auckland: MERGA.

Graven, M., Stott, D., Mofu, Z., & Ndongeni, S. (2015). Identifying stages of numeracy proficiency to enable remediation of foundation knowledge using the Learning Framework in Number. In D. Huillet (Ed.), *Proceedings of the 23rd annual conference of SAARMSTE* (pp. 69–83). SAARMSTE.

Graven, M., Venkat, H., Westaway, L., & Tshesane, H. (2013). Place value without number sense: Exploring the need for mental mathematical skills assessment within the Annual National Assessments. *South African Journal of Early Childhood Education, 3*(2), 131–143.

Hertzog, M., Ehlert, A., & Fritz, A. (2017). A competency model of place value understanding in South African primary school pupils. *African Journal of Research in Mathematics, Science and Technology Education, 21*(1), 37–48.

Houdement, C., & Chambris, C. (2013). Why and how to introduce numbers units in 1st and 2nd grades. In B. Ubuz, C. Haser, & M. Mariotti (Eds.), *Proceedings of the 8th CERME* (pp. 313–322). CERME.

Hurst, C. (2014). Developing pre-service teacher capacity to make appropriate choices of tasks and resources through diagnostic assessment of children’s work. In J. Anderson, M. Cavanagh, & A. Prescott (Eds.), *Curriculum in focus: Research guided practice. Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia* (pp. 295–302). Sydney: MERGA.

Hurst, C., & Huntley, R. (2017). Explicitly connecting mathematical ideas: How well is it done? In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 325–332). Melbourne: MERGA.

Hurst, C., & Hurrell, D. (2016a). Assessing children's multiplicative thinking. In B. White, M. Chinnappan, & S. Trenholm (Eds.), *Opening up mathematics education research. Proceedings of the 39th annual conference of Mathematics Education Research Group of Australasia* (pp. 336–343). Adelaide: MERGA.

Hurst, C., & Hurrell, D. (2016b). Sliding into multiplicative thinking: The power of the 'marvellous multiplier". In B. White, M. Chinnappan, & S. Trenholm (Eds.), *Opening up mathematics education research. Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia* (pp. 328–335). Adelaide: MERGA.

Jacobson, E., & Simpson, A. (2019). Prospective elementary teachers' conceptions of multidigit number: Exemplifying a replication framework for mathematics education. *Mathematics Education Research Journal, 31*, 67–88.

Jazby, D., & Pearn, C. (2015). Using alternative multiplication algorithms to 'offload' cognition. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 309–316). Sunshine Coast: MERGA.

Jeannotte, D., & Corriveau, C. (2019). Interactions between pupils' actions and manipulative characteristics when solving an arithmetical task. In U. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th CERME* (pp. 443–450). CERME.

Kortenkamp, U., & Ladel, S. (2013). Designing a technology based learning environment for place value using artefact-centric activity theory. In A. M. Lindmaier, & A. Heinze (Eds.), *Proceedings of the 37th conference of the International Group for Psychology of Mathematics Education* (Vol. 1, pp. 188–192). PME.

Lambert K., & Moeller K. (2019). Place value computation in children with mathematics difficulties. *Journal of Experimental Child Psychology, 178*, 214–225.

Larkin, K., Ladle, S., Kortenkamp, U., & Etzold, H. (2019). Developing student understanding of place value and supporting teachers' confidence in teaching place value via digital tools. *Australian Primary Mathematics Classroom, 24*(3), 21–26.

Litster, K., Moyer-Packenham, P., & Reeder, R. (2019). Base-10 blocks: A study of iPad virtual manipulative affordances across primary-grade levels. *Mathematics Education Research Journal, 31*, 349–365.

Longwe, J., Fauskanger, J., & Kazima, M. (2022). Teacher educators' word use when teaching student teachers how to teach place value to early years learners. *African Journal of Research in Mathematics, Science and Technology Education, 26*(3), 248–259.

Morais, C., & Serrazina, L. (2017). Models' use and adaptation aiming at conceptual understanding of decimal numbers. In T. Dooley, & G. Gueudet (Eds.), *Proceedings of the 10th CERME* (pp. 387–394). CERME.

Mostert, I. (2019). Number names: Do they count? *African Journal of Research in Mathematics, Science and Technology Education, 23*(1), 64–74.

Nguyen, H., & Gregoire, J. (2013). Re-examining the language supports for children's mathematical understanding: A comparative study between French and Vietnamese language. In B. Ubuz, C. Haser, & M. Mariotti (Eds.), *Proceedings of the 8th CERME* (pp. 1925–1934). CERME.

Nutchey, D., Grant, E., & English, L. (2016). Experiencing mathematics for connected understanding: Using the RAMR framework for accelerating students' learning. In B. White, M. Chinnappan, & S. Trenholm (Eds.), *Opening up mathematics education research. Proceedings of the 39th annual conference of the Mathematics Education Research group of Australasia* (pp. 495–502). Adelaide: MERGA.

Rogers, A. (2021). Computer based mathematics assessment: Is it the panacea? In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in mathematics education: Foundations and pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 329–337). Singapore: MERGA.

Schulz, A., & Walter, D. (2019). Practicing place value: How children interpret and use virtual representations and features. In U. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th CERME* (pp. 2941–2948). CERME.

Tsiapou, V., & Nikolantonakis, K. (2013). The development of place value concepts to sixth grade students via the study of the Chinese abacus. In B. Ubuz, C. Haser, & M. Mariotti (Eds.), *Proceedings of the 8th CERME* (pp. 2058–2066). CERME.

Wright, R. J., Martland, J., & Stafford, A. K. (2006). *Early numeracy: Assessment for teaching and intervention.* Sage Publications Ltd.

Young-Loveridge, J., & Bicknell, B. (2016). Developing early place-value understanding: A framework for tens awareness. In B. White, M. Chinnappan, & S, Trenholm, (Eds.), *Opening up mathematics education research. Proceedings of the 39th annual conference of Mathematics Education Research Group of Australasia* (pp. 640–647). Adelaide: MERGA.

Zembat, I., Yaşa, S., & Aslan, M. (2022). Teacher knowledge in the context of articulating arithmetic operations and place value. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the 12th CERME* (pp. 3735–3742). CERME.

# Learning to Notice Algebraically: The Impact of Designed Instructional Material on Student Thinking

Bridget Wadham
*Massey University*
b.wadham@massey.ac.nz

Emily Pearce
*Massey University*
e.pearce1@massey.ac.nz

Jodie Hunter
*Massey University*
j.hunter1@massey.ac.nz

In this paper, we explore how students' algebraic noticing's and explanations changed across a two-year period with the introduction of designed instructional material. The data in this report is drawn from n=53 Year 7-8 students' responses to a free-response assessment task across two different years. Analysis focused on how students noticed and explained algebraic relationships in pairs of equivalent equations. Findings indicate that with the introduction of designed instructional material, there was a shift in student noticing of number properties to identify equivalence between pairs of equations. However, identifying the distributive property of multiplication and developing generalisations about the algebraic relationships remained challenging for students.

Both in New Zealand and internationally, there has been increased attention to algebra and relationships as key learning areas of mathematics in research studies and policy documents (MoE, 2007; Schifter, 2017). In the New Zealand context, middle school students (aged 10-13) are expected to generalise the properties of multiplication and division with whole numbers. Despite this expectation, teachers often focus considerable attention and time on teaching their students how to calculate (Schifter, 2017), and rarely give opportunities for learning that focuses on algebraic structures (Arcavi et al., 2017). This approach results in students developing an over-reliance or compulsion to calculate (Hunter et al., 2022, Arcavi et al., 2017) unless an algebraic intervention (Blanton et al., 2015) or an "algebrafying" of the classroom occurs (Blanton & Kaput, 2003). Supporting teachers to implement sound research-based instructional approaches that focus on algebraic structures is an important aspect of positioning teachers to move beyond teaching calculation and focus on the algebraic nature of number. Previous studies have focused on design experiments or professional development in relation to early algebra and teacher change (Blanton, et al., 2015; Blanton & Kaput, 2003). However, in this study we address a gap in the field by focusing on the introduction of designed instructional material for teachers to use in the classroom. Specifically, we address the following research question:

- How do student responses to an assessment item involving noticing, and explaining algebraic structures change after provision of designed instructional materials to teachers?

## Literature Review

Understanding number properties, relationships and mathematical structure are vital elements of developing sound number sense, and the importance of this has been well documented across the last decades (e.g., Mason et al., 2009; Kaput, 2017; Carpenter et al, 2003). In the last ten years the field of early algebra has gained significant movement particularly in relation to a focus on algebraic thinking with young students, as opposed to the 'arithmetic-then-algebra' approach that is deeply institutionalized within educational structures (Kaput, 2017, p.5). An early algebraic thinking approach builds on students' natural ideas of patterning and relationships (Blanton et al., 2015) emphasising the complex kinds of mathematics that young students can achieve when provided with opportunity.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 517–524). Newcastle: MERGA.

Two of the big ideas underpinning early algebraic understanding are proposed by Blanton et al. (2015, p.43) as being 'equivalence, expressions, equations, and inequalities' and 'generalized arithmetic'. These ideas involve developing an understanding of the equals sign and equivalence, the properties of number (commutative, associative, distributive, identity, inverses), and the ability to reason with the structure of expressions and equations rather than calculating an 'answer'. If we consider the missing number equation $7 + 3 = \_\_ + 4$, a student relying on calculation may first compute $7 + 3 = 10$ and then reason that $6 + 4$ is also 10 (Blanton et al., 2015, p.51). In contrast, a student drawing on an equivalence or compensation approach, or who notices a relationship between both sides of the equal's sign; may be drawing on a form of relational or structural thinking (Carpenter et al., 2003; Mason et al., 2009).

Mason et al., (2009, p.12) states that "attention to structure runs through the whole of mathematics, and that shifts of attention make a difference to how mathematics is seen". Developing structural awareness allows students to move from arithmetic (calculation) towards algebraic thinking (hence a "shift of attention"). Students who notice and understand mathematical structure are typically comfortable applying number properties to different situations, can form generalisations and are reported by teachers as being more engaged within the classroom (Carraher et al., 2008; Gronow et al., 2022: Mason et al., 2009). In contrast, students who do not attend to structure may view the equals sign as a command to carry out a calculation (Carpenter et al., 2023), and as a result are likely to find it significantly more difficult to reason with algebraic concepts in the future. Gronow et al., (2022) found that some teachers may even believe that "low ability" students do not have the capability to notice mathematical structure, however, it is well reported by researchers that students can do this from a young age (Blanton & Kaput, 2003; Carraher et al., 2008).

### *Instructional Materials*

Instructional materials lie within the curriculum enactment process between the official and operation curriculum (Remillard & Heck, 2014). They refer to resources designed to support teachers with lesson instruction, and 'play a critical role in national education systems' (p.707). Within New Zealand, the MOE (2021) affirms that successful resources help teachers to understand what research is saying about effective teaching and how to put it into practice. However, for teachers to use instructional materials effectively, they must devote significant time and attention to develop a deep understanding of the mathematical concepts involved. Teachers must also hold the concept of 'explicitness' in the forefront of their minds whilst using the instructional materials, as this will ensure mathematical ideas are made clear to students (Leong et al., 2019). This concept of 'explicitness' is described by Selling (2016) as raising the collective awareness of the existence of mathematical concepts and practices and knowing why they are important in understanding mathematics. Mason et al., (2009) reports that it is not enough for teachers themselves to be aware of algebraic structure. They need to expect their students to justify and explain their actions using number properties that have been made explicit in the classroom. When a classroom has been 'algebrafied' and number properties and relational structure are made explicit to students through task design, mathematical practices, and substantive classroom conversations, then student outcomes show improvement (Blanton & Kaput, 2003). Whilst instructional materials can have considerable promise in supporting algebraic understanding, when used as a prescriptive tool by the teacher a surface level understanding may result for both teacher and students.

## Methodology

The data and participants of this study are drawn from a larger ongoing research project focused on schools involved in a professional learning and development research initiative entitled Developing Mathematical Inquiry Communities (DMIC). In this paper, we draw on data from a qualitative case study involving middle school students and their responses to an assessment item administered in two different school years after a taught unit. Analysis of student responses was used to identify common themes with a specific focus on identifying what changes occur in students' algebraic noticing across a two-year period.

## *Participants and Setting*

The participants were Year 7-8 students (aged 10-13) attending a low socio-economic middle school within New Zealand. All students present on the assessment day n=170 (2021) and n=157 (2022) completed a free-response assessment task (see data collection section). As this paper focuses on change over time, responses from Year 8 students in 2021, and Year 7 students in 2022 were removed along with any students who had left the school or did not complete both assessments due to Covid interruptions. This resulted in a cohort of n=53 students who completed an algebraic assessment task during both years. The cohort included students from the Pacific Nations (35%), Māori (30%), and NZ European (24%). It is important to emphasize that in the first year of the study, teachers individually designed a series of algebraic tasks to form a unit on number and algebra for their students. In the second year, the teachers utilized a research-based instructional unit provided within the DMIC PLD that was compiled and developed by the third author (see Figure 1). This instructional unit consisted of 14 contextualized and problematic tasks that drew students' attention to algebraic structures and relationships. Accompanying each task was information regarding big mathematical ideas (Randall, 2005), links to the New Zealand Curriculum, expected learning outcomes, and general notes that alerted teachers to important aspects regarding the teaching and learning of algebra as reported within research literature. Teachers were facilitated within the PLD to become familiar with this information and focus attention on the five teacher practices of anticipating, monitoring, selecting, sequencing, and connecting (Smith & Stein, 2018).

| | |
|---|---|
| **Task 7** | Can you work together in your group to solve these number sentences? Make sure that you develop an explanation and justification.<br><br>189 + 25 = __ + 26<br>85 – __ = 75 – 28<br>674 + 56 – __ = 671<br>24 x 16 = 48 x __<br>105 ÷ 15 = (45 ÷ 15) + (__ ÷ 15) |
| **Big Ideas** | Equations show relationships of equality between parts on either side of the equal sign. The properties of equality are: If the same real number is added or subtracted to both sides of an equation, equality is maintained; If both sides of an equation are multiplied or divided by the same real number (not dividing by 0), equality is maintained; Two quantities equal to the same third quantity are equal to each other. |
| **Curriculum Links** | **NA4-1:** Use a range of multiplicative strategies when operating on whole numbers.<br>**NA4-8:** Generalise properties of multiplication and division with whole numbers.<br>**NA4-7:** Form and solve simple linear equations. |
| **Learning Outcomes Students will be able to:** | Explain and justify relationships between numbers in an equation.<br>Write statements of equivalence in words and using notation.<br>Solve equivalence problems and explain and justify the solutions. |

| | |
|---|---|
| **Mathematical language** | Equivalent, equal sign. |
| **Sharing back/Connect** | Select student solution strategies that use relational reasoning.<br><br>**Connect:**<br><br>Ask students to generate conjectures related to the equivalence problems that build on the properties of equality. |
| **Teacher Notes** | • Students may initially treat the equals sign as an operator or indication to write the answer next.<br>• Students also may compute each side to work out whether they are equal.<br>• Notice students who use the relationships across the equals sign to see whether there is balance.<br>• Highlight to the students to look across the equals sign and find the relationships between numbers to the left and the numbers on the right.<br>• Notice students who use the relationships across the equals sign to see whether there is balance.<br>• Highlight the students relational responses (e.g., noticing the + 2 – 2 relationships).<br>• Press for use of arrows and notations to highlight the relationships. |

Figure 1. Example task from the instructional unit.

## *Data Collection*

Students were asked to complete a written free-response assessment task (see Figure 2) at the completion of the algebraic unit each year. This free-response task consisted of 12 individual equations that had equivalence to another equation through the distributive and associative number properties, or exponents. Under these equations were three prompts which encouraged students to describe and explain the number patterns, and to show if they work with other numbers (generalisation). The assessment task was launched by the teacher to ensure students knew what they were expected to do. Students then worked on the task individually within class time and were encouraged to explain and represent their thinking. The completed assessment tasks were collected, scanned by the research team, and stored securely.

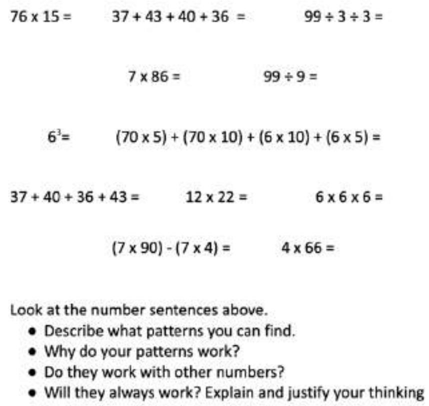



Figure 2. Free-response assessment task (Hunter et al., 2022).

### *Data Analysis*

Initially, student responses to the task were coded as either ‘not identifying’ or ‘identifying’ algebraic relationships. Responses coded as ‘not identifying’ were those in which students treated the task as a calculation exercise of individual equations. Responses coded as ‘identifying’ showed evidence of noticing or explaining one or more relationships between the six possible pairs of equations or items within the task. The samples coded as ‘identifying’ were then examined per item (equation pairs). Each item was assigned a code of 0-4 relating to the sophistication of the explanation given (see Table 1). Furthermore, items were coded as showing evidence of computational (C) or relational thinking (R) (Carpenter et al., 2003). For example, the student response “7 x 86 = (7 x 90) - (7 x 4). When you minus this (7 x 4) that will mean it will be 7 x 86. (90 - 4)” was coded as R3: explanation using relational thinking. The first and third author independently coded the samples. Any differences in coding were then discussed until a consensus was agreed on.

**Table 1**

*Examples of how Students Explained 7 x 86 = (7 x 90) - (7 x 4) with Codes*

| Code | Example of Code |
|---|---|
| (N) Not Identifying | “7x86=602” |
| (0) No explanation | Drew an arrow between the two number sentences |
| (1) Calculation Only | (7 x 90) - (7 x 4) = 7 x 86 because 7 x 80 = 560, 7 x 6 = 42 |
| (2) Low Level Explanation | “All I did to find the relationship between different equations is to find the answer using place value and see if the answers match up.” |
| (3) Explanation | “7 x 86 = (7 x 90) - (7 x 4). When you minus this (7 x 4) that will mean it will be 7 x 86. (90-4)” |
| (4) Partial Generalisation | “So you can do 90 x 7 - 28 to get 86 x 7.<br>Examples: 7 x 4 = 28 so 7 x 3 = 7 x 4 - 7” |

## Results and Discussion

This section will focus on identifying and describing several changes that occurred in students' algebraic thinking. Results indicate that a significant shift of attention occurred within the way students view the mathematical equations between 2021 and 2022 with the introduction of the designed instructional material. As shown on Table Two, this involved a significant shift in students viewing individual equations as a command to calculate towards identifying similarities between pairs of equations by using structural or relational reasoning. In the second year of data collection, 90.6% of students were able to ‘identify’ relationships between pairs of equations, in contrast to 22.6% of students during the first year (an increase of 68%). Findings from the 2021 sample are consistent with previous research, in which many students used calculations (Hunter et al., 2022; Schifter, 2017). However, the 2022 data indicates that the introduction of the material supported a shift with students away from solely calculation towards noticing and considering relational structure (Mason et al., 2009).

**Table 2**

*Percentage of Students Noticing Algebraic Relationships*

| | 2021 | 2022 | Change |
|---|---|---|---|
| Not identifying | 77.4% | 9.4% | -68% |
| Identifying | 22.6% | 90.6% | +86% |

**Table 3**

*Percentage of Students Identifying Algebraic Properties (all codes)*

| Item | 2021 | 2022 | Change |
|---|---|---|---|
| 76 x 15 = * | 5.7% | 41.5% | +35.8% |
| 37 + 43 + 40 + 36 = 37 + 40 + 36 + 43 | 15.1% | 81.1% | +66% |
| 99 ÷ 3 ÷ 3 = 99 ÷ 9 | 13.2% | 60.4% | +47.2% |
| 7 x 86 = (7 x 90) - (7 x 4) | 3.8% | 37.7% | +33.9% |
| $6^3$ = 6 x 6 x 6 | 22.6% | 81.1% | +58.5% |
| 12 x 22 = 4 x 66 | 7.5% | 41.5% | +34% |

*(70 x 10) + (70 x 5) + (6 x 10) + (6 x 5).

Also notable was a significant shift in the number of students identifying each pair of related equations (Table 3), with an increase from 33.9% of students in 2021 to 66% in 2022. The largest of these gains occurred in the number of students noticing the associative property of addition (+66%) and exponents (+58.5%), with 81.1% of the students being able to identify both these properties in 2022. Equation pairings involving the distributive and associative properties of multiplication also improved from 33.9% to 35.8%, however, this was a much smaller shift. Possible reasons for this could include that these number properties are more difficult for students to notice and explain (Hunter et al., 2022), or teachers themselves have a superficial understanding of these concepts and therefore do not make these explicit within their classrooms (Grownow et al., 2022; Mason et al., 2009).

Although results were generated for how students responded to each item, for the purpose of this paper, only 7 x 86 = (7 x 90) - (7 x 4) will be discussed in detail here (Table 4). This is because the item had the lowest percentage of change, so warrants closer inspection. In 2021, only two of the 53 students identified this relationship. This included one student who drew an arrow between the equations, and one student who gave a low-level explanation saying they both equal 602. In contrast, in 2022 there were 20 students who identified this relationship. Most commonly, students provided low-level explanations using calculation (n=6) or identified relational structure with no-response (n=7). These students drew arrows to show the equations were related or explained equality by relying on calculation means. This may indicate that whilst students' attention is shifting towards noticing relationships, explaining these relationships may take more time to develop. Or as described earlier, teachers may not be making these explanations explicit within their classrooms.

**Table 4**

*Number of Students Per Code Identifying the Distributive Property of Multiplication*

| | | No Response | Calculations Only | Low-Level | Explanation | Partial Generalisation |
|---|---|---|---|---|---|---|
| 2021 | C | 0 | 0 | 1 | 0 | 0 |
| | R | 1 | 0 | 0 | 0 | 0 |
| 2022 | C | 2 | 2 | 6 | 0 | 1 |
| | R | 7 | 0 | 0 | 2 | 0 |

*C=students gave indication of calculating, R=students only drew on relational structure.

## *Examples of Specific Students*

Three specific students' responses will be shown here to illustrate different ways in which change occurred between the years. This includes the two students who identified the item in 2021 (Table 5), and one who did not.

Student A responded in 2021 by drawing a line between 7 x 86 and (7 x 90) - (7 x 4) and gave no further explanation (see Figure 3). We cannot be sure about the reasons why the student connected these two equations, as they gave written explanations for other items. So, there is a chance these may have been connected by the process of elimination. In 2022, they provided supplementary evidence of calculations proving both are equal to 602.


2021 2022

Figure 3. Student A's responses.

Student B's explanation in 2021 stated the equations were "the same as" and the "answer: 602", indicating they solved the equation to prove equality (see Figure 4). Interestingly, in 2022 they repeated the previous response, however, they also gave an example of generalising this property to another example "(8 x 90) - (8 x 4) = 8 x 86", becoming the only one of the n=53 students who gave a partial generalisation of the distributive property of multiplication. We can surmise that whilst this student is either still reliant or compelled to calculate to confirm equality, they do realise number properties can be generalised.

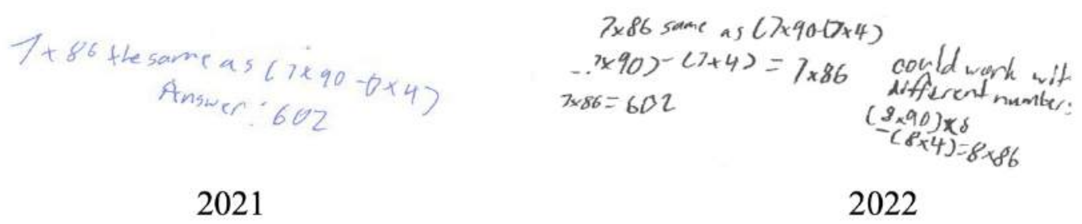



Figure 4. Student B’s responses.

Student C is an example of one of the n=51 students who did ‘not identify’ this item in 2021 and attempted to answer each individual equation (Figure 5). In contrast, they were one of the two students in 2022 who were able to give a relational explanation in 2022. As seen in Figure 4, the student used their understanding of structure to show that both equations are equivalent to 7x86 without drawing on any calculative means.

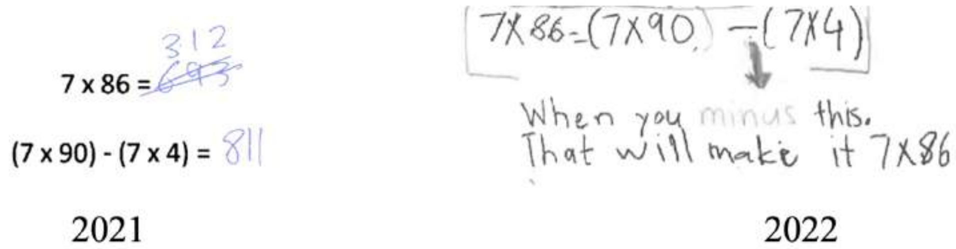



Figure 5. Student C’s responses.

In summary, the way in which each student noticed and reasoned with algebraic structure changed in different ways across the two-year period. This implies that the ways in which students develop an understanding of the distributive property is not constrained to a linear track of progression. They may in fact move back and forth between using calculation and relational means.

## Conclusion

We aimed to identify how students noticing and explaining of algebraic structures changed after teachers were provided with designed instructional materials. In summary, the results from 2021 showed students had little awareness of algebraic structure, and focused on solving individual equations (calculating), rather than noticing relationships between equations. In contrast, a shift of attention was seen in 2022, with many more students noticing algebraic structure, especially regarding the associative property and exponents. This indicates that the focus on calculating can be shifted by teachers who take the time to expose students to algebraic thinking within their classrooms. We surmise that providing research-based instructional materials could be a useful tool and show considerable promise in educating teachers about mathematical structure. This in turn will help to ‘algebrafy’ classrooms, support student outcomes and support student outcomes.

Despite this large shift, many students still appeared to experience difficulty in identifying the distributive property. Additionally, it appeared that generalising number properties to other instances, an expectation of the New Zealand Curriculum for this age cohort, was challenging. Future research would be helpful to investigate whether teachers are making generalisations explicit within their classrooms and how they can be supported to do so. Moreover, beyond instructional materials, how can teachers be supported to develop sufficient mathematical content knowledge to teach generalisation and number properties in their classrooms. We aim to gather further data in 2023, to contribute to the understanding of how different students' algebraic reasoning develops over time.

# References


Arcavi, A., Drijvers, P., & Stacey, K. (2017). *The learning and teaching of algebra: Ideas, insights, and activities.* Routledge.

Blanton, M., Stephens, A., Knuth, E., Gardiner, M., Isler, I., & Kim, S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education, 46*(1), 39–87.

Blanton, L., & Kaput, J. (2003). Developing elementary teachers' algebra eyes and ears. Teaching children mathematics, *10*(2), 70–77.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Heinemann.

Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM, 40*, 3–22.

Gronow, M., Mulligan, J., & Cavanagh, M. (2022). Teachers' understanding and use of mathematical structure. *Mathematics Education Research Journal, 34*, 215–240.

Hunter, J., Miller, J., Bowmar, A., & Jones, I. (2022). "It has the same numbers, just in a different order": Middle school students noticing algebraic structures within equivalent equations. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys*. *Proceedings of the 44th annual conference of the Mathematics Research Group of Australasia* (pp. 138–145). Launceston: MERGA.

Kaput, J. J. (2017). 1 What is algebra? What is algebraic reasoning? In James J. Kaput, David W. Carraher, & Maria L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–18). Routledge.

Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (2019). Making things explicit using instructional materials: A case study of a Singapore teacher's practice. *Mathematics Education Research Journal, 31*, 47–66.

Selling, S. K. (2016). Making mathematical practices explicit in urban middle and high school mathematics classrooms. *Journal for Research in Mathematics Education*, *47*(5), 505–551.

Smith, M. S., & Stein, M. K. (2018). *5 practices for orchestrating productive mathematics discussion*. National Council of Teachers of Mathematics.

Ministry of Education (2007). *The New Zealand curriculum*. Wellington: Learning Media. https://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum

Randall, CC (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *NCSM Journal,* Spring-Summer.

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *Zdm*, *46*, 705–718.

Schifter, D. (2018). Early algebra as analysis of structure: A focus on operations. In Kieran, C. (Ed.), *Teaching and learning algebraic thinking with 5 to 12-year-olds. The global evolution of an emerging field of research and practice* (pp. 309-328). New York: Springer.

# Introducing a Structured Problem-Solving Approach Through Lesson Study: A Case Study of One Fijian Teacher's Professional Learning

Indra Wati

*The University of the South Pacific*

S01001880@student.usp.ac.fj

Worldwide interest in Lesson Study (LS) and the opportunities offered for student learning through the use of a Structured Problem-Solving Approach (SPSA), as typically adopted in Japanese LS research lessons in mathematics, have left largely unanswered questions about the extent to which these can be replicated elsewhere. This paper presents a case study of one primary school teacher's learning experiences, and his views about LS and SPSA, as a result of participating in a project introducing SPSA through LS in three Fijian primary schools. The results reveal that engaging in the LS process was instrumental in supporting this teacher's implementation of SPSA in his mathematics classroom. The findings are important for teacher professional learning (PL) in Pacific cultural contexts.

Lesson Study (LS) is a professional learning approach originating in Japan. LS involves the careful planning, implementation and observation of a research lesson, followed by a post-lesson discussion and reflection by members of the planning team, observers and a 'knowledgeable other'. As teaching is socially and culturally situated, research is also socially and culturally situated. LS is based on collegial conversations focused on improving teaching. It sets out to ensure all students learn optimally, including those with learning challenges. Research lessons in mathematics in Japan, where LS originated, typically adopt a Structured Problem-Solving Approach (SPSA). A typical mathematics lesson using SPSA focuses on a single problem and consists of four phases: posing the problem, students solving the problem, comparing and discussing of student solutions–*neriage*–and summarising and reflecting on learning—*matome* (Shimizu, 1999).

While there has been worldwide interest in LS as a model for teacher professional learning and the use of SPSA as a means of engaging students in creative mathematical activity, questions about the extent to which these can be replicated elsewhere have been largely left unanswered (Groves, 2013; Groves et al., 2016).

This paper presents a case study of one primary school teacher's learning experiences (at his school), and his views about LS and SPSA, as a result of participating in a project introducing SPSA through LS in three Fijian primary schools. It addresses the research question:

- What are the opportunities and challenges in implementing SPSA through LS in Fijian primary schools? This study is significant because it explores the opportunities and challenges of implementing innovative pedagogy and a professional learning model that is foreign to Fijian teachers.

## Theoretical Framework

The study draws on Vyogotsky's (1978) sociocultural theories. Vygotsky posited that knowledge is socially constructed through social interaction and argued that it is a shared experience, rather than an individual experience. Teaching is viewed as a social activity, involving co-construction, mediation and scaffolding, and formative interaction (Bell, 2010). As teachers work through the LS process, there are multiple opportunities for them to reflect, analyse, decide on actions to be taken, evaluate, and share their understandings with other teachers. These conversations can take place in a social setting where teachers negotiate and discuss mathematics with more knowledgeable others (Takahashi, 2004). The collaborative practice and interaction with one another in a sociocultural environment during LS can enhance teachers' cognitive growth in terms of knowledge and effectiveness (Vygotsky, 1978; Warford, 2011).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 525–532). Newcastle: MERGA.

## Literature Review

LS has been adapted and implemented in many countries outside of Japan, with Fujii (2014) cautioning that there are many misconceptions regarding the implementation of LS in foreign countries—for example, implementing LS as a workshop; believing that SPSA is about solving a task; a focus on evaluating the teacher during the post-lesson discussion; and believing that the research lesson has to be re-taught. On the other hand, Takahashi (2014) talks about the importance of a knowledgeable other. He proposes that "final comments are important for effective LS and the best way to develop the ability to serve as a knowledgeable other is through participating in LS with colleagues" (Takahashi, 2014, p. 18).

Studies done on LS and SPSA outside Japan have reported several affordances and constraints. For example, findings from a study of the implementation of SPSA through LS in three primary schools in Victoria, showed that the meticulous planning procedure involved in Japanese LS gives teachers a chance to review and further their understanding of both the subject matter and the thinking processes of their students (Widjaja & Vale, 2013). In addition, Widjaja et al. (2017), reporting on the same project, concluded that teachers were becoming more adept at working collaboratively and orchestrating whole-class discussions based on anticipated student responses and targeted questions.

Based on their 20 years of experience, Lewis et al. (2019) identified some of the challenges in implementing LS in the USA. These included a culture of politeness regarding critique which can undermine inquiry as well as observers interfering in the lesson together with a reluctance to listen to students and collect data on student learning.

Despite the challenges, misconceptions, and successes identified in studies of the implementation of LS outside of Japan, it is nevertheless important to continue to implement LS in diverse settings to investigate its impact and whether it is transferrable to new cultural contexts while fully adopting the underlying principles of LS and SPSA in their authentic forms. Hence, this study set out to implement LS in its most authentic form as much as possible, highlighting the salient elements of LS and SPSA.

## Methodology

The overall project consisted of three introductory workshops, followed by three LS cycles in each school. Each LS cycle entailed extensive collaborative lesson planning of a research lesson, one member of the planning team teaching the research lesson, other participants across schools observing the research lesson in person or on video, an online cross school post-lesson discussion, and a subsequent focus group discussion (FGD) at the school. A total of nine research lessons were completed in this way. Interviews were also conducted with the headteachers at each school, together with a selection of the teachers. This paper focuses on one teacher who participated in the project, who taught the second (Cycle 2) research lesson at his school.

### *The School Context and the Teacher's Background*

Kini has a bachelor's degree and 24 years of teaching experience in middle primary schools in a semi-rural location. Note that all names used in this paper are pseudonyms. The school is comprised of 284 children–191 boys 93 girls–mostly iTaukei students or Indigenous Fijians. The school has nine teachers with a support staff.

### *Case Study Methodology*

The study reported in this paper employs a case study methodology (Cohen et al., 2018). The data collection process included video recording of the three introductory workshops, lesson planning sessions, research lessons, post-lesson discussions (PLDs), focus group discussions

(FGDs) and interviews, carried out over a period of approximately eighteen months. Field notes, photographs of student work samples and lesson plan samples were collected. The qualitative data analysis software Transana© was used to code and analyse video data.

All the activities were video recorded for analysis. Video recording of the research lesson presented by Kini was coded, using Transana. Codes and categories were generated to capture key aspects of the lesson in terms of its use of the SPSA approach. Twenty-three keywords were generated from viewing Kini's lesson multiple times. These were organised under four categories—namely organisation of the class, phases of the lesson, student activity, and teacher activity. Table 1 presents the phases of the study as applied to Kini's involvement, together with the corresponding data sources and data analysis techniques.

**Table 1**

*Phases of Kini's Involvement in Implementing SPSA Through LS*

| Phases | Activity | Data Sources | Data Analysis |
|---|---|---|---|
| 1 | Introductory workshop on LS and SPSA | Video recording & transcript of Focus Group Discussion | Content analysis |
| 2 | Collaborative planning for Kini's research lesson | Lesson plans (5 versions) | Comparison of first and final lesson plans |
| 3 | Implementation (and observation) of Kini's research lesson | Video recording & transcript of lesson (1 hour 15 minutes) | Coding of lesson video using Transana—generating 23 keywords under 4 categories |
| 4 | Post-lesson discussion and reflection—Kini, planning team & others from the three participating schools | Video recording & transcript of Post-Lesson Discussion | Content analysis of teachers' reflections on Kini's lesson |
| 5 | Focus Group Discussions after each LS Cycle | Video recording & transcript of Focus Group Discussion | Content analysis of teachers' reflections on LS and SPSA processes |
| 6 | One-on-one interview | Video recording & transcript of interview | Content analysis |

## Findings

The findings are organized in order of the study's phases showing how Kini navigated through the SPSA processes through LS.

### *Kini's Initial Experiences*

Kini was introduced to LS and SPSA through workshops using an online platform. Throughout the process Kini actively participated in an activity involving planning a research lesson in collaboration with participants from other schools. Kini valued the planning process in informing teacher's pedagogical approach as reflected in his comment, "lesson preparation is important as teachers will be able to come up with new strategies and ways of presenting the lesson" (FGD, 21 October 2021). Kini could see the potential for his students to learn mathematics by expressing themselves and believed that this could only happen if teachers "move away from the traditional way of teaching" (FGD, 21 October 2021).

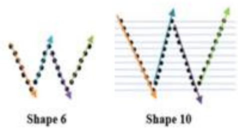
### *Lesson Planning*

Kini was an enthusiastic member of the planning team for all three research lessons conducted at his school—that is, in each of Cycles 1, 2 and 3. He also taught the Cycle 2 research lesson in his

Year 6 classroom. While planning the research lesson, Kini's team initially came up with four solutions which were focused on getting the correct answer, rather than focusing on students' strategies in deriving patterns through diagrams. The team members stated that they did not see, think, nor teach patterns in this manner and they had not thought of deriving rules or formulas using patterns. Excerpts of the initial and final lesson plans are shown in Table 2. The depth shown in the final lesson plan regarding solution 4 and arriving at the rule is a stark contrast to the team members' previous approaches in teaching mathematics. Kini realised this and added that "I still have a lot to learn, still have a lot to learn" (FGD 1, 25 March 2022).

**Table 2**

*Lesson Plan Comparison*

| Elements of SPSA | Initial Lesson Plan | Final Lesson Plan |
|---|---|---|
| Anticipated Student Solutions | Solution 4<br>6th Shape<br>= 4(6) + 1<br>= 24 + 1<br>= 25 | Solution 4<br>[Arrow down, arrow up, arrow down, arrow up]<br># of dots for shape 6 = 7 + 6 + 6 + 6 = (n+1) + n + n + n = n + 1 + 3n = 4n + 1<br><br># of Dots in W-Shapes:<br>Shape 6 – 7+6+6+6 = 7 + (3x6) = (6+1) + (3x6) = 7 + 18 = 25<br>Shape 10 – 11+10+10+10 = 11 + (3x10) = (10 + 1) + (3x10) = 11 + 30 = 41<br>Students can come up with a rule for number of dots in the nth Shape<br>where n represents the Shape Number)<br>Number of dots in nth shape = (n+1) + (3 x n)<br>Number of dots in $6^{th}$ Shape = (6+1) + (3x6) = 7 + 18 = 25<br>Number of dots in $6^{th}$ Shape = (10+1) + (3x10) = 11 + 30 = 41 |

*The Research Lesson*

Details of Kini's implementation of the research lesson are captured in the *keyword sequence map* produced from the Transana analysis described earlier. The keyword sequence map, as shown in Figure 1, uses keywords (codes) to capture details of the organisation of the class, the different phases of the lesson, and student and teacher activities during the lesson, and displays these against the passage of time during the lesson.

As can be seen in Figure 1, Kini's class organisation shifted from whole class to individual work, to whole class, and then to group work, in close alignment with the phases of the lesson.

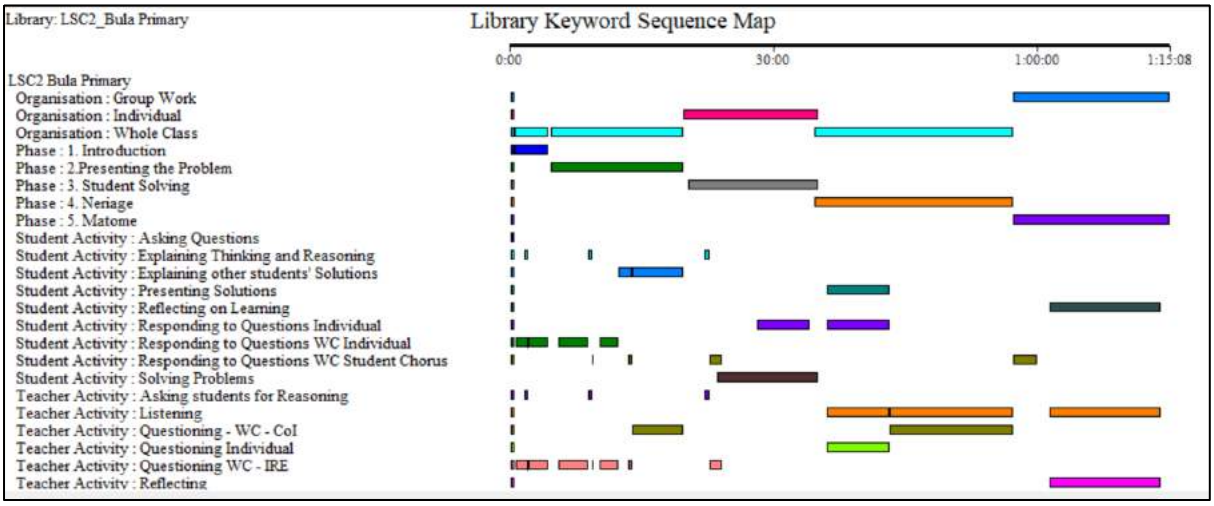



Figure 1. Keyword sequence map.

Kini was able to incorporate elements of SPSA into his lesson. For example, when presenting the lesson, he was able to engage a number of students in responding to questions as he tried to unpack a single problem. Moreover, the time taken for students to work on the problem individually was relatively brief, while the considerably longer *neriage* (comparing and discussing student solutions) phase involved students presenting their solutions while the teacher listened and involved as many students as possible in the discussion. Students were involved in explaining their thinking and reasoning and explaining other students' strategies. The teacher also asked questions and listened, identifying any gaps in students' learning and thinking. Kini stated that, "to wait is a virtue" (FGD 2, 22 June 2022). He also asked probing questions, prompting students to explain their reasoning, as well as asking scaffolding questions to enable students to think for themselves and come to a conclusion. In the process, Kini engaged the whole class, promoting the idea of a community of inquiry, rather than adopting a traditional *Initiate-Response-Evaluate* (*IRE*) model of questioning. He spent a considerable amount of time allowing students to reflect on their learning in pairs, groups and as a whole class. Kini set the tone for a new learning space for the students by concluding his lesson by saying, "I hope that's the way we're going to learn as we move forward, to share, discuss and discover things with other students" (Research Lesson, 19 May 2022).

Kini also showed an increased awareness of individual students' ownership of learning and how this was more inclusive of the average learner. This was captured in the FGD when Kini stated that, "I look at those lower bracket students or below average students, this approach really works well with them when they learn from fellow peers or other students" (FGD, 22 June 2022).

### *Post-lesson Discussion*

One of the key points highlighted by Kini while he was reflecting on his own teaching was that he missed out on some possible 'teaching moments'. For example, he overlooked that Rom's explanation was useful to other students because he arranged the magnets correctly to form the W-dot pattern as shown in Figure 2. Kini reiterated the importance of this, stating that he "should be careful to allow students to do their work on the board and display … it to other students, ensure other students are able to take that … up and … compare his work with the other two students" (PLD, 2 June 2022).

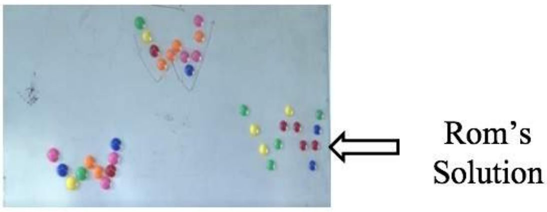



Figure 2. Student work on the board.

Kini also reflected on the fact that it would have been better if he had not deviated a little from the task at hand, when he focused more on the direction of the formation of the W-Dot shape, rather than the number of dots in the growing pattern. This affected the students while solving the problem, as reflected in his comment that the students were focusing on the direction they followed to create the W, rather than the dot formation and the number of dots used.

### *Learning Gains and Challenges*

By the end of Cycle 1 of LS, Kini had realized the challenges faced by the teachers of the three participating schools. He mentioned several challenges: 1) being observed when teaching a lesson was not easy; 2) working in a team and collaborating, 3) the demand for collective, voluntary commitment by the teachers was paramount for success (he thanked his team for making this commitment); and 4) the difficulty posed by extensive *neriage* in SPSA-based mathematics lessons to completing the lesson in the timetabled slot "to finish the lesson on time is challenging" (FGD 1, 25 March 2022). In spite of these challenges, Kini encouraged teachers to incorporate SPSA and to participate in LS, as he had observed positive changes in students. When the teacher incorporates SPSA, students use their prior knowledge, which builds students' confidence and excitement and curiosity to learn. According to Kini, SPSA provides a platform for these changes to manifest, "what I'm saying is, look at the excitement in the eyes or the faces of the students…let the student guess… 'What will happen today?' Oh, we going to do this today… coming to school is something exciting" (FGD 2, 22 June 2022). He also noticed that the teachers at his school were implementing aspects of SPSA in their classes. Kini observed that, "as I walk past, I can see students standing up, giving their answers … they are discussing among themselves. They stand up trying to correct the one standing at the board" (FGD 3, 3 November 2022).

## Discussion

This study examined the opportunities and challenges in implementing SPSA through LS in Fijian primary schools. The findings show that after active participation in LS and SPSA, the case study teacher demonstrated an emerging understanding of LS and SPSA. Some of the important findings in the lesson planning phase showed that this teacher benefitted from the collaborative process while anticipating student solutions. He echoed the power of good planning in all discussions (PLDs and FGDs). He asserted that his content knowledge and pedagogical skills were heightened in the process. He also realised the drawbacks of his current practice based on a traditional model of 'chalk and talk' and teacher telling. This resonates with finding from Groves (2013) in the Australian context, relating to the importance of anticipating students' solutions, and the need to plan for good questioning in order to elicit student responses and maximise the impact of SPSA's focus on sharing student solutions to develop students' higher order thinking. These aspects are also highlighted in the keyword sequence map of Kini's lesson, where he is asking probing and scaffolding questions to elicit student thinking and reasoning. These changes were seen in Kini's lesson due to consistent active engagement in the LS professional learning program. Kini planned, observed other teachers teach, and critiqued their lessons. He also implemented a research lesson and reflected on his own practice. These activities resonate with Widjaja et al.'s (2017) findings that enactment and reflection were crucial in facilitating teachers' professional learning.

As the study progressed, emerging understanding of LS processes became evident. For example, Kini was able to focus his comments on the students' learning rather than the teacher in PLDs. This resonates with one of the challenges Fujii (2014) highlighted, stating that foreign implementers should be wary of focussing on the teacher during PLD. Kini did not follow the lesson plan as a 'recipe' and navigated his lesson by his students' responses and the classroom situation. His professional learning took place over a period of 18 months during which he actively participated in all the processes.

The challenges identified by Kini (see above) were quite different from those stated by Lewis et al. (2019). For example, contrary to Lewis et al.'s concern that a "culture of politeness" regarding critique can undermine inquiry, Kini was quite critical during reflective post-lesson discussions as well as critical of his own lesson.

## Conclusion

In summary, Kini was introduced to SPSA through LS. In this study, at the end of the project, Kini had a good understating of SPSA and LS processes and demonstrated a willingness to learn. Through his active participation, Kini was able to integrate many of the elements of SPSA in his research lesson and at the same time made valuable contributions towards collaborative lesson planning and post-lesson discussions, embracing strengths of the research lessons and focusing on teaching and student thinking during the post-lesson discussions. He valued highly the collaborative lesson planning processes and the outcomes of engaging with the team to solve problems and anticipate student solutions. Kini was open to accept suggestions from his team members and researchers who assumed the role of knowledgeable others. He embraced the shift in the approach to teaching and learning of mathematics and professional learning that was situated in the classrooms. In the process, he decided to test a research lesson in his class. He brought about several changes in his practice regarding classroom organisation, his teaching approach, student activities and teacher activities. For example, normally he would pick a textbook, discuss an example to explain how to solve a problem, ask students to attempt similar exercises, have the teacher or students show their working on the board while students copied correct solutions as corrections. In this new approach, Kini's role as teacher shifted drastically. For example, in the *neriage* phase, he discussed student solutions on the board with students expected to explain their thinking and reasoning. Using LS and exploring SPSA in their classrooms resulted in challenges, such as time constraints, difficulties related to collaborative planning, and lessons being scrutinised. Teachers may be reluctant to modify how they teach, but when immersed in a new strategy that they find effective, their teaching habits begin to transform. "Leaving behind the lecture method requires a sophisticated pedagogical approach, which takes time to learn" (Takahashi, 2021, p. 5). Consequently, this study has shown glimpses of the effectiveness of LS as a PL model and SPSA as a teaching pedagogy with potential and relevance to the Pacific region.

## Acknowledgements

The *Using Lesson Study to Introduce a Structured Problem-Solving Approach* project is funded by a Pacific Scholarship for Excellence in Research and Innovation. I thank my PhD supervisors, Dr Hem Dayal, The University of the South Pacific, Honorary Associate Professor Susie Groves and Associate Professor Wanty Widjaja, Deakin University, for their support and encouragement in the preparation of this paper.

## References

Bell, B. (2010). Theorising teaching. *Waikato Journal of Education, 15*(2), 21–40.

Cohen, L., Manion, L., & Morrison, K. (2018). *Research methods in education.* Routledge.

Fujii, T. (2014). Implementing Japanese lesson study in foreign countries: misconceptions revealed. *Mathematics Teacher Education and Development, 16*(1), 65–83.

Groves, S. (2013). Implementing the Japanese problem-solving lesson structure, *Mathematics Education Research Group of Australasia,* (711–714).

Groves, S., Doig, B, Vale, V., Widjaja, W. (2016). Critical factors in the adaptation and implementation of Japanese Lesson Study in the Australian context. *ZDM Mathematics Education 48*(4), 501–512. https://doi.org/10.1007/s11858-016-0786-8.

Lewis, C., Friedkin, S., Emerson, K., Henn, L., & Goldsmith, L. (2019). How does lesson study work? Toward a theory of lesson study process and impact. In Huang, R., Takahashi, A., da Ponte, J. P. (Eds.), *Theory and practice of lesson study in mathematics: An international perspective* (pp.13–37). Cham: Springer. https: doi.10.1007/978-3-030-04031-4_2.

Shimizu, Y. (1999). Aspects of mathematics teacher education in Japan: Focusing on teachers' roles. *Journal of Mathematics Teacher Education, 2*(1), 107–116.

Takahashi, A., & Yoshida, M. (2004). Ideas for establishing lesson-study communities. *Teaching Children Mathematics, 10*(9), 436–443.

Takahashi, A. (2014). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 14–21.

Takahashi, A. (2021). *Teaching mathematics through problem-solving: A pedagogical approach from Japan*. Routledge.

Warford, M. K. (2011). The zone of proximal teacher development. *Teaching and Teacher Education, 27*(2), 252–258. https://doi.org/10.1016/j.tate.2010.08.008.

Widjaja, W., & Vale, C. (2013). Deepening teachers' understanding of content and students' thinking: The case of Japanese lesson study in Australian primary classrooms. In M. Inprasitha (Ed.), *The 6th East Asia regional conference on mathematics education (EARCOME 6): Innovations and exemplary practices in mathematics education* (Vol. 3, pp. 266–275). Phuket, Thailand: Center for Research in Mathematics Education.

Widjaja, W., Vale, C., Groves, S., & Doig, B. (2017). Teachers' professional growth through engagement with lesson study. *Journal of Mathematics Teacher Education, 20,* 357–383. https://doi.org/10.1007/s10857-015-9341-8.

Vygotsky, L. (1978). *Mind in society: Development of higher psychological processes.* Harvard University Press.

# A Tri-Nation Comparative Study of Place Value in Early Years Curricula Documents

Lise Westaway
*Rhodes University*
l.westaway@ru.ac.za

Silke Ladel
*University of Education, Schwäbisch Gmünd*
silke.ladel@ph-gmuend.de

Pamela Vale
*Rhodes University*
pamela.vale@ru.ac.za

Kevin Larkin
*Griffith University*
k.larkin@griffith.edu.au

Mellony Graven
*Rhodes University*
m.graven@ru.ac.za

Ulrich Kortenkamp
*Potsdam University*
ulrich.kortenkamp@uni-potsdam.de

In this paper we compare the early years mathematics curricula of Germany, South Africa, and Australia in relation to the place value concept. Place value is an important topic as it underpins much of the number work completed by learners in the early years of schooling. We found that there were differences between the three curricula that could be summarised using five themes: namely, number range, place value structure, role of the zero, influence of language, and use of materials. We argue that how the different curricula deal with these five themes influences the quality of learning provided and we highlight key areas of concern. In concluding we identify three important implications for our future research project.

## Introduction

The genesis of this article occurred in conversations that the authors held preparing a grant application, which aims to support the development of place value (PV) content knowledge (CK) and pedagogical content knowledge (PCK) for in- and pre-service teachers in Germany, South Africa, and Australia. It became immediately apparent that our respective national curricula differed greatly in the way PV is represented and in the way each national curricula is conceptualised. Consequently, it was necessary to commence our project with a comparative study of the three curricula. In this paper we provide a brief theoretical background in relation to teaching PV and outline our methodology in analysing the curricula. We then provide a brief overview of each curricula and identify five themes that differ between the respective curricula. We conclude by highlighting areas of concern within each curricula and with an outline of important areas for further investigation within our ongoing research project.

## Theoretical Background

PV involves the learning of several "big ideas" (Van de Walle, 2015, p. 247) and fundamental to these is the integration of early experiences of *counting* and *grouping* amounts in tens (and in tens of tens, and in tens of tens of tens, and so on). Building upon this knowledge is the activity of *bundling*, that is recognising a group of ten as a new object with a new name (e.g., one *ten* instead of a group of ten ones). As there are different ways to partition an amount into parts; that is bundles of different sizes, the acquisition of the *part-whole concept* is also necessary. Based upon the foundational knowledge of counting, grouping, bundling, and the part-whole concept is the concept of *positional notation*, which comprises the sub-concepts of *place value*, (the value of the bundle unit), the value of *digits*, (based both on the symbol and on its position), and finally the *number value*, (the application of the part-whole concept to each of the digits in a number).

Rogers (2012) notes that "despite the unchanging and recursive nature of our base-ten system, it seems some students never manage to fully unravel the hidden code that underlies place value"

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 533–540). Newcastle: MERGA.

(p. 648). Kortenkamp and Ladel (2013) reinforce this view, identifying that PV is a difficult topic to teach effectively. These findings are evident in the South African context, with research by Graven and Venkat (2021) indicating that poor mathematics performance was in part due to learners' "lack of understanding of PV and the base 10 number system" (p.24). Learners is the South African term used to refer to students or to children.

Given the difficulty for many learners in understanding PV, it is concerning that several researchers have indicated that understanding of PV is also problematic for some teachers. Research by Hopkins and Cady (2007) indicated teachers' difficulty in using different bases, the use of expanded notation, making conversions, and pictorial representations. The problem of a lack of CK and PCK of PV is often evident with pre-service teachers. Thanheiser and Melhuish (2019) indicate that many pre-service teachers come into mathematics content courses with knowledge of how to implement arithmetic procedures but without understanding the conceptual PV knowledge underlying them.

The availability of new digital tools in supporting learners in understanding PV is a more recent development. Cognisant of the need to utilise appropriate digital materials (see Larkin, 2016), we argue that the use of quality digital materials is beneficial in supporting the development of learners' conceptual understanding of PV. At the core of our argument is the knowledge that digital manipulatives provide learners with opportunities to interweave pictorial and symbolic representations, with the actions that they perform on them, to emphasise the underlying mathematical concepts by, for example, linking multiple external representations (Ladel, 2009).

Based on the review of the literature above, and on our analysis of the curricula from the three countries, the following research question guided our analysis.

- How do the analysed curricula differ in terms of what PV content is to be taught and how this content progressively builds?

Answering this question should enable the authors to identify implications for research and practice in the future.

## Method

We analysed the Australian Curriculum (Mathematics F-10 Version 9.0) (ACARA, 2022), the South African Curriculum and Assessment Policy Statement: Foundation Phase (Grades 1-3) Mathematics (SA.DBE, 2011) and the Bildungsstandards im Fach Mathematik für den Primarbereich. Beschluss der Kultusministerkonferenz vom 15.10.2004, i.d.F. vom 23.06.2022 (KMK, 2022). As Germany has different curriculum documents for each of the 16 federal states, we chose to analyse the two that are relevant for the in- and pre-service teachers we work with, namely, the Bildungsplan der Grundschule. Mathematik (MKJS BW, 2016) of Baden-Württemberg and the Rahmenlehrplan Teil C Mathematik (MBJS BB, 2015) of Brandenburg. Prior to the analysis, the sections pertaining to place value of the German curriculum were analysed by the German members of our team and translated into English by one of the German-speaking authors for further analysis. The Australian and South African curricula are published in English.

In the first step of analysis, each separate curriculum was analysed by a team member and relevant text was mapped to acknowledged key PV concepts (See Ladel et al., 2023 in work), including pre-concepts (part-whole, counting, and grouping) and PV sub-concepts (bundling [Base Ten], the decimal part-whole concept, and positional notation) as these are critical for flexible PV understanding. In the second step, a different team member (from a different country) also reviewed each curriculum and made comments for consideration. In the third step we discussed the comments as an entire research team and collated our findings in an Excel spreadsheet (See Figure 1 for an excerpt).

| | | p. 215 | p.216 | p. 217 |
|---|---|---|---|---|
| **grade/term** | | **Grade 2 Term 1** | | |
| **number range** | | **up to 25** | | |
| **Decimal Part-Whole** | **Standard partitioning** | Place Value<br>T4 - Recognise the place value of numbers 11 to 19. (decompose two-digit numbers into ten and ones e.g. 18 is 10 and 8)<br><br>"In Term 1, learners work with a higher number range and continue to:<br>- count and group to make a group of tens and loose ones; and<br>write 18 = 1 ten and 8 loose ones<br>13 = 10 and 3."<br><br>"Ten is 1 ten that contains 10 ones. Regular ten and one words (24 is 2 groups of 10 and 4 ones or 2 tens and 4 ones) need to be used regularly to establish a language that symbolises decomposing and composing."<br><br>Building up and breaking down<br>This is one of the most important techniques in the Foundation Phase (learners will also use decomposing frequently in the Intermediate Phase). Using this technique allows learners to split (decompose) and recombine numbers to help make calculations easier. during this term learners will:<br>• break up numbers using place value;<br>• break up numbers using multiples of 10; and<br>• break up into number pairs e.g. pairs that make 20. (p. 227) | "Learners should continue to manipulate concrete apparatus by grouping to form ten ones and understanding that 10 is one group of ten loose ones"<br><br>"Base tens blocks (part of the Dienes blocks) can be introduced to develop the idea of a ten as a single entity and that: 10 ones make 1 ten; 20 ones make 2 tens; and 16 ones make 1 ten and 6 loose ones" | "Learners should be engaged in written activites that build and consolidate:<br>- the concept of groups of ten and loose ones/units; and<br>**- the value of a digit."**<br>**"During this term learners will break up numbers using place value" (p. 226)** |
| | **Non Standard partitioning strong** | | | **"Equivalent representations**<br>During this term the focus is showing equivalent representations for the same number. **Twenty should be described as 2 tens (using the bundles or groups of objects) or 2 groups of tens (Kev - what is difference?).** It is important to show learners that 20 can look different. So show 20 loose objects, one group of ten and 10 loose ones and 2 groups of ten."<br><br>**NOTE (Lise)**: The focus is the language to be used - "Ten is 1 ten that contains 10 ones. Regular 'ten and one' words (24 is 2 groups of 10 and 4 ones or 2 tens and 4 ones) need to be used regularly to establish a language that symbolises decomposing and composing". |
| | **Non Standard partitioning not strong** | | | |

Figure 1 Excerpt from the South African Curriculum Excel spreadsheet.

## *Brief Overview of the Three Curricula*

The nationwide German curriculum (KMK, 2022) briefly lists the competencies learners should acquire till the end of Grade 4. This is a result of a reorientation of the curriculum design from an *input orientation* to an *output orientation* (Klieme et al., 2003). PV is only directly mentioned as "Understanding number representations and number relationships". In addition, there are also implicit mentions that we will discuss later; however, these might not be readily identified by teachers. As indicated earlier, the nationwide German curriculum is substantiated in special curricula within each of the 16 federal states. The curriculum of Baden-Württemberg (MKJS BW, 2016) defines the competences for Grade 1/2, with the number range up to 100, and for Grade 3/4, with the number range up to 1 000 000. The curriculum of Brandenburg (MBJS BB, 2015) refers not to grades, but to levels that span several years of schooling, however, the Brandenburg curriculum refers to the same number ranges. As with the national curriculum, the curricula of each state remains superficial, for example, "children are able to use the decimal place value system and recognize its structure (ones, tens, hundreds, bundles, unbundles)" (MKJS BW, 2016). Special didactical material is barely mentioned in this part of the curricula ("number and operation"), but is seen instead as an extra competence—"working with mathematical objects and tools".

South Africa has a highly prescriptive curriculum. Each phase of the schooling system has its own curriculum document for mathematics. The Foundation Phase (Grades R to 3) document specifies what needs to be taught in each of the four terms in the school year. Although the number range increases in each term, the concepts and skills remain the same. PV is taught from Grade 1 and these learners are required to decompose numbers in accordance with the number range for the grade; namely "decompose numbers into multiples of 10 and ones/units" (SA.DBE, 2011) up to 20.

In Grade 3 learners are required to "decompose three-digit numbers up to 999 into multiples of 100, multiples of 10 and ones/units … [and] identify and state the value of each number" (SA. DBE, 2011). Clarification notes provide guidelines on how to teach PV and are included in the curriculum. The curriculum suggests that teachers use concrete and semi-concrete materials (e.g., bundles of sticks, PV cards and number lines) to support learners in developing an understanding of PV; however, no mention is made of the use of digital resources.

The Australian curriculum (ACARA, 2022) is quite prescriptive in terms of PV content descriptors, indicating the range of numbers that learners work with, and then less prescriptive in providing elaborations that teachers may utilise. In terms of the range of numbers that learners are expected to work with, in Foundation learners "name, represent and order numbers including zero to at least 20"; however, there is no mention of PV. In years 1, 2 and 3 learners "recognise, represent, and order numbers to at least (120, 1000, beyond 10 000) using physical and virtual materials, numerals, and number lines (Year 1 and 2) and naming and writing conventions in Year 3. The use of digital (virtual) materials is also recommended throughout the early school years.

This brief synopsis suggests that there is significant variability in terms of curricula structures, curricula specifications, number ranges, and materials used to teach PV across the three countries.

### *Cross Curricula Comparison of the Three Curricula*

In this section we compare the three curricula documents according to the five distinguishing features that emerged from the analysis: namely number range; PV structure; the role of zero; the influence of language in learning PV; and the use of materials in PV learning.

### *Number Range*

Of initial note across the first four years of the three curricula is the difference in the number range that teachers are required to focus on each year (See Table 1).

**Table 1**

*Number Range Per Year Group Across the Three Countries*

| Australia | Germany | South Africa |
|---|---|---|
| Foundation: to at least 20 | | Reception: up to 10 |
| Year 1: to at least 120 | Grades 1-2: up to 100 | Grade 1: Up to 19<br>Term 3: 11-15; Term 4: 11-19 |
| Year 2: to at least 1 000 | | Grade 2: Up to 99<br>Term 1: 11-25; Term 2: 11-50; Term 3: 11-75; Term 4: 11-99 |
| Year 3: 10 000 and beyond | Grades 3–4: up to 1 000 000 | Grade 3: Up to 999<br>Term 1: up to 99; Term 2: up to 500; Term 3: up to 700; Term 4: up to 999 |

The number range in both the Australian and South African curricula is significantly lower than Germany (Year 3/4 learners working up to 1 000 000) with Year 3 learners in Australia working beyond 10 000 and Grade 3 learners in South Africa working with numbers up to 999. In both the Australian and South African curricula learners are effectively "adding a place each year"; however, the South African curriculum is more prescriptive as it specifies the number range for each term in each grade, for example, in Grade 2, the number range is up to 25, 50, 75 and 99 in terms 1, 2, 3 and 4 respectively. The adding of a "place" each year is in our view problematic for the development of learners' PV understanding. Limiting learning to one new place each year means that it is not easy for learners to develop an understanding that the decimal PV system is based on groupings of 10 (i.e., $10^0$, $10^1$, $10^2$, $10^3$ etc.) and that it has a repeated naming pattern (e.g., 10 ones is the same as 1 ten, ten tens are the same as 1 hundred…). As the PV principle can be repeated at any place, including decimals, restricting PV understanding to a fixed number of places seems counter intuitive.

There are several inconsistencies regarding the number range in the curricula of South Africa, for instance, while Grade 2 learners are required to calculate up to 99, examples in the clarification notes for teachers often exceed this upper limit with Grade 2 learners required to "partition two-digit numbers in multiples of tens and ones" (SA. DBE, 2011). With the examples provided including "12 tens and 8 ones; 18 tens and 4 ones" (SA. DBE, 2011). These notes imply that South African Grade 2 learners should recognise the PV of numbers beyond 99. The different language used to specify the number range in the curricula is also worth noting. In the South African and German curriculum, the words "up to" are used to indicate the end point for each grade, while in the Australian curriculum, the words, "to at least", suggest that teachers are encouraged to go beyond the specified number range.

### *Place Value Structure*

Greater emphasis is given to understanding base-ten PV patterns in the Australian curriculum. For example, in Year 2, learners are required to read and write "numerals, and saying and ordering two-, three- and four-digit numbers using patterns in the number system", while in Grade 3 learners are required to use "the repeating pattern of place value names and spaces within sets of 3 digits to name and write larger numbers: ones, tens, hundreds, ones of thousands, tens of thousands, hundreds of thousands, ones of millions, tens of millions" (ACARA, 2022). This is strengthened by requiring learners to predict and name "the number that is one more than 99, 109, 199, 1009, 1099, 1999, 10 009 … 99 999 and discussing what will change when one, one ten and one hundred is added to each" (ACARA, 2022). Likewise, the structure of the base-ten PV system is emphasised in Germany where learners "are able to use the decimal place value system and recognise its structure (units, tens, hundreds, bundles, unbundles)" (MKJS BW, 2016).

In the German curriculum there is much attention given to bundling and unbundling as it is repeatedly referred to throughout the curriculum document. There is also explicit attention given to "establishing the relationship between unbundling and transferring" (MKJS BW, 2016) in which the written algorithms for the four operations are linked conceptually to PV concepts. Multiple decompositions of numbers are also included in the curriculum from as early as Grade 1-2 in the German curriculum, allowing learners the opportunity to engage with the concepts of part-whole relationships. It is problematic, though, that many aspects of teaching PV are hidden in arithmetic strategies and are not mentioned explicitly in the curriculum, as not all teachers will be aware of the opportunities to teach PV when teaching arithmetic strategies.

Partitioning in non-standard and standard ways is explicitly stated in the Australian curriculum: "partition, rearrange, regroup and rename two- and three-digit numbers using standard and non-standard grouping" (ACARA, 2022). This includes "renaming numbers in different ways for example, renaming 245 as 24 tens and 5 ones or 2 hundreds and 45 ones" (ACARA, 2022). In the elaborations in the curriculum documentation, it is mentioned that learners should use the PV chart and move materials from one place to another to show this renaming. Learners thus have multiple opportunities to explore PV through both standard and non-standard partitioning of numbers. It is to us astonishing that, in the German curriculum, non-standard partitioning is not mentioned explicitly at all. Implicitly it is included (e.g., in "understand oral and semi-written calculation strategies for the four basic arithmetic operations and use them flexibly") (KMK 2022); however, the need for non-standard partitioning remains largely unacknowledged, and again teachers might miss this aspect in their teaching.

In the South African curriculum, there is mention of both standard and non-standard partitioning. As an example, in the Grade 2 curriculum it is mentioned that learners should "show different arrangements of numbers, for example, 35 can be shown as 35 loose ones, 3 tens and 5 loose ones and 2 groups of tens and 15 loose ones". This is, however, only one of two examples of non-standard partitioning. Learners are exposed very early to standard PV partitioning of numbers into tens and ones (Grade 1, Term 3), and later into hundreds, tens, and ones (Grade 3, Term 2) and this standard partitioning is emphasised in the curriculum and national workbooks. The dominant strategy used for calculating in the South African curriculum is 'breaking down and building up', for example 346 + 154 is broken down into 300 + 40 + 6 + 100 + 50 + 4. Compared to Australian learners, South African and German learners are exposed to less exploration around the concept of PV and are required to use standard partitioning almost exclusively from Grade 1. This limits their opportunity to fully conceptualise the power of the decimal system.

### *The Role of the Zero*

It is important that the role of the zero is clearly understood in PV. In the Australian and South African curricula, the zero is introduced as a number in the Foundation and Grade R years respectively, whilst in Germany (Baden-Württemberg) it is only mentioned in Grade 3. In South Africa, Grade R teachers are encouraged to, "point out that zero means 'nothing'". The special meaning of the digit zero in the number 10 is not made explicit, even though learners in Grade R work with numbers to 10. This is a missed opportunity, and the emphasis on the zero meaning "nothing" may introduce misconceptions related to PV understanding later where the digit zero plays a significant role rather than being 'nothing'. In Year 2 (Australia), learners "recognise the role of a zero digit in place value notation" (ACARA, 2022) and Grade 3 (Germany) learners engage with "the special meaning of the number 0" (MKJS BW, 2016). In South Africa, learners only encounter this in the final term of Grade 3 (the end of the Foundation Phase), when they are required to "recognise 0 as a place holder in two and three-digit numbers". In all three curricula it is interesting to note that the emphasis on the special role of the zero in PV does not appear earlier when learners first encounter numbers beyond ten and multiples of ten.

### *Language in the Learning of Place Value*

Both the German and Australian curricula make explicit mention of language-related issues in the teaching and learning of PV. In the German curriculum, teachers are asked to pay attention to "which errors in speech or spelling are due to misconceptions about place value or linguistic difficulties (for example, language of origin, mixing up tens and ones)" (MKJS BW, 2016). In the German language, two-digit numbers are expressed by naming the ones and then the tens, for example, *einundzwanzig* (one and twenty) and thus this so-called 'number inversion' of tens and ones could be understandable if a learner's language of origin has a different system of naming numbers. This note in the curriculum is relevant and necessary. The Australian curriculum makes explicit mention of language in an elaboration note in which it is explained that Year 1 learners should come to recognise that "numbers are used in all languages and cultures but may be represented differently in words and symbols" as well as including that in Year 3 learners should compare "the Hindu-Arabic numeral system to other numeral systems" (ACARA, 2022). In contrast there is no mention of the influence of language on PV understanding in the South African curriculum. This is a missed opportunity given that South Africa has eleven official languages, and that these languages have several differing number-naming conventions regarding PV (including ones that follow the German convention of naming ones before tens). This is particularly the case given that in the Foundation Phase, mother tongue instruction is promoted; however, from the Intermediate Phase, Grades 4-6 onwards, most learners transition to learning in English despite it being the home language of less than 10% of learners (Robertson & Graven, 2019).

### *Learning Materials*

It is interesting to note the different teaching and learning materials that are mentioned in the three curricula and the differences in what is emphasised when comparing them. In the Australian curriculum there is explicit mention of virtual materials, which is absent in the South African curricula and only mentioned as "the students use mathematical tools appropriately (e.g., drawing tools, digital tools)" (KMK 2022) in the German curriculum. The other physical materials that are mentioned in the Australian curriculum include hundred charts, number lines and large collections of recycled materials to represent large quantities. There is also mention made of PV charts. In Germany, the PV chart is also mentioned, as well as the number line. In addition, base ten blocks are also mentioned, and the requirement that learners "represent numbers up to 1 000 000 in different ways" (MKJS BW, 2016) regarding the use of these materials. The materials mentioned in the South African curriculum include sticks (to create bundles of 10), connecting cubes, an abacus, PV cards and base ten blocks. Interestingly, base ten blocks are only mentioned for use when learners are working with numbers smaller than 100, which means that it is only the tens and ones that would be used. It is therefore no different to using bundles of sticks and, consequently, the power of the representation of ones, tens, hundreds, and thousands, offered by these blocks, is lost.

## Conclusion and Implications

Above we have noted several 'missed opportunities' and areas of concern. The German and Australian curricula are similar in providing a greater number range, while the South African curriculum is very prescriptive and limits the number range that learners are expected to work with. This restriction of number ranges regarding the learning of PV may reduce learner opportunities to see the patterns of PV and the structure of our decimal number system. The curricula also vary in terms of recommending when learners should be introduced to zero as a place holder, the use of physical or virtual materials to support learners, and in the approach they take towards the importance of language in teaching PV. In the case of the South African curriculum there is no mention of digital resources and, given that the covid pandemic introduced many teachers and learners to the use of digital resources for teaching and learning, we consider this as a missed opportunity. In terms of moving forward, our analysis of the curricula suggest these avenues for research and practice.

- What are the implications for research, and for teacher education and practice, based on how PV is presented in the respective curricula?
- Investigating ways in which in-service and pre-service teachers can be provided opportunities to develop understanding of the prerequisite concepts and sub-concepts underpinning the teaching of PV and how these are presented in various curricula.
- How to authentically embed the use of appropriate materials, including virtual manipulatives, to support PV learning, including opportunities to experiment with these materials.
- Further research into the crucial role language plays within PV learning, particularly in contexts where the structure of the PV system is different between the mother tongue and the language of instruction.

## Acknowledgment

Part of this work is based on research supported by the SA National Research Foundation (Grant No. 74658).

# References

Australian Curriculum, Assessment and Reporting Authority. (2022). *The Australian curriculum* (version 9.0). ACARA. https://v9.australiancurriculum.edu.au/

Bailey, J. (2015). Supporting lower-achieving seven- and eight-year-old children with place value understandings. *Australian Primary Mathematics Classroom, 20*(3), 3–9.

Bicknell, B., Young-Loveridge, J., & Simpson, J. (2017). Using quotitive division problems to promote place-value understanding. *Australian Primary Mathematics Classroom, 22*(2), 28–32.

Graven, M., & Venkat, H. (2021). Piloting national diagnostic assessment for strategic calculation. *Mathematics Education Research Journal*, *33*(1), 23–42. https://doi.org/10.1007/s13394-019-00291-0

Klieme, E., Avenarius, H., Blum, W., Döbrich, P., Gruber, H., Prenzel, M., Reiss, K., Riquarts, K., Rost, J., Tenorth, H.-E., & Vollmer, H. J. (2003). *Zur entwicklung nationaler bildungsstandards. Eine expertise*. pedocs. http://nbn-resolving.de/urn:nbn:de:0111-pedocs-209019

Kultusministerkonferenz [KMK] (2022). *Bildungsstandards im fach mathematik für den primarbereich*. Beschluss der Kultusministerkonferenz vom 15.10.2004, i.d.F. vom 23.06.2022., https://www.kmk.org/fileadmin/veroeffentlichungen_beschluesse/2022/2022_06_23-Bista-Primarbereich-Mathe.pdf

Kortenkamp, U., & Ladel, S. (2013). Designing a technology based learning environment for place value using artefact-centric Activity Theory. In A. M. Lindmaier, & A. Heinze (Eds.), *Proceedings of the 37th conference of the International Group for Psychology of Mathematics Education* (Vol. 1, pp. 188–192). PME.

Ladel, S. (2009). *Multiple externe repräsentationen (MERs) und deren verknüpfung durch computereinsatz. zur bedeutung für das mathematiklernen im anfangsunterricht*. Verlag Dr. Kovac, Hamburg.

Ladel, S., Larkin, K., & Kortenkamp, U. (2023). *The Concept of Place Value* (In work).

Larkin, K., Kortenkamp, U., Ladel, S., & Etzold, H. (2019). Using the ACAT Framework to evaluate the design of two geometry apps: An exploratory study. *Digital Experiences in Mathematics Education, 5*(1), 59–92. 10.1007/s40751-018-0045-4

Ministerium für Kultus, Jugend und Sport Baden-Württemberg. (2016). *Bildungsplan der grundschule. bildungsplan 2016. Mathematik.* Neckar-Verlag GmbH. https://www.bildungsplaene-bw.de

Ministerium für Bildung, Jugend und Sport des Landes Brandenburg [MBJS BB]. (2015). *Rahmenlehrplan für die jahrgangsstufen 1-10 der Berliner und Brandenburger schulen. Teil C Mathematik*. https://www.bildungsplaene-bw.de

Rogers, A. (2012). Steps in developing a quality whole number place value assessment for years 3-6: Unmasking the “experts”. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics education: Expanding horizons. Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia* (pp. 648–655). Singapore: MERGA.

Robertson, S. A., & Graven, M. (2019). Exploratory mathematics talk in a second language: A sociolinguistic perspective. *Educational Studies in Mathematics*, *101*(2), 215–232. https://doi.org/10.1007/s10649-018-9840-5

South African. Department of Basic Education (SA.DBE). (2011). *Curriculum and assessment policy statement mathematics grade 1–3*. Department of Basic Education.

Thanheiser, E., & Melhuish, K. (2019). Leveraging variation of historical number systems to build understanding of the base-ten place-value system. *ZDM Mathematics Education, 51*, 39–55. 10.1007/s11858-018-0984-7

Van der Walle, J. A., Karp, K. S., & Bay-Williams. (2015). *Elementary and middle school mathematics: Teaching developmentally* (9th ed). Pearson.

# Exploring the Impact on Practice of Secondary Teachers' Beliefs and Attitudes Towards 21st Century Skills and Mathematical Proficiency

Rachael Whitney-Smith
*University of Notre Dame Australia*
rachael.whitney-smith@my.nd.edu.au

Lorraine Day
*University of Notre Dame Australia*
lorraine.day@nd.edu.au

In this paper we report on an aspect of the findings of a larger three-phase study exploring the factors that influence teachers implementing pedagogies that cultivate students' STEM capabilities and 21st century skills. Data were collected through an online questionnaire, semi-structured interviews, focus groups and case studies. This paper will focus on the findings from the first phase of this study and initial analysis of focus groups and semi-structured interviews data. Preliminary findings show that participants hold mixed beliefs concerning student proficiency in mathematics and there are common factors that influence decisions concerning the use of pedagogical practices that support students' mathematical proficiency and the development of students' 21st century skills. These factors include teachers' personal beliefs and attitudes, perceived time and curriculum constraints, student behaviour and students' academic ability.

For more than two decades, policy makers, industry representatives, academics and key stakeholders in education have promoted 21st century skills as an essential component of what students need to learn to be successful in a global society (Goos et al., 2017; Griffin et al., 2012; OECD, 2018, 2019, 2021; Trilling & Fadel, 2009). International mathematics curriculum documents and assessment frameworks such as PISA (OECD, 2018b) and TIMSS (Thomson et al., 2021) have explicitly referenced the importance of providing learning and assessment opportunities for students to demonstrate proficiency in mathematics, especially emphasising their mathematical thinking, reasoning and problem-solving skills (OECD, 2018b; Siemon et al., 2014; Siemon et al., 2018).

Mathematical proficiency is a key aspect of the Australian Curriculum: Mathematics (ACARA, 2022; ACARA, 2017a; Siemon et al, 2018) and refers to a person's ability to understand, communicate and apply mathematical concepts and procedures flexibly, to reason with and solve problems in a variety of contexts (Siemon et al., 2018; Sullivan, n.d.). The National Research Council define mathematical proficiency in a holistic manner including five interrelated components: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition; describing these as the interrelated strands of proficiency (ACARA, 2022b; Kilpatrick et al., 2001). The revised Australian Curriculum: Mathematics Version 9.0, emphasises the importance of all students developing proficiency with the mathematics they are learning, with an expectation of proficiency embedded into content descriptions and achievement standards (ACARA, 2022b).

The term 21st century skills has been universally used to describe the set of abilities and competencies that are essential to thrive in the modern world (Bialik et al., 2015; Trilling & Fadel, 2009; OECD, 2018a; Whitney-Smith et al., 2022). There exist many frameworks through which to define these skills but essentially, they hold much commonality (Bialik et al., 2015; Greenstein, 2012; Griffin et al., 2012; OECD, 2021; Whitney-Smith et al., 2022). Most existing frameworks include reference to the key skills of critical thinking, creativity, collaboration, and communication, coined the 4Cs of 21st century Skills (Greenstein, 2012; OECD, 2021). Several frameworks also include reference to problem-solving, digital literacy, global citizenship and meta-cognition (Bialik et al., 2015; Griffin et al., 2012. 21st century skills are encompassed within the Australian Curriculum: General Capabilities (AC: GC) (ACARA, 2022; Whitney-Smith et. al., 2022). The capabilities encompass more than knowledge and skills, they include behaviours and dispositions (ACARA, 2017b, 2022).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 541–548). Newcastle: MERGA.

Teacher beliefs and attitudes towards mathematics can be influenced by the philosophical beliefs they hold about education and impact on decisions made in the planning, programming, and practice of teaching, mathematics (Beswick, 2012, 2016, 2019; Marshman & Goos, 2018; Martinez-Sierra et.al., 2019; Siemon et. al., 2014). There is no consensus on a universal definition for what is a belief (Beswick, 2019; Martinez-Sierra et.al., 2019). For this study, a belief was defined as a set of attitudes, opinions, or convictions that an individual holds to be true about the world, themselves, or others (Martinez-Sierra et. al., 2019). Teacher beliefs about mathematics and mathematics education have been researched extensively (Beswick, 2012, 2019; Marshman & Goos, 2018; Martinez-Sierra et. al., 2019) and play an influential role in the decision-making processes associated with classroom practice (Beswick, 2019; Martinez-Sierra et. al., 2019).

This paper will focus on the specific research questions of the wider study that investigate teachers' beliefs and attitudes towards mathematical proficiency, the role of mathematics in the development of students' 21st century skills, and how teacher beliefs and attitudes, along with other influential factors, impact on teachers' practice.

## Methodology

This study involves the integration of both quantitative and qualitative methods adopting an explanatory sequential mixed methods design structure (Creswell, 2015; Creswell & Plano Clark, 2011), through the three phases, using the following approach, as seen in Figure 1. The adoption of a three-phase design was driven by the study's aim to explore and explain teacher beliefs, attitudes and practices towards the role mathematics plays in developing students' STEM capabilities and 21st century skills.

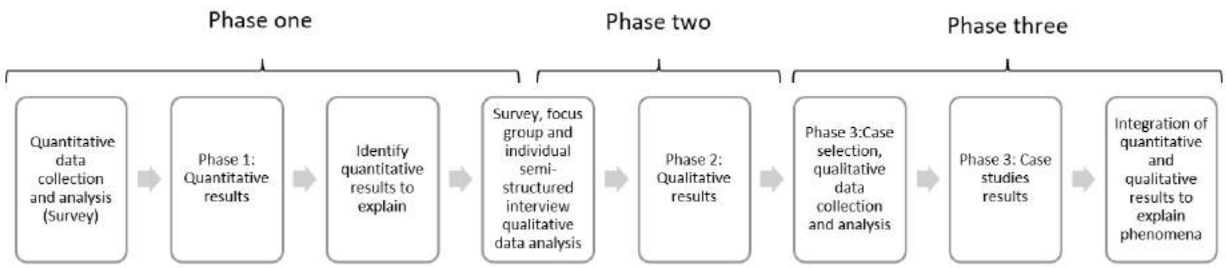



Figure 1. Mixed methods procedural diagram adapted from Creswell (2015).

### *Data Collection*

The data being reported on in this paper were collected in Phases one and two through the following means.

- Phase one: involved an online questionnaire which provided a representative sample (*n*=60). The survey tool consisted of 19 questions, seven relating to teacher background information, six using a five-point Likert scale to elicit ordinal responses and six yes/no responses that provided opportunity for qualitative explanation for the participants' response.
- Phase two: involved purposefully selected semi-structured interviews (*n*=2) and focus groups (*n*=4). Both interview and focus group participants were asked a series of 14 questions about their beliefs, attitudes and practices towards mathematics education.

### *Participants*

This study was conducted in Western Australia involving both regional and metropolitan teachers of Years 7-10 mathematics.

**Table 1**

*Teaching Background of Phase One Survey Participants (N=60)*

| | Responses | | | |
|---|---|---|---|---|
| Current teaching position | Classroom teacher (mathematics) | Head of department (mathematics) | Head of department (other) | No response |
| | 45 (75%) | 12 (20%) | 1 (1.7%) | 2 (3.3%) |
| Teaching experience | <2 years | 3-10 years | 11-20 years | 21 + years |
| | 8 (13.3%) | 23 (38.3%) | 13 (21.7%) | 14 (23.3%) |

*Note.* One participant was currently teaching mathematics at TAFE and was therefore excluded from the data set in any further analysis.

Focus groups were recruited via the survey instrument and through responses to the original recruitment emails sent to school principals. They consisted of four to six participants, consistent with the recommended focus group size (Creswell, 2015), three involving teachers from the same school and one involving teachers from two neighbouring schools from the same region of Western Australia. Being unsuccessful in recruiting further participants from two participating schools, two independent semi-structured interviews were conducted. The background of focus group and interview participants varied from highly experienced heads of learning area, to graduate, early career and out of field teachers. The school demographics of the participating teachers included both metropolitan ($n$=4), and regional ($n$=3); high (1186) and low (895) Index of Community Socio-educational Advantage (ICSEA); selective ($n$=2) and non-selective ($n$=5) and schools with a significant number of students from First Nations (27%) and non-English speaking (26%) backgrounds.

### *Data Analysis*

Initial findings from each phase informed the next in a quantitative (Phase one) → Qualitative (Phases two & three) sequential design (Creswell, 2015). This paper focuses on the quantitative analysis of survey data and preliminary findings of Phase two that draw on the analysis of qualitative survey data, focus group and semi-structured interview data using qualitative methods such as thematic analysis (Teddlie & Tashakkori, 2009). For the focus groups and semi-structured interviews, this involved systematically transcribing audio recordings, collating responses into tables and identifying patterns and themes. Codes or labels were identified for an idea or phenomenon such as "time constraints", "students' academic ability" and "student behaviour". Codes were mapped to aspects of participants responses and used to facilitate the identification of patterns, themes and meaning (Jackson & Bazeley, 2019; Krueger, 1998).

## Results

The following results stem from the analysis of Phase one and an initial analysis of Phase two data as part of the larger study. An analysis of the survey data, including qualitative responses, produced interesting findings regarding teacher beliefs and practices. When asked about their beliefs concerning whether all students are capable of learning mathematics, most 51 (91.1%) participants responded agree or strongly agree, with five (8.9%) participants responding they were undecided or disagreed. In response to whether they believed all mainstream students are capable of reasoning mathematically, most 48 (84.2%) participants agreed or strongly agreed, with six (10.5%) undecided and three (5.3%) disagreeing. Participants were also asked whether they believed that mathematical reasoning was a crucial component of being considered proficient in mathematics, 45 (81.8%) either agreed or strongly agreed, two (3.6 %) disagreed and eight (14.5%) were undecided. Thirty-seven (62.7%) responded "*Yes*" and 22 (37.3%) responded "*No*" when asked whether they taught mathematics content to all their classes through the proficiency strands of conceptual understanding, procedural fluency, authentic problem-solving and mathematical reasoning. Table 2 provides a sample of the qualitative responses given for some of the participants Yes/No responses and their assigned code.

**Table 2**

*Do You Teach Mathematical Content Through the Proficiency Strands with all Your Classes?*

| Y/N | Please explain your response | |
|---|---|---|
| No | The foundational level class would struggle with higher order thinking | A |
| No | It is easier with students who are motivated | SB |
| No | Some topics I run out of time to accurately move through each task, often only hitting understanding and fluency and not getting to the rich tasks | T, TS |
| No | I have a bottom Year 8 and we are really focussed on understanding and fluency | A |
| No | I look at the content descriptors as a first point of call as this is what the judging standards refer to. The content is packed and it is hard to review everything | CCC |
| Yes | I try to, it can be a challenge as to which behaviours you are observing and developing | SB |
| Yes | Attempt to do so but not always successful especially with my lower ability levels | A |
| Yes | Time constraints sometimes force me to jump to procedures and return to understanding if we have time (especially with classes with more gaps in their foundational understanding) | T, A |
| Yes | Although I would like to achieve a level of reasoning in all of my classes it can often be hard to achieve with some students as they are still trying to understand the concepts. I usually try to encourage my lower-level classes to feel proficient in the basics of a topic and then we can explore some problem-solving or real-world applications. Not every student will gain the understanding behind the problem-solving task, but some mathematical reasoning is at least being introduced to them. | A |
| No | I don't do this consciously or explicitly. If it occurs, it is incidental based on the questions in the textbook. | R |
| No | Conceptual understanding, procedural fluency Yes. Authentic problem-solving, mathematical reasoning, No. | TS |

*Note.* A=Academic; SB=Student Behaviour; T=Time; CCC=Curriculum Content Coverage; R=Resources; TS=Taught Separately.

Question 14 asked whether the mathematical ability of students influenced the pedagogical approach teachers adopted, with 48 (80%) participants responding "*Yes*" and nine (15%) responding "*No*". Qualitative responses supporting a response of "*Yes*" included reference to using more practical activities and concrete materials with low ability classes, with several responses claiming more behaviour problems existed in their streamed low ability classes, causing a reluctance for adopting more innovative teaching practices. One participant responded, "*For example, it's easier to do more problem-solving/rich tasks and group work with extension class because students are interested, motivated and enjoy these type of lessons and tend to take lesser time than students who need to be taught how to work together and work things out.*" Another responded, "*low ability students need more Explicit Direct Instruction (EDI) and use concrete materials. They are unlikely to benefit from things like flipped classroom.*" This response was similarly presented by a number of respondents such as, "*Weaker classes I use more manipulatives and more able classes more* *modelling and simulation activities*", "*I would use more hands-on materials with students who are struggling and more direct instruction breaking things down*" and "*Lower ability students spend more time learning skills rather than having a discussion or working with how they understanding these skills*." One "*Yes*" response provided the explanation, "*the more mathematical knowledge a student has attained, the less it is necessary to motivate their learning by demonstrating explicitly how the mathematics they are learning is used in the real-world, they begin to appreciate the power and beauty of abstract mathematics*".

In terms of beliefs, attitudes and practices relating to 21st century learning, participants were asked whether they address the Australian Curriculum: General Capabilities within their mathematics teaching programs, 48 (85%) responded "*Yes*" and nine (15%) responded "*No*". Table 3 provides a sample of the qualitative responses given for some of the participants Yes/No responses and their assigned code.

**Table 3**

*Do You Address the Australian Curriculum: General Capabilities Within Your Mathematics Teaching Programs?*

| Y/N | Please explain your response | |
|---|---|---|
| No | Try to. Numeracy is easier than most. Big focus in schools for literacy, very little for numeracy. | NA, L, N |
| No | Some of the capabilities in most lessons—literacy/numeracy, ICT but probably not all | NA, L, N |
| No | This is hard to explicitly do | C |
| Yes | Literacy and numeracy are a given | L, N |
| Yes | Some are much harder than others to address but I always looking for opportunities to link the lesson work to GCs | C |
| Yes | Numeracy is addressed when teaching maths | N |

*Note.* C=Challenging; NA=Not All; L=Literacy; N=Numeracy.

Table 4 provides the Likert scale responses concerning the impact of certain factors on participants' pedagogical choices.

**Table 4**

*What Impact do the Following Factors Have on your Pedagogical Choices?*

| | Extreme | Large | Medium | Small | Nil |
|---|---|---|---|---|---|
| Teacher beliefs and attitudes | 12 (20%) | 27 (45%) | 11 (18.3%) | 6 (10%) | 4 (6.7%) |
| Behaviour management | 9 (15%) | 23 (38.3%) | 16 (26.7%) | 6 (10%) | 4 (6.7%) |
| Students' academic levels | 11 (18.3%) | 26 (43.3%) | 16 (26.7%) | 3 (5%) | 4 (6.7%) |
| Time constraints | 16 (26.7%) | 25 (41.7%) | 14 (23.3%) | 1 (1.7%) | 4 (6.7%) |
| Curriculum Year level | 8 (13.3%) | 21 (35%) | 18 (30%) | 8 (13.3%) | 5 (8.3%) |

These findings from Phase one, influenced the following Phase two focus group and interview questions, '*Do you believe that proficiency in mathematics is an achievable expectation for all students in junior secondary mathematics*?', '*Do you teach through the proficiency strands with all of your mathematics classes and why or why not*?', '*Who or what influences decisions when it comes to selecting a teaching and learning approach for teaching your lower secondary mathematics classes*?', '*What role do you see mathematics plays, as a school subject, in developing students' General Capabilities?'* and '*Do you consider all three-dimensions of the Australian Curriculum when planning, teaching and assessing students in your lower secondary mathematics classes?'* The initial analysis of Phase two data collected during focus groups and interviews show some emerging themes, consistent with themes emerging from the qualitative survey data such as, time constraints, curriculum and assessment implications, perceived student ability levels, and classroom behaviour management.

## Conclusions and Implications

Teachers' beliefs and attitudes need to support the role of mathematics in developing students' 21st century skills and the provision for learning opportunities for all students to develop proficiency with the mathematics they are learning, if the curriculum is to be implemented as intended (ACARA, 2022; OECD, 2021). Given the larger study in which this research sits has adopted a mixed methods approach, findings from the qualitative data collected in Phases two and three of this study will be used to explain the quantitative results of Phase one (Creswell, 2015; Creswell & Plano Clark, 2011; Teddlie & Tashakkori, 2009). Phase one data showed that most teachers 53 (88%) pedagogical choices are influenced by the academic ability of the students in their class, which was also an emerging theme in participant responses during Phase two focus groups and interviews. In Phase one, although 53 (88.3%) surveyed participants responded that they agree or strongly agree mathematics is an integral part of 21st century learning, and 53 (88.3%) either agreed or strongly agreed that the use of real-world and authentic contexts is important, when asked how often they use real-world contexts 29 (48.3%) participants responded either sometimes or rarely and 45 (75%) participants responded that they either sometimes, rarely or never use authentic problem-solving (other than word problems). Time constraints, classroom behaviour, students' academic ability and the pressure of student assessment in mathematics are some of the early themes emerging from the Phase two analysis, in response to questions relating to the factors that influence participants planning and teaching of mathematics. Early findings from Phase two data analysis also show an apparent disparity between several of the respondents' beliefs and attitudes concerning what should happen to what happens in practice. This is consistent with findings from previous studies that suggest an inherent disparity between teachers' classroom practices and their perceptions of best practice (Beswick, 2012; Clark & Lomas 2016).

## References

Australian Curriculum, Assessment and Reporting Authority. (2017a). *The Australian curriculum: Version 8.4.* ACARA. https://www.australiancurriculum.edu.au/

Australian Curriculum, Assessment and Reporting Authority (2017b). *The Australian curriculum: General capabilities.* ACARA. https://www.australiancurriculum.edu.au/f-10-curriculum/general-capabilities/

Australian Curriculum, Assessment and Reporting Authority. (2022). *The Australian curriculum: Version 9.0.* ACARA. https://v9.australiancurriculum.edu.au/

Bailik, M., Fadel, C., Trilling, B., & Goff, J. (2015). *Skills for the 21st century: What should students learn?* Center for Curriculum Redesign.

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics, 79*(1), 127–147.

Beswick, K. (2019). The role of knowledge and beliefs in helping learners to progress their mathematical understanding. *Journal of Mathematics Teacher Education, 22*(2), 125–128.

Clarke, D., & Lomas, L. (2016). Changes in teachers' knowledge and beliefs about mathematics and mathematics teaching: A case study. In B. White, M. Chinnappan, & S. Trenholm (Eds.), *Opening up mathematics education research. Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia* (pp. 391–398). Adelaide: MERGA.

Creswell, J. (2015). *A concise introduction to mixed methods research.* Sage Publications.

Creswell, J., & Plano Clark, V. (2010). *Designing and conducting mixed methods research* (2nd ed.) Sage Publications.

Goos, M., Vale, C., Stillman, G., Makar, K., Herbert, & Geiger, V. (2017). *Teaching secondary school mathematics: Research and practice for the 21st century*. Allen & Unwin.

Greenstein, L. (2012). *Assessing 21st century skills: A guide to evaluating mastery and authentic learning*. Corwin.

Griffin, P., McGraw, B., & Care, E. (Eds.) (2012). *Assessment and teaching of 21st century skills.* Springer. https://doi.org10.1007/978-94-007-2324-5

Jackson, K., & Bazeley, P. (2019). *Qualitative data analysis with NVIVO.* (3rd Edition) Sage Publications.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* National Academy Press.

Krueger, R. (1998). *Analyzing and reporting focus group results: Focus group kit 6*. Sage Publications.

Martinez-Sierra, G., Garcia-Garcia, J., Valle-Zequeida, M., & Dolores-Flores, C. (2019). High-School mathematics teachers' beliefs about assessment in mathematics and the connections of their mathematical beliefs. *International Journal of Science Education, 18*, 485–507.

Organisation for Economic Cooperation and Development (2018a). *The future of education and skills education 2030: The future we want.* OECD Publishing.

Organisation for Economic Co-operation and Development. (2018b). *PISA 2021 mathematics framework*. OECD Publishing. https://www.oecd.org/pisa/sitedocument/PISA-2021-mathematics-framework.pdf

Organisation for Economic Cooperation and Development. (2019). *The future of education and skills education 2030: OECD learning compass 2030.* OECD Publishing. https://www.oecd.org/education/2030-project/contact/OECD_Learning_Compass_2030_Concept_Note_Series.pdf

Organisation for Economic Cooperation and Development. (2021). *What students learn matters: Towards a 21st century curriculum.* OECD Publishing. https://doi.org/10.1787/d86d4d9a-en

Siemon, D., Callingham, R., Day. L., Horne, M., Seah., R., Stephens, M., & Watson, J. (2018). From research to practice: The case of mathematical reasoning. In Hunter, J., Perger, P., & Darragh, L. (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 40–49). Auckland: MERGA.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R., & Warren, E. (2014). *Teaching mathematics: Foundations to middle years*. Oxford Univesity Press.

Sullivan, P. (n. d.). Using the proficiencies from the Australian mathematics curriculum to enrich mathematics teaching and assessment. https://www.australiancurriculum.edu.au/media/6610/using-the-proficiencies-to-enrich-mathematics-teaching-and-assessment.pdf

Thomson, S., Wernert, N., Buckley, S., Rodrigues, S., O'Grady, E., & Schmid, M. (2021). *TIMSS 2019 Australia. Volume II: School and classroom contexts for learning*. Melbourne: Australian Council for Educational Research (ACER). https://doi.org/10.37517/978-1-74286-615-4

Trilling, B., & Fadel, C. (2009). *21st century skills: Learning for life in our times*. Jossey-Bass.

Bishop, A., FitzSimons, G., & Seah, G. W. T. (1999). Values in mathematics education: Making values teaching explicit in the mathematics classroom. *Proceedings of the 1999 annual conference of the Australian Association for Research in Education*. Melbourne. http://www.aare.edu.au/99pap/bis99188.htm

Davey, G., & Pegg, J. (1991, July). Angles on angles: Students' perceptions. *Proceedings of the 14th annual conference of the Mathematics Education Research Group of Australasia*. Perth: MERGA.

Forgasz, H. J., & Leder, G. C. (1998). Affective dimensions and tertiary mathematics students. In A. Olivier, & K. Newstead (Eds.), *Proceedings of the 22nd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 296–303). PME.

Galbraith, P., Haines, C., & Pemberton, M. (1999). A tale of two cities: When mathematics, computers, and students meet. In J. M. Truran, & K. M. Truran (Eds.), *Making the difference. Proceedings of the 22nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 215–222). Adelaide: MERGA.

Mitchelmore, M. C., & White, P. (1995, April). Development of the angle concept by abstraction from situated knowledge. *Paper presented at the annual meeting of the American Educational Research Association*. San Francisco, CA. (ERIC Document Reproduction Service No. ED 416 066)

Outhred L., Farrell, J., & Cotter, K. (1996, July). How well do student teachers integrate science and mathematics concepts? A pilot study. *Paper presented to working group 1 at the 8th International Congress on Mathematical Education*, Seville, Spain.

Sowey, E. R. (2020). Statistics education that inspires long-term retention of learning: A qualitative model. *Statistics Education Research Journal, 19*(1), 1–28. https://doi.org/10.52041/serj.v19i1

Sullivan, P., Warren, E., & White, P. (2000). Students' responses to content specific open-ended mathematical tasks. *Mathematics Education Research Journal, 12*(1), 2–17.

Truran, J., & Truran, K. (1998). Using clinical interviews in qualitative research. In J. A. Malone, B. Atweh, & J. R. Northfield (Eds.), *Research and supervisions in mathematics and science education* (pp. 41–63). Lawrence Erlbaum.

# Searching for, Sifting Through, and Selecting Curriculum Materials for Mathematics Planning During Practicum

Susanna Wilson
*University of Canterbury*
sue.wilson@canterbury.ac.nz

This paper describes results from a case study about how a primary pre-service teacher (PST) used curriculum materials (CMs) when planning for a mathematics lesson during her final practicum. The data is drawn from a doctoral study (in progress) and results show how the PST initiated an active process of searching for and sifting through CMs on a familiar website to make selections for a lesson. Selections were based on several aspects, including the mathematics focus of her lesson, curriculum connections, her chosen teaching approach and mathematical representation for teaching multiplication. Implications for mathematics Initial Teacher Educators (ITEs) are discussed.

Planning for teaching is a complex process which occurs for teachers and PSTs at a psychological and a practical level. Psychologically it involves teachers thinking about and making decisions for lessons, then translating these into practical actions for teaching which are recorded on planning documents (Clark & Peterson, 1986; John, 2006). Shulman (1987) describes teacher and PST planning as a process of pedagogical reasoning where,

> An idea is grasped, probed, and comprehended by a teacher, who then turns it about in his or her mind, seeing many sides of it. The idea is shaped or tailored until it can in turn be grasped by students (p. 13).

In the field of mathematics education there is widespread recognition that what teachers and PSTs think about when planning for lessons is also complex, due to the many aspects of knowledge that need to be considered. Examples of these include what mathematics ideas to teach and what pedagogical approaches to use (Ball, 2000). These decisions are made with students in mind, particularly how lesson content can connect with their experiences, contexts, and interests (Grossman & Thompson, 2008).

Due to the complexity and importance of planning processes, planning is a core component of ITE programmes, including mathematics education courses. Course and assessment work often includes planning experiences which approximate the planning practices of more experienced teachers with the aim of supporting PSTs to learn how to plan in preparation for mathematics teaching during practicum (Grossman & Thompson, 2008). An important part of these experiences is how to search for and select CMs such as hard copies of textbooks, teacher guides, student texts, and internet sites. A rationale for this is that PSTs often rely heavily on CMs as the base for their lessons, looking to them for guidance about what to teach and how to do this (Grossman & Thompson, 2008). Unlike more experienced teachers PSTs are only beginning to build up a repertoire of CMs to use when planning, relying on those gained from course work and practicum experiences (Ensor, 2001). During practicum they can spend a considerable amount of time finding CMs for their lessons, and with limited experience can have difficulty making selections when planning (Mutton et al., 2011).

Within ITE courses it is important then, that PSTs are taught how to select CMs and sift through these with purpose before making selections for their mathematics lessons (Amador & Earnest, 2019). This is particularly important in countries like New Zealand (NZ) where there are no mandated CMs, meaning PSTs can choose what they like, and often use the internet as a main source (Caniglia & Meadows, 2018). Knowing how they are searching for these and how they select them for teaching during practicum would help ITEs provide targeted support for PSTs during course work. Unfortunately, examining how they do this is challenging, because PSTs thinking and decision-making processes during planning are often "invisible" (Choy et al., p.3). Whereas

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 549–556). Newcastle: MERGA.

practicum teaching can be observed and lesson plans analysed, PSTs mental planning processes are harder to access (Kavanagh et al., 2020). Additionally, while there is some research about how teachers use CMs when planning for mathematics teaching, there is very little about how PSTs do this (Earnest & Amador, 2019; Wilson & McChesney, 2018). This means there is a research gap about these important planning practices. This paper offers a contribution to this limited field by presenting results about how one PST planned for mathematics teaching during practicum. The specific research question is:

- How did one final year PST search for, sift through, and select CMs when planning for mathematics lessons during practicum?

## Background Literature

### *Beginning Teacher's Use of Curriculum Materials for Planning*

In their study of beginning teachers (BTs) and how they used CMs when planning English lessons, Grossman and Thompson (2008) found searching for these was time consuming. Similar to the teachers in Ensor's study (2001), the BTs did not have a collection of CMs built up from experience, so had to develop this during the early stages of their career. Common sources of CMs were their teaching colleagues, libraries, and the internet, and included teacher texts, student books, and units from internet sites. They searched through these and made selections by looking for CMs which aligned with what they had to teach, and how they wanted to do this. Lesson overviews were popular because they provided a scaffold for their lessons. Some BTs initially stuck closely to published lesson suggestions when planning, eventually adapting these as they became familiar with learner needs. This was described as "playing around" with lesson suggestions (Grossman & Thompson, 2008, p.7).

### *Teacher's Use of Curriculum Materials for Mathematics Planning*

Likewise, Kaufmann et al. (2002) examined how BTs used CMs; in their case they focussed on mathematics planning. Rather than having to search for CMs, the BTs were given a mathematics textbook and a teacher's guide. They began planning by reading the textbook lesson suggestions and teacher notes, searching for ideas for their lessons. Some also followed the lesson suggestions as written, others reordered activities to suit their lesson structures, and others made adaptations, for example, changing activities to better suit their students. The BTs also added tasks such as problem-solving tasks to add depth to their lessons. Overall, their choices of the CMs were influenced by their beliefs about effective mathematics teaching, for example, that mathematics lessons should include activities where students use critical thinking skills when finding solutions to problems. Similarly in her study, but with more experienced mathematics teachers, Superfine (2008) found the teachers also substituted suggested activities by providing alternative tasks for students. One example included adding in practice problems, instead of word problems, believing this was a better way for some students to learn the mathematics ideas that were central to her lesson. Importantly, in both studies the BTs and the more experienced teachers demonstrated agency when planning. They had the flexibility and the authority to make decisions about what to choose, adapt, substitute, and add from the CMs for their lessons. The needs of their students were an important consideration when making these decisions.

### *PSTs Use of Mathematics Curriculum Materials for Mathematics Planning*

In one of the few studies which examined how PSTs used mathematics CMs when planning Amador and Earnest (2019) directed PSTs to use a textbook to plan a mathematics lesson plan during course work. While planning a fractions lesson one group of PSTs read the lesson suggestions in the textbooks and analysed these looking for word problems to use with students. They also decided to adapt these for their lessons. One example included changing the context in a word problem to a context that was more familiar to students. The original context was a rectangular brownie which students had to split into equal parts, and this was changed to a rectangular chocolate bar. The PSTs thought that making this change would help students connect with the fractions concept at the centre of their lessons. Amador and Earnest (2019) caution that while this decision was well intentioned, the PSTs focus on adapting the context distracted them from making decisions about how to teach the mathematics concepts in the lesson. They suggest that when planning and making adaptations PSTs need to prioritise the important mathematics concepts they need to teach and only use real life contexts to support this learning.

In another study which examined the planning practices of first year PSTs during practicum, Wilson and McChesney (2018) found these novice teachers had to spend time searching widely for CMs for their lessons. Most of these PSTs searched the internet for CMs preferring to use websites they trusted, such as *nzmaths* (Ministry of Education, n.d). This was because they were familiar with the website and knew how to navigate within it to find CMs for their mathematics lessons. Aspects within CMs that they searched for included the mathematics focus, national curriculum connections, and like the PSTs in Amador and Earnest's (2019) study, activities which had real life contexts. They also searched for teacher notes which included suggestions for how to teach specific activities, and solutions for problems which they could learn before teaching their lessons.

## Research Design

The doctoral study is a qualitative study which used an interpretive methodological approach, specifically a multiple case study (Yin, 2014). This approach was chosen because it enabled the author to carry out an in-depth investigation into the planning practices of the PSTs (Cohen et al., 2011). There are four individual cases in the study, where a case is a PST completing their final five-week practicum of their three-year undergraduate primary teaching degree. The four cases were selected from a group of volunteers and were purposely chosen because of the year level they were teaching (the author wanted a range of age groups), and their proximity to the author so that data could be easily collected during the study. The author was their mathematics education lecturer but did not teach or assess the participants' work during the data collection period. Ethical consent was granted for the study. Data collection methods included one focus group interview with all participants, and self-audio recorded think-aloud episodes which each PST carried out at three different stages during practicum. Additionally, individual interviews were carried out with each PST after practicum, and their mathematics lesson plans were analysed.

The data for this paper is from the first case study, a participant called Kate (a pseudonym). The data sources used are Kate's first audio recorded think-aloud episode which she recorded in the second week of practicum. During this episode, Kate talked aloud as she planned, describing what she was thinking about and deciding for her lessons. This recording was transcribed and returned to Kate for checking before being analysed. The lesson plan from this episode was also analysed, along with Kate's interview transcript. The analysis process followed an iterative process of coding and categorising the data into themes (Miles et al., 2018). Codes were developed by identifying and highlighting key words and commonly occurring planning decisions and actions that Kate made while planning. These codes were collated into categories which were then organised into themes within the case. The use of multiple data sources enabled the author to corroborate and strengthen these themes during the analysis process. The results from one of these themes, about how Kate searched for, sifted through and selected CMs when planning her first practicum mathematics lesson are now described.

# Results

## *Kate's Practicum Setting*

Kate's practicum setting was a rural school, with year three and four students (ages seven and eight). These students were organised into two mixed ability groups, and she had to teach multiplication, specifically multiplication as repeated addition. Her mentor teacher (MT) gave her "free reign" to teach how she wanted and to choose resources and tasks for her lessons. To help make these decisions, Kate observed her MT teaching mathematics lessons in week one to gather information to help with her planning decisions for week two lessons. She noticed students spent a lot of time listening during her MTs lessons, so decided to choose a word problem approach which she had used on a previous practicum. This approach allowed students to use and apply multiplication skills, for example skip counting, as well as arrays which she had observed them learning in week one. Kate also wanted them to talk and work together when solving the word problems.

## *Searching for Curriculum Materials*

Kate began planning by searching for curriculum materials for her lessons and going straight to *nzmaths*, describing this as "jumping into *nzmaths*". In the interview she explained that she trusted the CMs on this site because they were authored by NZ mathematics educators and she had used it during previous practicums, so was familiar with its content. She searched for specific multiplication resources and tasks and found a unit of work called *Arrays Hooray*. She read it and noticed it used multiplication word problems and the multiplication representation of arrays. She had chosen both aspects for her lessons so decided to save the unit to look at later and continued searching on *nzmaths* for other CMs.

She returned to the *nzmaths* homepage and began another search for possible resources and tasks, this time finding an online copy of a teacher guide called *Book 6: Teaching multiplication and division* (Ministry of Education, 2008). She paused, before deciding she should have a look at this because it was a "Ministry book". She read the book and noticed a task in the teacher notes which had a word problem, "Kayla had four bags of marbles. There are six marbles in each bag. How many marbles does Kayla have?" (p. 4). This problem focussed on multiplication as adding equal sets and included an image of bags of marbles. Kate recognised this as being a different representation to arrays, and considered using it in her lesson saying, "hmm, it would be good to step away from doing arrays, and to throw in something different", but decided she didn't want "to confuse students with something new". She continued looking through the book and noticed two other tasks, *Three's Company* (p.12) which taught multiplication as skip counting, and *Animal Arrays* (p. 15) which used arrays. She thought back to the tasks in the *Arrays Hooray* unit she had already found and decided they aligned better with her chosen approach because they used word problems. She also saved *Book 6* and continued searching.

She returned to the homepage again, did another search for multiplication resources and tasks, and found the *Problem-Solving Activities* section. She focussed on the level two tasks which aligned with the NZC (2007) objectives and chose one called *Sharing Lollies* because it was labelled as multiplication activity. Kate read and analysed the activity and decided it focussed on fractions and division, so decided not to use it. However, she also saved it as a possible activity for future use.

## *Sifting Through Curriculum Materials*

In the interview Kate described this initial searching through *nzmaths* as a process of "researching" which involved "navigating through the website", and "sifting" through what she found for suitable resources and tasks for her lessons. "Navigating" meant moving between different sections on the website, "sifting" meant analysing what she found and deciding whether to keep or discard it for future use. Possibilities were saved as she searched and sifted, and these decisions were influenced by how they aligned with the word problem approach and the multiplication representation she had chosen. However, other options were also saved, such as the problem-solving tasks, and the *Book 6* tasks, as Kate anticipated what she might need in future lessons.

*Searching Within the Unit Plan and Making Adaptations*

Kate's next planning decision was to select the *Arrays Hooray* unit as the base for her first lesson. She opened this and began a second phase of searching and sifting through this resource, again, deciding what to use and what to adapt from it. This involved a more focussed evaluation of the contents of the unit, which Kate did by reading the unit carefully, and making selections based on her prior decision to use the word problem approach and arrays as the multiplication representation. She re-read the word problems, searching and sifting through these looking for examples which used multiplication with single digit numbers, deciding these were the number values that were suitable for her students. She selected one which was, "Tame has an orange orchard with 6 rows of trees. In each row there are 8 trees. How many trees does Tame have altogether?" (p.2) but decided to modify it by changing Tame's name to Tom, who was one of her students, the orange orchard to an apple orchard, and the multiplication factors to "five lots of four". Kate made these changes to connect the problem to her students. She explained students liked hearing their names in the problems, an apple orchard was more familiar than an orange orchard and working with groups of five was easier than groups of eight. A copy of the new problem that Kate wrote on her lesson plan is produced in Figure 1.

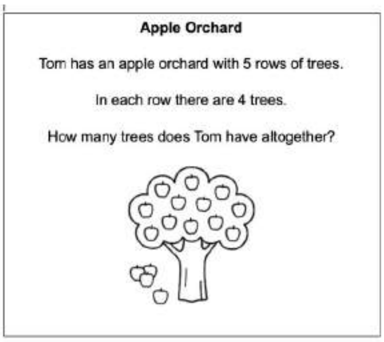



Figure 1. The apple orchard problem.

Kate continued searching through the unit and found teachers' notes with information about how to use the arrays, how students might solve the word problems, and useful mathematics terms for use in the lesson. Kate read these notes and commented that they were "very helpful" so decided to select this information and copy it onto her lesson plan for use during her lesson. This action completed her searching, sifting and selecting process for her first lesson.

## Discussion

The results show that given "free reign" to choose the CMs for her mathematics lessons Kate chose one source, that is, the *nzmaths* website. This is contrary to the suggestions made by Grossman and Thompson (2008) that novice teachers access a range of sources of CMs. Throughout her planning session she did not look at any hard copies of teacher or student guides and did not mention the need to find other sources. This indicates that she trusted *nzmaths* to provide her with a sufficient range of CMs, and that these would be suitable for her lessons. This was like the PSTs in Wilson and McChesney's (2018) study who also found that PSTs opted to use *nzmaths* as a source of CMs for mathematics planning and trusted it because it was authored by the Ministry of Education which gave it status. The author as Kate's mathematics teacher educator often used this site during course work and was not surprised to see Kate using it, however, did not expect it to be the only source as there are many hard copies of CMs available for NZ teachers to use, as well as a vast range of websites. This suggests a range of hard copy CMs were not available to Kate in her practicum setting, which led to her decision to only use an internet site. Her sole use of *nzmaths* suggests she decided that the site contained all she needed for her lessons and therefore did not need to search for other sites. This decision made it relatively easy for her to find CMs for her lesson, which questions the suggestion by Mutton et al, (2011), that PSTs can have difficulty finding these resources during practicum.

Kate searched and navigated within the site using the information she received from her MT to begin this process. Knowing the curriculum level and the mathematics focus of multiplication enabled her to carry out a targeted search for CMs for her lessons. These clear parameters meant Kate was only presented with CMs that aligned with this information which meant she did not have to search throughout the site herself looking for lesson options. She was also able to trust that the CMs she was presented with connected with her lesson focus in some way. Finetuning her navigating process also saved her time. In addition to the CMs she was presented with Kate also chose to broaden her search to find other possibilities. Her searches through the problem-solving section on *nzmaths* and within *Book 6* are example of this. This shows she did not want to be restricted to only using the website suggestions, preferring to have some control over her choices of CMs. Her prior decisions to use word problems and the representation, along with the curriculum level and mathematics focus of her lesson helped guide her search for these other CMs. Similarly, the teachers in the study by Kaufmann et al. (2002) looked beyond the CMs they were working with to find different options for their lessons. In both cases, these teachers and Kate wanted to ensure they had a range of CMs for their lessons, which shows they were considering their students and the kinds of activities that would promote effective learning of the mathematics ideas in their lessons. Both groups of PSTs were also confident to move beyond the main CMs they were using to find other options for their lessons.

Kate's next level of searching involved sifting through her selections using her pedagogical choices of the teaching approach (the word problems) and the mathematical representation (the array) to guide her final selection of CMs for her lesson. These aspects along with the curriculum alignment and lesson focus informed her decision to choose the unit plan, because it contained all of these aspects. Grossman and Thompson (2008) suggest that novice teachers often choose and rely on published lesson scaffolds to provide ideas for pedagogical approaches for lessons. However, Kate did this the other way around by choosing CMs that connected with her pre-selected pedagogical decisions. In this way the CMs did not direct her teaching, instead due to her clarity about how she wanted to teach she was able to discard or keep CMs based on her pedagogical decisions.

Kate's clarity about how she wanted to teach also guided her evaluation of the word problems within the unit. One of her final actions was to read these closely to determine if she needed to make changes to them for her lesson. During this process of searching through the problems she again used her pedagogical decisions to help her decide what to keep from the problems and what to change. Her checking of the multiplication factors and the context within the orange orchard problem are examples of this. Kate's decision to change the factors to numbers her students could work with shows she was thinking about the importance of ensuring the numbers she chose would be accessible for her students, ensuring they were not too hard or too easy. Additionally, her decision to also check that the mathematics in the problem could be represented using the array shows that she was prioritising using this tool as an important part of her lesson. Her decision to change the context to the more familiar apple orchard context was made to ensure students could connect with the mathematics within the word problem. While this was important, the other aspects of the mathematics focus and using the representation seemed to be a priority during her decision-making process. Unlike the PSTs in Amador and Earnest's (2019) study her adaptations had a broader focus beyond merely changing the context for students. Again, her clear view about what she had to teach and how she what wanted to do this, guided her decisions, this time when making adaptations to the word problems. These final actions also helped her feel confident that the problems from the unit were ready to use in her lesson and that they were accessible to the students in her practicum setting.

During Kate's final look through the unit plan she noticed important information for teachers about how to use the array representation, what mathematical terms to use, and possible learner solution strategies. Her decision to copy these onto her lesson plan, shows she valued this information for teaching. While these aspects did not influence her choice of CMs they were selected as an important part of her lesson. This indicates that teachers' notes may be another aspect PSTs look for when making choices of CMs for their lessons and is worthy of future investigation.

## Implications and Conclusion

The author acknowledges that this paper is limited to reporting the results from one case study, however the identification of the three phases of Kate's planning process, and the specific aspects she used to make CM selections for her lessons, makes visible how she used CM when planning during practicum. This information is useful for mathematics ITEs because it is an authentic example of a planning process originating from a PST and could therefore be used within course work to inform the design of a planning process, particularly related to the searching and selecting of online CMs and their use during planning. In this way processes like Kate's could be used to guide rehearsals during course work where PSTs could practise with guidance from ITEs, how to search for, sift through, and select CMs for mathematics lessons. This could also include making adaptations to activities for students. This would help PSTs prepare for carrying out similar processes during practicum, where they, like Kate may have to find their own CMs and work independently when preparing their mathematics lessons for teaching. Kate's familiarity with *nzmaths* from course work and previous practicums allowed her to easily navigate within it, which had the added benefit of saving her time when planning. Becoming familiar with the site during course work would also support PSTs to carry out efficient planning processes during practicum, where a challenge is often having enough time to complete planning for mathematics lessons.

Kate's pedagogical choices for her mathematics lessons helped her search, sift and select CMs and make final adaptations and selections for her lessons. It seems that the teaching of effective pedagogical approaches for mathematics teaching within ITE courses not only benefits PSTs understanding of how to teach mathematics, but also supports them to critically analyse and select appropriate CMs for their lessons. Superfine (2008) agrees that this is the case for more experienced teachers, so it is important that PSTs are also supported to do this early in their careers. As suggested by Caniglia and Meadows (2019) this is particularly important, given the variety and extent of hard copy and online CMs available for all teachers to choose from.

Finally, Kate is one case from a larger study, and the author was impressed by her sense of agency and her confidence to initiate this planning process which enabled her to choose and adapt CMs for her mathematics lessons. Going forward, the identification and naming of these planning practices will be used to examine how the other PSTs in the larger study worked with CMs as they planned mathematics lessons during practicum.

## References

Amador, A., & Earnest, D. (2019). Launching forth: Pre-service teachers translating elementary mathematics curriculum into lessons. *Mathematics Education Research Journal*, *31*, 301–323. https://doi.org/10.1007/s13394-018-0254-6

Ball, D. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, *51*(3), 241–247. https://doi.org/10.1177/0022487100051003013

Clark, C., & Peterson, P. (1986). Teachers thought process. In M. Wittrock (Ed,). *Handbook of research on teaching,* (3rd ed., pp. 255–296). Macmillan.

Caniglia, J., & Meadows, M. (2018), Pre-service mathematics teachers' use of web resources. *International Journal of Technology in Mathematics Education, 25*(3),17–34.

Choy, B. H., Dindyal, J., & Yeo, J. B. W. (2022). Making visible a teacher's pedagogical reasoning and actions through the use of pedagogical documentation. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys*. *Proceedings of the 44th annual conference of the Mathematics Research Group of Australasia* (pp. 138–145). Launceston: MERGA.

Cohen, L., Manion, L., & Morrison, K. (2011). *Research methods in education* (7th ed.). Routledge.

Ensor, P. (2001). From pre-service mathematics teacher education to beginning teacher: A study in recontextualising. *Journal for Research in Mathematics Education, 32*(3), 296–320. https://doi.org/10.2307/749829

Grossman, P., & Thompson, C. (2008). Learning from curriculum materials: Scaffolds for new teachers. *Teaching and Teacher Education, 24(*8*)*, 2014–2026. https://doi.org/10.1016/j.tate.2008.05.002

John, P. (2006). Lesson planning and the student teacher: Re-thinking the dominant model. *Journal of Curriculum Studies, 38*(4), 483–498.

Kauffman, D. (2002, April 1–5). A search for support: Beginning elementary teachers' use of mathematics curriculum materials. In *Annual meeting of the American Educational Research Association*. New Orleans. https://projectngt.gse.harvard.edu/publications/search-support-beginning-elementary-teachers-use-mathematics-curriculum

Kavanagh, S. S., Conrad, J., & Dagogo-Jack, S. (2020, Mar). From rote to reasoned: Examining the role of pedagogical reasoning in practice-based teacher education. *Teaching and Teacher Education, 89*, 102991. https://doi.org/10.1016/j.tate.2019.102991

Miles, M. B., Huberman, A. M., & Saldana, J. (2018). *Qualitative data analysis: A methods sourcebook* (4th ed.). Sage Publications.

Ministry of Education. (n.d.). nzmaths. https://nzmaths.co.

Ministry of Education. (2007). *The New Zealand curriculum*. Wellington: Learning Media. https://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum

Ministry of Education. (2008). *Book 6: Teaching multiplication and division*. Learning Media.

Mutton, T., Hagger, H., & Burn, K. (2011). Learning to plan, planning to learn: The developing expertise of beginning teachers. *Teachers and Teaching, 17*(4), 399–416. https://doi.org/10.1080/13540602.2011.580516

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*, 1–22.

Superfine, A. (2008). Planning for mathematics instruction: A model of experienced teachers' planning processes in the context of a reform mathematics curriculum. *The Mathematics Educator, 18*(2), 11–22.

Wilson, S., & McChesney, J. (2018). From course work to practicum: Learning to plan for mathematics teaching. *Mathematics Teacher Education and Development, 20*(2), 96–133.

Yin, R. (2014). *Case study research design and methods* (5th ed.). Sage Publications.

# Round Tables

# Gender Diversity and Mathematics: Implications and Directions For Future Gender Research

Helen Forgasz
*Monash University*
helen.forgasz@monash.edu

Gender diversity has been officially recognised by the Australian government. No longer is the male/female binary the sole means of gender identification. The census, birth certificates, and government data, for example, are now reported with three gender identifiers: male, female, and (in general) gender diverse. While controversy abounds in the literature about the terminology used for gender identification, the official terms adopted by Australian government authorities are pragmatic and understood in society.

With respect to education, each state/territory already has prepared (or will soon) guidelines for schools on issues related to gender diverse students. To date, Victoria, through the Victorian Curriculum and Assessment Authority (VCAA), appears to be the only state/territory in which enrolments and achievements of students completing the Victorian Certificate of Education (VCE) are reported for the three gender categories since 2017.

## *The Session*

In this session, I will share some of the data on the representation of gender diverse people in society. I will also present some of the research findings on gender diverse students and their schooling experiences, teachers' and pre-service teachers' views about gender diverse students, as well as the limited research results on gender diverse students that are specifically focussed on mathematics and science.

The VCE enrolment data for two of the three mathematics subjects offered at the VCE level by gender (male/female/gender diverse) will be presented; the numbers of gender diverse students were too small to include the third subject. The enrolment patterns identified will be discussed.

Drawing attention to the known experiences of gender diverse students and their presence in mathematics classrooms—a situation that cannot be ignored—is an important starting point. With time, gender diversity in schools, and in mathematics classrooms in particular, will require more attention to be paid to the needs of these students. But, very little is known about their needs in mathematics classrooms, or of the obstacles they do or might face. There are minimal data available to work with. The necessity for knowledge on how best to facilitate the mathematics learning of all students, how best to be inclusive in general, and gender inclusive in particular, in the mathematics classroom are areas wide open for research.

What are the implications for gender diverse students of what is being done in mathematics classrooms today? Are changes needed? If so, what kinds of changes? What are the implications for those undertaking gender research in mathematics education? In the session, I aim to open discussion to explore these issues, examine possible ways forward, and identify avenues for future research.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia (*p. 559). Newcastle: MERGA.

# Mathematical PlayWorld: A New Practice of Teaching Young Children's Mathematics in Play

Liang Li
*Monash University*
liang.li@monash.edu

Leigh Disney
*Monash University*
leigh.disney@monash.edu

Empirical studies have been undertaken to further understand children's mathematical learning in play-based settings (Worthington et al., 2020; Magnusson & Pramling, 2018; Worthington & van Oers, 2016; Poland & van Oers, 2007;). However, there is little know on how imaginative play can be intentionally interwoven into children's mathematical teaching programs. Thus questioning how preschool teachers can design programs which foster young children's mathematical problem-solving and thinking in imaginative play.

To better understand children's mathematical problem-solving process and actions and how this is supported by joint imaginative play between children and teachers, this paper draws upon the Vygotskian concepts of play and imagination to study how imaginative play can be designed to promote children's mathematical problem-solving and thinking. The lens in this study centred upon how imaginative play becomes a meaningful context to motivate children's exploration of the concept of measurement in play.

In extending on previous literature, the study specifically reports a new practice, drawing upon Fleer's (2018) Conceptual PlayWorld approach, which supports children's mathematical thinking and learning, titled a "Mathematical PlayWorld" (Disney & Li, 2022; Li & Disney, 2023). This approach starts with a selected story, where children and teachers build emotional connections with the story characters, taking character roles while they enter the playworld space to investigate an emotionally charged mathematical problem. Through the visual narrative methodology, 14 hours of video observation, focus group discussion and reflective interviews have been used to analyse two teachers and a group of 11 preschoolers' joint imaginative play in the *Mathematical PlayWorld*. Within the joint imaginary situation, children actively engaged with concepts of measurement to solve the mathematical problem of "*how to design a broom to fly to the witch's sister's birthday party…how long does it need to be*?". As children constructed a broom out of large blocks, their understanding of measurement allowed them to make comparisons of the attributes to be measured, such as long and short and allowed teachers to introduce concepts such as comparing, ordering, and matching while imagining the length of the new broom.

This study found that Mathematical PlayWorld creates motivating conditions to support children's mathematical learning in imagination and support the quality of intentional mathematics teaching in play-based early years settings. We argue that joint imaginative play should be promoted to support children's mathematics learning as it motivates children's active exploration of mathematics concepts, thus increasing the quality of mathematics education.

## References

Disney, L., & Li, L. (2022). Above, below, or equal? Exploring teachers' pedagogical positioning in a playworld context to teach mathematical concepts to preschool children. *Teaching and Teacher Education, 114* (10) p. 103706.

Fleer, M. (2018). Conceptual PlayWorlds: The role of imagination in play and learning. *Early Years, 41*(4), 353–364. https://doi.org/10.1080/09575146.2018.1549024

Li, L., & Disney, L. (2023). Young children's mathematical problem solving and thinking in a playworld. *Mathematics Education Research Journal*, *35*(1), 23–44. https://doi.org/10.1007/s13394-021-00373-y

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia (*p. 560). Newcastle: MERGA.

# Where have the Very Young Children Gone in Mathematics Education Research?

Laurinda Lomas
*Macquarie University*
laurinda.lomas@mq.edu.au

Audrey Cooke
*Curtin University*
audrey.cooke@curtin.edu.au

Chrissy Monteleone
*Australian Catholic University*
chrissy.monteleone@acu.edu.au

Belinda Trewartha
*University of South Australia*
belinda.trewartha@unisa.edu.au

Ann Downton
*Monash University*
ann.downton@monash.edu.au

There is growing national interest in children's early learning competencies, and the impact context and curriculum have on enriching these. Recently-published documents and emerging initiatives point to stakeholder and political investment in the birth to 8 years period. These include the Early Years Royal Commission in South Australia led by Hon. Julia Gillard, NSW Early Years Commitment (2023), and the updated Early Years Learning Framework of Australia (EYLF2.0; AGDE, 2022). The Gillard Early Years Royal Commission focuses on the first 1000 days of a child's life, acknowledging their cruciality and reinforcing the salience of in-depth inquiry into this pivotal period. Considering this present and imminent attention, it is reasonable to ask about the contribution of mathematics research and the extent to which it is woven into such important conversations.

With this query in mind, we were curious about the presentation of early years research in Australia in the 2018-2022 MERGA conference proceedings. A brief analysis suggests that less than ten percent of presentations focused on children's early learning in mathematics. Of that ten percent, only one paper per year focused on very young children. We understand that in Australia, the early years of *school* have a strong presence in mathematics education research. However, there appears to be less research emphasis on very young children, aged birth to 4. Is this the case? If so, where have all the very young children gone?

As part of this round table, we invite MERGA members to contribute to our wonderings, including their research, experiences and ideas. In this process we would like to explore potential synergy between what participants share, and early years research.

Our focus questions for this session include:

- Are very young children the missing thread in mathematics education research?
- What relationships could be generated between early years mathematics research and the broader MERGA research?
- What might be the implications of a research gap?

## References

Australian Government Department of Education (2022). *Belonging, being and becoming: The early years learning framework for Australia* (V2.0). Australian Government Department of Education for the Ministerial Council.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia (*p. 561). Newcastle: MERGA.

# Technology-Enhanced Mathematics Retraining for Quality Teaching

Elena Prieto-Rodriguez
*The University of Newcastle*
elena.prieto@newcastle.edu.au

Sabrina Syed
*The University of Newcastle*
sabrina.syed@newcastle.edu.au

As part of the NSW Teacher Supply Strategy, the NSW Department of Education, in partnership with the University of Newcastle, launched the Mathematics Retraining Program in January 2023 to support experienced teachers to upskill in mathematics. The key objectives of the program are to provide teachers who originally trained in areas other than mathematics with subject content knowledge and pedagogical tools to be effective mathematics teachers; to deliver a learning experience that is flexible, relevant, and research-informed; create a community of practice through enhanced technologies to connect these teachers with leaders in mathematics education in NSW, Australia and globally and to maximise NESA accreditation outcomes in the minimum time possible. The program offers a fully funded master's program, graduate diploma and graduate certificate at the University of Newcastle. The NSW Department of Education and the University of Newcastle assist teachers in choosing the most suitable course based on their prior experience and career goals. To evaluate this program, the University of Newcastle is conducting a mixed-method research to support quality assurance and assess program efficacy for the core outcomes of interest. The specific research questions are:

- How do participants' professional identities develop as they participate in the program?
- How do participants' mathematics teaching practices develop as they participate in the program?

This research uses the six domains described by Hanna et al. (2019) to measure teacher identity formation along with the Quality Teaching Model (Gore et al., 2021) as a research lens for observations of practice. The six domains are ‘motivation’, ‘self-image’, ‘self-efficacy’, ‘task perception’, ‘commitment’, and ‘job satisfaction’. The Quality Teaching Model is an evidence-based pedagogical framework that features teaching practices linked to improved student outcomes and can represent 3 dimensions and 18 elements of pedagogy. Three quantitative surveys and two semi-structured interviews (per year) with program participants, support officers and mentors will be conducted during the degree. In addition, each participant will have three lessons assessed via expert coding using the Quality Teaching Model. Statistical and qualitative data analyses will be conducted throughout as the data is collected to ensure the timely discovery of essential findings, which will be leveraged for the continuous improvement of the program.

## References

Gore, J. M., Miller, A., Fray, L., Harris, J., & Prieto, E. (2021). Improving student achievement through professional development: Results from a randomised controlled trial of Quality Teaching Rounds. *Teaching and Teacher Education*, *101*, 103297. https://doi.org/https://doi.org/10.1016/j.tate.2021.103297

Hanna, F., Oostdam, R., Severiens, S. E., & Zijlstra, B. J. H. (2019). Domains of teacher identity: A review of quantitative measurement instruments. *Educational Research Review*, *27*, 15–27. https://doi.org/10.1016/j.edurev.2019.01.003

(2023). In B. Reid-O’Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia (*p. 562). Newcastle: MERGA.

# Inclusive Mathematics Education: Supporting Students who are Hard of Hearing or Deaf/deaf

Kate Quane
*University of South Australia*
kate.quane@unisa.edu.au

Lorraine Gaunt
*Charles Sturt University*
lgaunt@csu.edu.au

Belinda Trewartha
*University of South Australia*
belinda.trewartha@unisa.edu.au

Tom Porta
*University of Southern Queensland*
*tom.porta@usq.edu.au*

An aim of the Australian Mathematics Curriculum is for students to be "confident, proficient and effective users and communicators of mathematics" (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2022, Aim). One crucial aspect of communication in mathematics is the ability to articulate ideas, strategies, and solutions. This requires students to be able to communicate coherently and fluently, often in spoken form. In this way, students are engaging in the act of oracy. Research suggests, children who are hard of hearing or D/deaf (HH/D/d) face unseen challenges in their education, including difficulty in communicating (Hyde et al., 2003) and keeping up with the demands of school (Antia et al., 2009), particularly in mathematics.

Numerous studies have reported on the achievement gap that exists between students who are HH/D/d and their peers who are hearing (Antia et al., 2009; Chen & Wang, 2021; Hyde et al., 2003). This matters because Hyde et al. (2003) report that 83% of students who are HH/D/d attend inclusive education settings rely on spoken language to communicate. Thus, students who are HH/D/d are reported to have "delayed language acquisition" which ultimately impacts on students' mathematical understanding and problem-solving skills (Hyde et al, 2003, p.56).

Similarly, Bakker et al. (2021) raises the concern that some teaching practices in mathematics "perpetuate inequality" (p. 10). Our aim for this roundtable is to elicit participants' perspectives of inclusive mathematics education practices, inviting participants to share models for inclusion and how these practices may influence the teaching and learning of mathematics, particularly oracy which would support students who are HH/D/d. These models will help frame the next stage of our research, examining the physical, linguistic, cognitive, social, and emotional skills of oracy from a mathematics education perspective, that better support the teaching and learning of children who are HH/D/d.

## References

Antia, S. D., Jones, P. B., Reed, S., & Kreimeyer, K. H. (2009). Academic status and progress of deaf and hard-of-hearing students in general education classrooms. *Journal of Deaf Studies and Deaf Education*, *14*(3), 293–311.

Australian Curriculum, Assessment and Reporting Authority. (2022). *Mathematics: Understand this learning area*. ACARA. https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/mathematics

Bakker, A., Cai, J., & Zenger, L. (2021). Future themes of mathematics education research: an international survey before and during the pandemic. *Educational Studies in Mathematics*, *107*(1), 1–24. https://doi.org/10.1007/s10649-021-10049-w

Chen, L., & Wang, Y. (2021). The contribution of general cognitive abilities and specific numerical abilities to mathematics achievement in students who are deaf or hard-of-hearing. *Journal of Developmental and Physical Disabilities*, *33*(5), 771–787.

Hyde, M., Zevenbergen, R., & Power, D. (2003). Deaf and hard of hearing students' performance on arithmetic word problems. *American Annals of the Deaf*, *148*(1), 56–64.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia (*p. 563). Newcastle: MERGA.

# Enabling Beginning Secondary Mathematics Teachers to Flourish: Teacher Identity Development During School-Based Placements

Bronwyn Reid O'Connor
*University of Sydney*
bronwyn.reidoconnor@sydney.edu.au

Ben Zunica
*University of Sydney*
benjamin.zunica@sydney.edu.au

Eddie Woo
*University of Sydney*
eddie.woo@sydney.edu.au

Teacher recruitment and retention is a critical concern in Australia (AITSL, 2021). Teacher shortages are a particular concern in secondary mathematics (Shah et al., 2022) despite significant funding dedicated to supply initiatives in Australia. Majority of such funding is dedicated to encouraging more teachers to train in secondary mathematics. However, there is little intervention planned for ensuring beginning teachers are positioned to remain in the profession, despite being more likely to leave the profession than established teachers (Wyatt & O'Neill, 2021). In response, we have commenced the *Flourishing Mathematics Teachers Project* in NSW to address this important gap in current interventions.

This project aims to explore the development of pre-service secondary mathematics teachers' identities to better understand the circumstances that enable them to flourish in their career. A flourishing mathematics teacher is defined as one who possesses high levels of agency, autonomy, interest, curiosity, and enjoyment as well as a growth mindset (Dweck, 2000; Ryan & Deci, 2017). Specifically, we seek to explore how school-based placements influence preservice mathematics teacher identity development, with the aim of identifying incubator environments that foster positive mathematics teacher identity development. We focus on placements as they are key opportunities for identity formation (Bobis et al., 2020).

In this roundtable we will present some preliminary findings from the initial phase of the study and discuss its next stages. This roundtable serves as an opportunity for others involved in teacher retention, and mathematics teacher identity research to collaborate. Participants will be invited to share their experiences concerning the qualities of positive placement experiences, and what constitutes effective mentorship.

## References

Australian Institute for Teaching and School Leadership. (2021). Teaching Futures: Background Paper. https://www.aitsl.edu.au/teachingfutures

Bobis, J., Khosronejad, M., Way, J., & Anderson, J. (2020). "Sage on the stage" or "meddler in the middle": Shifting mathematics teachers' identities to support student engagement. *Journal of Mathematics Teacher Education*, *23*(6), 615–632.

Dweck, C. (2000). *Self-theories: Their role in motivation, personality and development*. Philadelphia, PA: Taylor & Francis.

Ryan, R. M., & Deci, E. L. (2017). *Self-determination theory: Basic psychological needs in motivation, development, and wellnes*s. Guilford Publications.

Shah, C., Richardson, P. W., Watt, H. M., & Rice, S. (2022). 'Out-of-field' teaching in mathematics: Australian evidence from PISA 2015. In *Out-of-field teaching across teaching disciplines and contexts* (pp. 71–96). Springer, Singapore.

Wyatt, J. E., & O'Neill, M. (2021). Investigation of early career teacher attrition and the impact of induction programs in Western Australia. *International Journal of Educational Research*, *107*, 101754.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia (*p. 564). Newcastle: MERGA.

# How Do Primary Pre-Service Teachers Plan and Document Rich Mathematics Learning Experiences using a Zoo?

Shaileigh Rowtcliff
*Flinders University of South Australia*
shaileigh.rowtcliff@flinders.edu.au

Timna Garnett
*Flinders University of South Australia*
timna.garnett@flinders.edu.au

Rich learning in the primary mathematics classroom is not a new concept and is an area that has been championed by many education researchers over the years (Grootenboer, 2009; West, 2018). One way in which we can explore how to inspire authentic learning and rich mathematics tasks in primary pre-service teachers planning, is to use field trips or situated learning opportunities as part of their initial teacher education. "Situated learning places a student in a setting that is often outside of the classroom such as a … zoo, museum, laboratory or natural area" (McCormas, 2014, p.98).

Field trips to the zoo can be used to enrich the curriculum, make connections between the real world and the school classroom, and provide students with a meaningful learning experience (Scott & Matthews, 2011). However, not all teachers are confident with teaching … knowledge and skills, or with teaching with technology, … when visiting a zoo (Kisiel, 2013). This study is formed as an action research (AR) study to examine how primary pre-service teachers (PST) learn and think when documenting their experience in science, technologies, and mathematics (STEM) topics using multimodal documentation techniques (Salehi et al., 2012) at a zoo. This focus of this round table is the mathematics aspect of STEM.

In September-October 2022, the first stage of the project was conducted by taking PST to the Adelaide Zoo. A range of activities were conducted which focused on how the environment can be used to enhance teacher planning of mathematics for primary years. The student's learning was documented through a digital tool during three sessions include a pre-zoo workshop, the experience at the zoo and a post-zoo workshop. Later the PSTs used their learning to plan rich mathematical tasks and critically reflect on the experience as part of a final assessment.

At this round table, we will present one example of how pre-service teachers planned and documented rich mathematics learning experiences using a zoo. Participants will be invited to share their experiences of planning and documenting rich learning experiences with primary pre-service teachers and their experiences of using zoos or other field trips in mathematics education.

## References

Grootenboer, P. (2009). Rich mathematical tasks in the Maths in the Kimberley (MITK) Project, Symposium, Mathematics Education Research Group of Australasia.

McCormas, W. (2014). *Situated learning. The language of science education*. Springer Link.

Kisiel, J. (2013). Introducing future teachers to science beyond the classroom. *Journal of Science Teacher Education*, *24*(1), 67–91.

Salehi, S., Kim, J., Meltzer, C., & Blikstein, P. (2012). Process pad: A low-cost multi-touch platform to facilitate multimodal documentation of complex learning. *Proceedings of the Sixth International Conference on Tangible, Embedded and Embodied Interaction.*

Scott, C. M., & Matthews, C. E. (2011). The "science" behind a successful field trip to the zoo. S*cience Activities: Classroom Projects and Curriculum Ideas*, *48*(1), 29–38.

West, J. (2018). Stimulating mathematical reasoning with simple, open-ended tasks. *Australian Primary Mathematics Classroom*, *23*(1).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia (*p. 565). Newcastle: MERGA.

# Short Communications

# Exploring Possible Mechanisms Supporting Transfer of Spatial Reasoning Training to Measurement and Geometry Achievement

Jonathan Adams
*University of Canberra*
jonathan.adams@canberra.edu.au

The relationship between the two is complex, however, and the mechanisms supporting transfer from spatial training to mathematics achievement are not well understood.

Adams et al. (2022), found that a spatial training intervention delivered by classroom teachers improved Year 11 students' performance on Measurement and Geometry problems. The present study builds on this work by utilising semi-structured interviews to explore possible mechanistic pathways which may have supported transfer in this context. Interview responses and student work samples were analysed using a novel approach combining Garofalo and Lester's (1985) cognitive-metacognitive framework with response mapping techniques (Stillman & Galbraith, 1998) to examine how students' drawn representations supported students' reasoning throughout the problem-solving process.

This presentation discusses preliminary findings and possible implications for future research exploring transfer from spatial training to mathematical achievement.

## References

Adams, J., Resnick, I., & Lowrie, T. (2022). Supporting senior high-school students' measurement and geometry performance: Does spatial training transfer to mathematics achievement? *Mathematics Education Research Journal.* https://doi.org/10.1007/s13394-022-00416-y

Garofalo, J., & Lester, F. K. (1985). Metacognition, Cognitive Monitoring, and Mathematical Performance. *Journal for Research in Mathematics Education, 16*(3), 163–176. https://doi.org/10.2307/748391

Stillman, G. A., & Galbraith, P. L. (1998). Applying mathematics with real world connections: Metacognitive characteristics of secondary students. *Educational Studies in Mathematics, 36*(2), 157–194.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 569). Newcastle: MERGA.

# How Kura Kaupapa Māori Supports Learners to Develop Positive Mathematics Identities

Melina Marama Amos
*University of Auckland*
melina.amos@auckland.ac.nz

Tony Trinick
*University of Auckland*
t.trinick@auckland.ac.nz

Lisa Darragh
*University of Auckland*
l.darragh@auckland.ac.nz

The Māori language of New Zealand was considered endangered due to colonial policies, and Kura Kaupapa Māori (KKM) schools were established to resist colonisation and revitalise the language. The KKM education model places the child and their whānau (family) at the centre, embracing the knowledge they come with and working with the community to support them to reach their full potential from a te ao Māori perspective (Smith, 2012; Tocker, 2015). KKM aims to support Māori children to live successfully in the Māori and globalised worlds, including pursuing careers in STEM fields; such a career choice requires developing a positive mathematics identity.

Mathematics Identity is socially formed. It involves the narratives, discourses, and behaviours people employ to define who they are concerning mathematics and in relationship with other simultaneously lived identities (Darragh & Radovic, 2018). The research on Mathematics Identity considers how mathematics learners are perceived and treated by others and how the local teaching practice is defined. Yet there is a lack of research on how KKM graduates develop positive mathematics identities (MLI) and transition into STEM careers. This study aims to explore how KKM schooling has influenced students to pursue STEM careers and to investigate Māori learners' self-reported mathematics identity in relation to approaches to teaching and learning in New Zealand society.

The study design incorporates narrative inquiry methodology in the form of pūrākau to capture the participants' lived experiences. Pūrākau were traditional stories but are now used in contemporary ways to create new knowledge based on Māori culture, worldviews and inspiration, which are at the core of Māori identity. Pūrākau as methodology also draws from and responds to the wider historical, social and political research contexts. The study will recruit six graduates from KKM schools, their family members, and teachers who were integral to their identity development. The findings can lead to strategies to support schools and families to build positive mathematics identities in KKM children. Initial findings from interviews with core participants, teachers, and family members will be shared.

## References

Darragh, L., & Radovic, D. (2018). Mathematics learner identity. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer International Publishing. https://doi.org/10.1007/978-3-319-77487-9_100017-1

Smith, G. H. (2012). The politics of reforming Māori education: The transforming potential of kura kaupapa Māori. In H. Lauder, & C. Wylie, *Towards successful schooling* (pp. 73–87), Routledge.

Tocker, K. (2015). The origins of kura kaupapa Māori. *New Zealand Journal of Educational Studies, 50*(1), 23–38.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 570). Newcastle: MERGA.

# Where is the Mathematics in Teacher Designed STEM Tasks?

Judy Anderson
*The University of Sydney*
judy.anderson@sydney.edu.au

Kate Wilson
*The University of Sydney*
kwil0054@sydney.edu.au

Promoting integrated STEM education to engage students in learning the STEM subjects and highlight the potential for pursuing a STEM career implies the need to redesign curriculum from siloed subjects to some new arrangement. While some providers have developed integrated STEM tasks, to meet the needs of students, teachers need to become designers of STEM tasks. Such tasks need to enthuse students, purposefully connect the STEM subjects, retain STEM subject integrity, and allow for deep learning. Designing tasks that include all four STEM subjects in an authentic way is challenging although some suggest this is not necessary if at least two of the STEM subjects are represented (e.g., Kelley & Knowles, 2016). Roerhig et al. (2021) argued the integration of mathematics can be difficult and should not be limited to a tool to the service of science and engineering. This argument might encourage teachers to avoid integrating mathematics into STEM tasks whereas providing students with integrated STEM learning experiences that foreground mathematics helps to promote the importance of using mathematics to solve important problems. Maass et al., (2019) advised mathematics could play a powerful role in integrated STEM tasks using three interdisciplinary practices—twenty-first century skills, mathematical modelling and education for responsible citizenship.

To support teacher design of integrated STEM tasks, a professional learning program for primary school teachers, organised by the first author and colleagues (Anderson & Tully, 2020; Way et al., 2022), provided advice and mentoring support as teachers designed tasks to trial with their students. Since teachers had varying experiences of designing integrated STEM programs, the types of tasks ranged from shorter, design focused with mathematics as a tool to measure lengths or time, to longer inquiries based on driving questions where mathematics could be used to investigate patterns and propose solutions to more challenging questions. Examples of the range of tasks will be presented.

## References

Anderson, J., & Tully, D. (2020). Designing and evaluating an integrated STEM professional development program for secondary and primary school teachers in Australia. In J. Anderson, & Y. Li (Eds.), *Integrated approaches to STEM education: An international perspective* (pp. 403–426). Singapore: Springer.

Kelley, T. R., & Knowles, J. G. (2016). A conceptual framework for integrated STEM education. *International Journal of STEM Education*, *3*(1), 1–11. https://doi. org/10.1186/s40594-016-0046-z.

Maass, K., Geiger, V., Romero Ariza, M., & Goos, M. (2019). The role of mathematics in interdisciplinary STEM education. *ZDM Mathematics Education*, *51*(6), 869–884. https://doi.org/10.1007/s11858-019-01100-5

Roehrig, G. H., Dare, E. A., Ellis, J. A., & Ring-Whalen, E. (2021). Beyond the basics: A detailed conceptual framework of integrated STEM. *Disciplinary and Interdisciplinary Science Education Research*, *3*:11, 1–18. https://doi.org/10.1186/s43031-021-00041-y

Way, J., Preston, C., & Cartwright, K. (2022). STEM 1, 2, 3: Levelling up on primary schools. *Education Sciences*, *12*, 827. https://doi.org/10.3390/educsci12110827

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 571). Newcastle: MERGA.

# The Nature of Multiplication Constructs, Representations, and Strategies in the South African and Australian Curriculum

Tammy Booysen
*Rhodes University*
t.booysen@ru.ac.za

Lise Westaway
*Rhodes University*
l.westaway@ru.ac.za

The curriculum serves as a support tool and guiding framework to teachers on possible ways to sequence mathematics concepts (Makgato & Ramaligela, 2012) as well as certain values and education pedagogies. Research on the curriculum of different countries is important as it develops an understanding of the mathematics concepts and what needs to be taught (Makgato & Ramaligela, 2012). This paper was led by the research question:

- What is the nature of multiplication constructs, representations, and strategies in the South African and Australian curriculum?

This paper takes the form of a document analysis of the Australian curriculum and the South African curriculum. The aim of the paper is to examine the nature of the multiplication constructs, representations, and strategies in both curriculums using the Charalambous, Delaney, Hsu and Mesa (2010) analytical framework. For this paper, I focused on 3 aspects on the analytical framework namely the constructs, the representation, and the strategies for the data analysis process. These three categories have clear links in the South African and Australian curriculum. In the theoretical framework by Charalambous, Delaney, Hsu and Mesa (2010) the term multiplication *construct* refers to addition, equal groups, arrays, scaling/comparisons, counting, cartesian product then there are multiplication problems without a construct. The *representation* of a multiplication problem can either be in the form of equal groups, linear equations, arrays, numberlines, multiplication tables, counting, bar/scaling, 10x10 tables, cartesian product. There was also a category with no representation. The theoretical framework suggests that the following *strategies* can be used in multiplication namely, commutativity, multiplication tables, multiplication by 1 and 0, counting, adding/subtracting, distributivity, doubling, bar-models, modelling/ concretization/ drawings, metacognition, estimation, mnemonics, and associativity.

The findings suggest that the dominant constructs such as counting, equal groups, and arrays are in both the South African and the Australian curricula. Counting, equal groups and numberlines are the common mode of representation in multiplication and modelling, concretisation, drawing is the dominant strategy used in both South African and Australian curriculum. Notably, the South African curriculum is more prescriptive. For example, it provides teachers with many clarification notes on how to implement the curriculum. Whereas the Australian curriculum is less explicit and provides room for teachers to used their pedagogical content knowledge.

## References

Charalambous, Y., Delaney, S., Hsu, Y. and Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning*, *12*(2), 117–151 DOI: 10.1080/10986060903460070

Makgato, M., & Ramaligela, S. M. (2012). Teachers' criteria for selecting textbooks for the technology subject. *African Journal of Research in Mathematics, Science and Technology Education*, *16*(1), 32–44. https://doi.org/10.1080/10288457.2012.10740727

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 572). Newcastle: MERGA.

# Digital Technologies in the Australian Curriculum: Mathematics

Jennifer Chalmers
*University of South Australia*
jennifer.chalmers@unisa.edu.au

Lisa O'Keeffe
*University of South Australia*
lisa.okeeffe@unisa.edu.au

Vitomir Kovanovic
*University of South Australia*
*vitomir.kovanovic@unisa.edu.au*

Shane Dawson
*University of South Australia*
*shane.dawson@unisa.edu.au*

Digital technologies have been proposed as one potential solution to improve student engagement in mathematics in the middle years. However, the value of digital technologies as a tool for supporting engagement, as well as the degree of implementation remains dependent on teacher, leadership or school-specific motivations.

To reflect the growing significance of digital technologies in society and prepare students to prosper in a digital world (Nascimbeni & Vosloo, 2019), the Australian Curriculum has embedded digital literacy as a general capability. The aim of digital literacy is to support students to critically identify and use digital technologies, adapt to new ways of thinking and working mathematically and raise awareness of how students can protect themselves and others in digital environments (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2021). Given the importance of digital literacy to society, it is anticipated there will be increasing emphasis on digital technologies in each future iteration of the Australian Curriculum.

In this presentation, we share findings from a content analysis examining the ways in which digital technologies are represented in the Australian Curriculum: Mathematics (AC:M). Language, location and context were analysed to identify intentions, assumptions and implications about digital technologies made within the AC:M. This content analysis was conducted on versions 8.4 (ACARA, n.d.-a) and 9 (ACARA, n.d.-b) of the AC:M to reveal changes in how digital technologies are being recommended to support teaching and learning in mathematics.

While this research is in progress, the preliminary findings indicate an increase in the use and variety of digital technology terminology in version 9. However, references to digital technologies are still limited in compulsory sections of the AC:M such as the content descriptions and achievement standards. It is likely that individual teacher choice and preferences are still major influencers in determining if and how digital technologies are used in mathematics.

## References

Australian Curriculum, Assessment and Reporting Authority. (n.d.-a). *Foundation to year 10 curriculum: Mathematics* (version 8.4). ACARA. https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Australian Curriculum, Assessment and Reporting Authority. (n.d.-b). *Foundation to year 10 curriculum: Mathematics* (version 9.0). ACARA. https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/mathematics

Australian Curriculum, Assessment and Reporting Authority. (2021). *General capabilities: Digital literacies (previously ICT): Consultation—introductory information and learning continua*. ACARA. . https://www.australiancurriculum.edu.au/media/7024/gc_digital_literacy_ict_capability_consultation_curriculum.pdf.

Nascimbeni, F., & Vosloo, S. (2019). *Digital literacy for children: Exploring definitions and frameworks* (p. 41). UNICEF. https://www.unicef.org/globalinsight/media/1271/file/%20UNICEF-Global-Insight-digital-literacy-scoping-paper-2020.pdf

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 573). Newcastle: MERGA.

# Teaching Mathematics Through Problem Solving for Low-Readiness Learners

Ban Heng Choy
*National Institute of Education, Nanyang Technological University*
banheng.choy@nie.edu.sg

Despite the consensus on the importance of problem solving in school mathematics, it is not always clear whether problem solving should be the "end result" or "means through which mathematical concepts, processes, and procedures are learnt" (Lester, 2013, p. 246). Common approaches, or what Lester (2013) referred to as an "ends approach" (p. 246), to feature problem solving in classrooms include creating a series of specialized lessons to teach problem solving (Toh et al., 2016) and using problems positioned at the end of each chapter in textbooks as opportunities for students to apply their newly learned concepts to more novel problems (Jäder et al., 2019). One important merit of this approach is the explicit emphasis of the four-phase problem-solving model of George Pólya—understanding the problem, devising a solution plan, carrying out the plan, and looking back at the solution (Pólya, 1945). Such an approach provides a clearer guideline for students to follow instead of leaving it for them to grapple with the problem. However, providing students with clearly articulated steps to follow may hinder students' development of problem solving skills and dispositions (Goulet-Lyle et al., 2019). For instance, in Singapore, teachers often demonstrate a step-by-step approach to solving problems for their students, especially for those who are struggling with mathematics or low readiness students (Kaur et al., 2019). Moreover, textbooks often present problem solving as following a solution template, which is not helpful (Jäder et al., 2019). Therefore, while providing students a step-by-step approach to problem solving may reduce the complexity of the problem-solving tasks and hence offer them more access to these problems; students are consequently denied of the opportunity to make sense of the mathematics and less likely to persist in problem solving when they encounter new problems. In this short communication, I will share the initial ideas of a new project, aimed at teaching mathematics through problem solving, and invite feedback from participants on the type of problems and they are used in the classrooms to teach mathematics.

## References

Goulet-Lyle, M.-P., Voyer, D., & Verschaffel, L. (2019). How does imposing a step-by-step solution method impact students' approach to mathematical word problem solving? *ZDM*, *52*(1), 139–149. https://doi.org/10.1007/s11858-019-01098-w

Jäder, J., Lithner, J., & Sidenvall, J. (2019). Mathematical problem solving in textbooks from twelve countries. *International Journal of Mathematical Education in Science and Technology*, *51*(7), 1120–1136. https://doi.org/10.1080/0020739x.2019.1656826

Kaur, B., Toh, T. L., Lee, N. H., Leong, Y. H., Cheng, L. P., Ng, K. E. D., Yeo, K. K., Yeo, B. W. J., Wong, L. F., Tong, C. L., Toh, W. Y. K., & Safii, L. (2019). *Twelve questions on mathematics teaching*. National Institute of Education, Nanyang Technological University.

Lester, F. K. J. (2013). Thoughts about research on mathematical problem-solving instruction. *The Mathematics Enthusiast*, *10*(1 & 2), 245–278.

Pólya, G. (1945). *How to solve it*. Princeton University Press.

Toh, T. L., Tay, E. G., Leong, Y. H., Quek, K. S., Toh, P. C., Dindyal, J., Ho, F. H., Kang, K. H., & Yen, Y. P. (2016). *Mathematical problem solving for everyone: Infusion and diffusion (MinD).* NIE Research Brief Series (Issue No. 16–011). National Institute of Education.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 574). Newcastle: MERGA.

# Using Barad's 'Apparatus' to Reconceptualise the Young Preverbal Child's Mathematical Engagement in Their Environment

Audrey Cooke
*Curtin University*
audrey.cooke@curtin.edu.au

Barad's (2007) conceptions of apparatus can be used in an early learning environment is explored here. His description of apparatus can provide a valuable way to conceptualise early learning environments and, subsequently, identify further opportunities for preverbal children to engage with mathematical understandings. Barad views apparatuses as providing objectivity. In early childhood education, the environment can be viewed as an apparatus through which children's understandings can become evident. That is, the environment serves as a form of tool through which early childhood educators can 'see' children's understandings when engaging with their environment. In this way, it is as if the environment is the measuring device—and the early childhood educator needs to be able to 'read' what the 'measurement' is. For very young preverbal children, this becomes more important as understandings cannot be communicated via language.

However, the measuring apparatus needs to be finely tuned—that is, the environment needs to be carefully considered and created and conceived—and the early childhood educator well versed in interpreting the information provided via the apparatus—in other words, being able to analyse what the preverbal child's engagement with the environment, and the intra-action (where intra-action refers to the constant state of becoming that occurs due to the individual and the world within which they exist being both cause and effect [Barad, 2007]) between the child and other matter within the environment, shows about the child's mathematical understandings. As part of this fine tuning, recognition of agency in all elements of the environment is needed so that early childhood educators strengthen their identification of mathematical engagement through attending to how the environment may enable the young child's learning (Smythe et al., 2017). Likewise, a consideration of intra-actions of the elements of the environment and how these themselves contribute to the mathematical understandings that can be developed (Smythe et al. 2017) is needed. This impacts the early childhood educator in two ways—highlighting the intra-active aspect of the very young child's engagement and changing the focus from the educator as the pre-eminent force in the environment for children engaging with mathematics to the foregrounding of all elements within the environment.

Reconceptualising the environment as apparatus can emphasise how the recognition of the intra-action of the elements of the environment can provide another way to interpret what early childhood educators see in terms of young children engaging in mathematical thinking (Björklund et al. 2020). An examination of how using the ideas of apparatus enable early childhood educators to interrogate the early learning environment in terms of the impact on very young preverbal children's opportunities to demonstrate and engage with mathematical thinking is provided, together with how this may be used to assist preservice early childhood educators in developing their understandings of mathematical education in early childhood.

## References

Barad, K. (2007). *Meeting the universe half way: Quantum physics of the entanglement of matter and meaning*. Duke University Press.

Björklund, C., van den Heuvel-Panhuizen, M., & Kullberg, A. (2020). Research on early childhood mathematics teaching and learning. *ZDM, 52*, 607–619. https://doi.org/10.1007/s11858-020-01177-3

Smythe, S., Hill, C., MacDonald, M., Dagenais, D., Sinclair, N., & Toohey, K. (2017). *Disrupting boundaries in education and research*. Cambridge University Press. https://doi.org/10.1017/9781108234931

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 575). Newcastle: MERGA.

# Unpacking the 'M' in Integrated STEM Tasks: A Systematic Review

Gabi Cooper
*Griffith University*
gabi.cooper@alumni.griffithuni.edu.au

Ban Heng Choy
*National Institute of Education, Nanyang Technological University*
banheng.choy@nie.edu.sg

Despite the promise of integrated STEM for authentic learning, the notion of integration is problematic, and it remains unclear whether such integrated approaches can result in significant learning in STEM disciplines. This is particularly true for mathematics, which is often underrepresented in integrated STEM tasks (English, 2016; Fitzallen, 2015; Maass et al., 2019), considered an "accessory" discipline in many cases. In this systematic review, we investigate the issue of integrating mathematics by unpacking the centricity of STEM tasks (Teo et al., 2021), analysing the connections between mathematics and the other STEM disciplines (Tan et al., 2019), and highlighting the different faces of mathematics presented in these such tasks (Devlin, 2000). Our findings suggest that although current STEM tasks anchored in mathematics are problem-centric, they embed relatively weak inter-disciplinary connections. Analyses also revealed less emphases on mathematics as a way of knowing in comparison to other mathematical faces presented. We discussed possible implications and suggestions for strengthening the 'M' in STEM through designing tasks to promote more authentic integrated STEM learning experiences. A suggested task design element to consider is the implementation of mathematics as a creative medium and way of knowing, in combination with considering user-centric and solution-centric approaches. This would offer different affordances for mathematics content, thereby increasing the strength of the interdisciplinary connections underpinned in the tasks and fostering more authentic integrated STEM experiences for learners.

## References

Devlin, K. (2000). The four faces of mathematics. In M. J. Burke, & F. R. Curcio (Eds.), *Learning mathematics for a new century: 2000 yearbook of the National Council of Teachers of Mathematics* (pp. 16–27). NCTM.

English, L. D. (2016). STEM education K–12: Perspectives on integration. *International Journal of STEM Education*, *3*(1). https://doi.org/10.1186/s40594-016-0036-1

Fitzallen, N. (2015). STEM Education: What does mathematics have to offer? In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 237–244). MERGA.

Maass, K., Geiger, V., Ariza, M. R., & Goos, M. (2019). The role of mathematics in interdisciplinary STEM education. *ZDM*, *51*(6), 869–884. https://doi.org/10.1007/s11858-019-01100-5

Tan, A.-L., Teo, T. W., Choy, B. H., & Ong, Y. S. (2019). The S-T-E-M Quartet. *Innovation and Education*, *1*(1). https://doi.org/10.1186/s42862-019-0005-x

Teo, T. W., Tan, A. L., Ong, Y. S., & Choy, B. H. (2021). Centricities of STEM curriculum frameworks: Variations of the S-T-E-M Quartet. *STEM Education*, *1*(3), 141–156. https://doi.org/10.3934/steme.2021011

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 576). Newcastle: MERGA.

# Perfect for Whom? Producing Mathematics Learner Identities Via Online Instructional Platforms

Lisa Darragh
*University of Auckland*
l.darragh@auckland.ac.nz

Students developing positive relationships with mathematics is a core challenge within mathematics education. Yet enabling student learner identity becomes increasingly complex with the growth in use of online instructional platforms in mathematics programmes. In this short communication I share early results of case study research into students' mathematics identities as produced in two secondary mathematics classrooms using Education Perfect and other online resources for learning mathematics.

Online instructional platforms such as *Education Perfect* and *Mathletics* were already commonplace in mathematics classrooms of Aotearoa New Zealand before the COVID-19 pandemic (Darragh & Franke, 2021) and distance learning only increased their use worldwide (Williamson et al., 2020). Online platforms typically require an annual subscription, provide comprehensive curricula content, and rely on automated data-analytics to guide an individualised learning pathway. Yet research into technology in mathematics education appears more focussed on technological tools that enhance teaching and learning rather than casting a critical eye on platforms powered by artificial intelligence. One way to look critically at these platforms is to consider how they influence students' identities as learners of mathematics. In my study I use a performative notion of identity (Butler, 1988) and define identity as a socially produced way of being in relation to mathematics, enacted and recognised in various contexts (Darragh & Radovic, 2018). I contend that the context of online instructional platforms provides a stage for enacting identity that may constrain performances in different ways to other common classroom settings.

In the wider study I plan to use a case study approach to investigate five learners in each of four *different* classrooms and schools. Aotearoa New Zealand demonstrates a wide diversity of mathematics teaching and learning experiences, and thus the study aims to similarly capture some of this diversity. The two case study schools to be discussed in this communication are both at secondary level, yet one is very traditional (Year 9 to 13) whilst the other (Year 7 to 10) uses innovative learning environments (ILEs) and partners with Apple Australia in the use of I-pads. The teachers in the two case studies have very contrasting opinions of the use and value of platforms such as *Education Perfect.* I expect there to be illuminating similarities and differences in the way in which student learner identity is produced by each context.

## References

Butler, J. (1988). Performative acts and gender constitution: an essay in phenomenology and feminist theory. *Theatre Journal*, *40*(4), 519–531. http://www.jstor.org/stable/3207893

Darragh, L., & Franke, N. (2021). Online mathematics programs and the figured world of primary school mathematics in the digital era. *Mathematics Education Research Journal*, *online fir*. https://doi.org/https://doi.org/10.1007/s13394-021-00384-9

Darragh, L., & Radovic, D. (2018). Mathematics learner identity. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer International Publishing. https://doi.org/10.1007/978-3-319-77487-9_100017-1

Williamson, B., Eynon, R., & Potter, J. (2020). Pandemic politics, pedagogies and practices: Digital technologies and distance education during the coronavirus emergency. *Learning, Media and Technology*, *45*(2), 107–114. https://doi.org/10.1080/17439884.2020.1761641

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 577). Newcastle: MERGA.

# Identifying and Developing Mathematics in Australian Indigenous Languages: A Functional Typological Approach

Cris Edmonds-Wathen
*Charles Darwin University*
cris.edmonds-wathen@cdu.edu.au

This presentation reports on the design of and initial stages of a project which aims to improve the mathematical learning of Indigenous language speaking students by delivering mathematics education in Australian Indigenous languages. This project is investigating how the language structures needed for school mathematics occur in Indigenous Australian languages in order to guidelines to assist schools to initiate or extend mathematics programs in Australian Indigenous languages. It will add to what is known about the variation in how languages are and can be used mathematically from a systematic cross-linguistic perspective.

The core of the project is three typologically diverse case studies at three sites: Areyonga School (Pitjantjatjara), Groote Eylandt Bickerton Island Primary College Aboriginal Corporation (Anindilyakwa) and Murrupurtiyanuwu Catholic Primary School (Tiwi), which have past or current history of Indigenous Australian language mathematics programs. Their diverse linguistic and cultural environments address a theoretical focus on diversity. Collaborating with Indigenous educators and elders, the project will develop a learning progression, assessment tools and sequence of mathematics lessons in each language. This will be iteratively investigated and refined throughout each case study.

This project is applying a novel functional typological approach to the identification and development of diverse languages for mathematics teaching and learning (Edmonds-Wathen, 2019). Typology is a field of linguistics which aims to compare and describe languages in a framework-neutral manner (Nichols, 2007), hence enabling languages to be contrasted on equal terms, rather than from analytic perspectives that privilege one language over another. A functional typological approach to mathematics register development thus takes mathematics as the semantic and functional field of interest (Edmonds-Wathen, 2019).

The chosen initial foci are foundational logical reasoning, classification, and spatial concepts (such as identification of similarity and difference and ways of describing roundness or straightness). As tasks and tools will be developed collaboratively at each site, they will vary and be specifically contextualized. For example, the environment surrounding Areyonga is full of geometrical shapes in the form of rocks that fall from the highly stratified escarpments and rockhills. The language of geometrical shapes developed thus far in Pitjantjatjara therefore draws on that used to describe rocks and stone artefacts, and the body, which people metaphorically extend to describe the anatomy of geographical features. A triangle is a *kanti*—a triangular quartz piece traditionally used as a blade. Rockhills are described as having a *mulya* 'nose' where a ridge ends and slopes downwards. The *kanti* 'triangle' therefore has *mulya maṉkurpa* 'three noses', or three corners. This presentation includes early data on language development work, the design of initial lessons and, from Areyonga, their implementation in the classroom.

## References

Edmonds-Wathen, C. (2019). Linguistic methodologies for investigating and representing multiple languages in mathematics education research. *Research in Mathematics Education, 21*(2), 119–134.
Nichols, J. (2007). What, if anything, is typology? *Linguistic Typology, 11*(1), 231–238.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 578). Newcastle: MERGA.

# Exploring Primary Teacher Education Students' Self-perception of Readiness to Teach Mathematics

Kylie-Ann Fitzgerald
*University of Notre Dame Australia*
kylie.fitzgerald@nd.edu.au

Thuan Thai
*University of Notre Dame Australia*
thuan.thai@nd.edu.au

Gregory Hine
*University of Notre Dame Australia*
gregory.hine@nd.edu.au

The tertiary education of Teacher Education Students (TES) is pivotal in their professional preparation and formation as qualified educators in specific learning areas, including mathematics. Graduate teachers will require an understanding of pedagogical content knowledge, and more specifically, the knowledge of ways to teach mathematical content in a way that assists student understanding (Shulman, 1987). However, there is currently a lack of understanding around the best ways to prepare teachers in the teaching of mathematics in initial teacher education programs (Thai & Hine, 2019). As argued by Norton and Allen (2020), there is a lack of research in TES readiness to teach primary school mathematics. Although there has been an attempt to address this lack of research in secondary TES' perceptions of readiness to teach mathematics (Hine & Thai, 2019), there continues to be a lack of research in TES' readiness to teach primary school mathematics.

This short communication will report on the preliminary findings of an investigation on TES' self-perception and readiness to teach primary mathematics. The aim of this research is to bring awareness to the areas within primary mathematics education that TES require further training in, and to help initial teacher education providers prepare TES to become confident and competent primary school mathematics teachers. Analysis of the pre-course questionnaire revealed that TES felt 'somewhat ready' to teach primary mathematics, although, they were considerably more positive in teaching content rather than pedagogy. In particular, TES indicated that they required further training in engaging and providing differentiated instruction for primary students in mathematics. The follow-up questionnaire, conducted after a semester long course on mathematical pedagogy showed that the majority of TES indicated they were more 'ready' to teach primary mathematics in terms of pedagogy and content.

*Keywords:* mathematics education, teacher education students, self-perception, readiness, initial teacher education.

## References

Hine, G., & Thai, T. (2019). Pre-service secondary mathematics teachers' self-perceptions of readiness to teach secondary school mathematics. *Mathematics Teacher Education and Development*, *21*(2), 64–86.

Norton, S., & Allen, J. (2020). Pre-service primary teachers have a say on genericism in mathematics curriculum preparation. *Curriculum Perspect*ives, *40*, 159–172.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–23.

Thai, T., & Hine, G. (2019, September). Exploring primary teacher education students' self-perception of readiness to teach primary mathematics. In *Proceedings of the educator scholar conference held in Sydney, Australia.* University of Sydney.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 579). Newcastle: MERGA.

# What's the Problem? Implementing School Mathematics Curriculum Reform

Merrilyn Goos
*University of the Sunshine Coast*
mgoos@usc.edu.au

Margaret Marshman
*University of the Sunshine Coast*
*mmarshma@usc.edu.au*

Anne Bennison
*University of the Sunshine Coast*
abenniso@usc.edu.au

Emily Ross
*The University of Queensland*
emily.ross@uq.edu.au

School curriculum change is a consistent feature of the Australian educational landscape which creates challenges for teachers. Curriculum change also presents opportunities for engaging teachers in professional learning and for researching factors that support or hinder curriculum implementation. The development, implementation, and reviews of the *Australian Curriculum: Mathematics* provides the context for the research and development project we are conducting with a group of primary and secondary schools in Queensland. This project aims to support schools to interpret, plan, and implement the new mathematics curricula more effectively.

Remillard and Heck (2014) defined curriculum as "a *plan for the experiences* that learners will encounter, as well as the *actual experiences* they do encounter, that are designed to help them reach specified mathematics objectives" (p. 707, original emphasis). Their model of the curriculum policy, design, and enactment system points to mediating factors that inevitably influence elements of the *official* curriculum, as specified by governing authorities, and the *operational* curriculum enacted in classrooms. These identified influencing factors include; views of individuals and groups wielding power; research on learning, teaching, and assessment; teacher knowledge, beliefs, and practices; teachers' access to resources and support; contextual opportunities and constraints; and a range of student characteristics and cultural resources.

We are working with middle-level school leaders who were asked to propose a small action research project that investigates a "problem" or area for improvement within mathematics teaching in their school, with support from our project team. In this short communication, we seek to understand some unanticipated outcomes of the project by addressing the research question: *How do schools represent curriculum implementation problems and perceptions of their professional development needs?* Using Bacchi's (2009) question "What's the problem represented to be?", we analyse interviews with mathematics leaders in two schools to explore factors that mediate and influence curriculum enactment (i.e. the *operational* curriculum). Findings revealed that the professional learning requirements identified by these school leaders did not always align with those of classroom teachers in their schools.

## References

Bacchi, C. (2009). *Analysing policy: What's the problem represented to be?* Pearson Australia.

Remillard, J., & Heck, D. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education, 46*, 705-718. https://doi.org/10.1007/s11858-014-0600-4

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 580). Newcastle: MERGA.

# Capturing Conceptual Changes with Dynamic Digital Representations

Amelia Gorman
*The University of Sydney*
agor3358@uni.sydney.edu.au

Jennifer Way
*The University of Sydney*
jennifer.way@sydney.edu.au

Janette Bobis
*The University of Sydney*
janette.bobis@sydney.edu.au

In this presentation, we give an overview and rationale of the research design of a study that has commenced in 2023. The aim for this study is to explore what conceptual changes emerge while children in middle-primary grades engage with dynamic digital representations of decimal fractions. Unlike with static representations, students are able to actively interact with the mathematics content embedded in dynamic digital representations by making conjectures and testing them immediately to experience the concept from various perspectives and receive feedback on their thinking (Orrill & Polly, 2013).

This study will be conducted using a qualitative methodology, in which the researcher will use task-based interviews combined with microgenetic methods to collect and analyse data (Chinn & Sherin, 2014). Six participants in total from one Year 4 classroom (aged 9-10 years) in NSW will be invited to participate in this study. Data collection for this study will consist of three key phases; an initial task-based interview; four proceeding task-based interviews each focussing on a different dynamic digital representation that feature the three key concepts of decimals, *place value*, *decimal density* and *relative magnitude,* which are considered vital for consolidating number sense, since these properties unify all numbers (Siegler, Thompson & Schneider, 2011); and a final task-based interview. A synthesis of data across each task-based interview will occur, where the video-audio recordings and screen-captures are interpreted using a combination of inductive and deductive approaches to repeated viewings. The aim of the approach is to detect learning changes by determining what specific shifts in the learner's attention have occurred and how they have transpired (Voutsina, George & Jones, 2019). The method of investigating moment-to-moment processes of learning will allow inferences to be made, using the researcher-designed 'Framework for Analysis', about what features of each dynamic digital representation prompted attention shifts and how these translate to changes in conceptual understanding of decimal fractions.

## References

Chinn, C. A., & Sherin, B. L. (2014). Microgenetic methods. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 171–190). Cambridge University Press.

Orrill, C. H., & Polly, D. (2013). Supporting mathematical communication through technology. In D. Polly (Ed.), *Common core mathematics standards and implementing digital technologies* (pp. 22–36). Hershey, PA: IGI Global.

Siegler, R. S., Thompson, C., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*(4), 273–296.

Voutsina, C., George, L., & Jones, K. (2019). Microgenetic analysis of young children's shifts of attention in arithmetic tasks: underlying dynamics of change in phases of seemingly stable task performance. *Educational Studies in Mathematics, 102*(1), 47–74.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 581). Newcastle: MERGA.

# Mathematics Instruction Online: Finding the Right Level of Challenge in Emergency Remote Teaching

Anita Green
*Monash University*
anita.green1@monash.edu

Differentiation can be hard to define, however it involves maximising the learning opportunities for each student (Tomlinson, 2005). For example, teachers could enact differentiation by adapting teaching approaches, instruction, resources, or classroom activities. Effective differentiation can be difficult to achieve for many teachers (Eysink et al., 2017). During the Covid-19 pandemic teachers and students moved to teaching and learning online for a period of time know as 'emergency remote teaching' or ERT (Hodges et al., 2020). Teachers were required to adapt their teaching and learning practices to suit the online learning environment. As part of a larger study investigating what teachers are cued into noticing in the primary mathematics classroom, two semi-structured interviews were conducted with eight teacher participants late in 2020. One interview was conducted during ERT and the second interview was conducted a few weeks after returning to the face-to-face classroom. Teachers in this study found differentiating more challenging in the online environment. In this short communication I will share the participants experiences of adapting their teaching practices for ERT and the difficulties they faced differentiating effectively. All teachers in this study reported using levelled groups in the online environment in order to ensure the right level of challenge was provided for each student. This presentation will discuss why participants chose this approach in the online environment and how this differed from their regular teaching practice in the face-to-face classroom which involved the use of more open-ended, rich tasks.

## References

Eysink, T. H., Hulsbeek, M., & Gijlers, H. (2017). Supporting primary school teachers in differentiating in the regular classroom. *Teaching and Teacher Education, 66*, 107–116.

Hodges, C. B., Moore, S., Lockee, B. B., Trust, T., & Bond, M. A. (2020). The difference between emergency remote teaching and online learning. Educause. https://er.educause.edu/articles/2020/3/the-difference-between-emergency-remote-teaching-and-online-learning

Tomlinson, C. A. (2005). Grading and differentiation: Paradox or good practice? *Theory into Practice, 44*(3), 262–269.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 582). Newcastle: MERGA.

# Pre-Service Teachers Use of a Pedagogical Framework to Notice Students' Mathematical Thinking

Mark Gronow
*Central Queensland University*
m.gronow@cqu.edu.au

Michael Cavanagh
*Macquarie University*
michael.cavanagh@mq.edu.au

Joanne Mulligan
*Macquarie University*
joanne.mulligan@mq.edu.au

Teachers' noticing of students' learning is essential for all teachers of mathematics (Choy & Dindyal, 2017). Although noticing is a complex process, it can help pre-service teachers (PSTs) make sense of the mathematical content knowledge and the pedagogy to teach it. Noticing students' mathematical thinking requires PSTs to attend to students' thinking, interpret their understandings and decide how to respond to these understandings (Jacobs et al., 2022).

PSTs can benefit from learning a pedagogical framework that supports their noticing of students' mathematical thinking when teaching mathematics (Choy, 2016). The CRIG pedagogical framework (Gronow et al., 2020) is presented as an instrument for PSTs to notice students' mathematical thinking. The framework comprises four mathematical components (Connections, Recognising patterns, Identifying similarities and differences, and Generalising and Reasoning).

In this short communication, we present the results from a two phased study, where four PSTs (two primary and two secondary mathematics) engaged in a professional learning program and classroom support to implement the CRIG pedagogical framework in mathematics lessons. Data collected over two separate three-week periods included audio recordings of professional learning workshops, video recordings of lessons, and audio recordings of interviews with PSTs. This study describes how the PSTs' understanding and use of the CRIG pedagogical framework supported their noticing of students' mathematical thinking.

We identify the affordances and challenges that the PSTs experienced when implementing the CRIG pedagogical framework into their mathematics lessons. We draw comparisons between the interpretation and implementation of CRIG in the primary and secondary school settings.

## References

Choy, B. H. (2016). Snapshots of mathematics teacher noticing during task design. *Mathematics Education Research Journal*, *28*(3), 421–440.

Choy, B. H., & Dindyal, J. (2017). Noticing affordances of a typical problem. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st conference of the International Group for the Psychology of Mathematics Education* (pp. 249–256). Singapore: PME.

Gronow, M., Mulligan, J., & Cavanagh, M. (2020). Teachers' understanding and use of mathematical structure. *Mathematics Education Research Journal, 34*(2), 215–240. https://doi.org/10.1007/s13394-020-00342-x

Jacobs, V. R., Empson, S. B., Jessup, N. A., Dunning, A., Pynes, D., Krause, G., & Franke, T. M. (2022). Profiles of teachers' expertise in professional noticing of children's mathematical thinking. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-022-09558-z

Stockero, S. L., Rupnow, R. L., & Pascoe, A. E. (2017). Learning to notice important student mathematical thinking in complex classroom interactions. *Teaching and Teacher Education, 63*, 384–395.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 583). Newcastle: MERGA.

# Student Responses to a Cognitive Activation Pedagogical Approach

Ciara Harding
*The University of Sydney*
char6275@uni.sydney.edu.au

Janette Bobis
*The University of Sydney*
janette.bobis@sydney.edu.au

Jennifer Way
*The University of Sydney*
jennifer.way@sydney.edu.au

Cognitive activation (Klieme et al., 2009; Lipowsky et al., 2009) is a burgeoning construct that has gained international recognition since its inclusion in the 2012 Programme for International Students Assessment (PISA) questionnaire. Despite the growing momentum, there is still uncertainty about what practices are entailed within a cognitive activation approach. This uncertainty is considered a reason for the minimal uptake of cognitive activation pedagogies within mathematics classrooms (Le Donne et al., 2016). The aims of our study are to unpack the various component practices encompassing a cognitive activation instructional approach and to examine how students with varying engagement and achievement levels respond to it.

In this presentation, we will expound upon seminal definitions of cognitive activation, examine some of the finer-grained practices incorporated within the construct and introduce aspects of an on-going study. The study will utilise a case study methodology to examine student responses to cognitive activation instruction in a Year 6 mathematics classroom. Quantitative data obtained from the Motivation and Engagement Scale (Martin, 2006) and a researcher-constructed content knowledge test will be used to categorise students based on their engagement and achievement levels. All students will then participate in a series of six intervention lessons designed using a cognitive activation approach. Lessons will be video recorded to help analyse students' responses to the various cognitive activation practices. Semi-structured interviews with ten case study students with various engagement and achievement levels will be conducted. Data will be interpreted using inductive approaches to detect similarities and differences between students who share analogous engagement and achievement characteristics.

## References

Klieme, E., Pauli, C., & Reusser, K. (2009). The Pythagoras study: Investigating effects of teaching and learning in Swiss and German mathematics classrooms. In T. Janik, & T. Seidel (Eds.), *The power of video studies in investigating teaching and learning in the classroom*. New York, NY: Waxmann Publishing Co.

Le Donné, Fraser, P., & Bousquet, G. (2016). Teaching strategies for instructional quality insights from the TALIS-PISA link data. In *Teaching strategies for instructional quality insights from the TALIS-PISA link data*. OECD Publishing.

Lipowsky, K., Pauli, C., Drollinger-Vetter, B., Klieme, E., & Reusser, K. (2009). Quality of geometry instruction and its short-term impact on students' understanding of the Pythagorean Theorem. *Learning and Instruction*, *19*(6), 527–537. https://doi.org/10.1016/j.learninstruc.2008.11.001

Martin, A. (2006). Examining a multidimensional model of student motivation and engagement using a construct validation approach. *British Journal of Educational Psychology*, *77*(2), 413–440.

OECD (2013). PISA 2012 *Assessment and analytical framework mathematics, reading, science, problem solving and financial literacy*. Paris: OECD Publishing. http://dx.doi.org/10.1787/9789264190511-en

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 584). Newcastle: MERGA.

# Exploring Secondary Mathematics Teachers' Motivation to Attend Voluntary Professional Learning

Gregory Hine
*University of Notre Dame Australia*
gregory.hine@nd.edu.au

Rachael Whitney-Smith
*University of Notre Dame Australia*
*rachael.whitney-smith@my.nd.edu.au*

The significance of this study rests upon the belief that the effect of quality teaching impacts upon student learning (Hattie, 2009). In recognition of this claim, schools and school systems have increasingly focused on the implementation of quality teaching practices and professional learning (PL) that support these practices (Labone & Long, 2016). There appears to be limited research and literature published on Australian secondary school teachers' motivation for participating in PL. Most current research appears to have explored professional development of Australian mathematics teachers (Goos et al., 2018), pre-service teachers (Hine, 2016), and out-of-field teachers (Goos et al., 2019). Furthermore, summaries of research have explored theoretical and evaluative approaches to PL, the nature of teacher capabilities, and characteristics of effective PL programs (Beswick et al., 2016).

The overarching guiding question to be explored is: What are the reasons secondary school mathematics teachers (SSMTs) participate in voluntary PL? A secondary question to be explored is: What are the type(s) of professional knowledge underpinning SSMTs' stated motivation to participate in voluntary PL courses. Over three years, as many as 73 SSMTs participated in pre-course surveys, post-course surveys, and individual interviews. Situated within a conceptual framework (Ball et al., 2008), qualitative responses were analysed *a posteriori* via open coding. While participants reported their motivations primarily as developing knowledge in mathematics content and pedagogy, there were claims of growing in confidence in these knowledge domains. Such claims in interviews were largely connected to the identification of topics requiring practice and consolidation prior to teaching.

## References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching—what makes it so special? *Journal of Teacher Education*, *59*(5), 389–407. DOI: 10.1177/0022487108324554

Beswick, K., Anderson, J., & Hurst, C. (2016). The education and development of practicing teachers. In K. Makar et al. (Eds.), *Research in mathematics education in Australia 2012–2015*. DOI 10.1007/978-98-10-1419-2_16

Goos, M., Bennison, A., & Proffitt-White, R. (2018). Sustaining and scaling up research-informed professional development for mathematics teachers. *Mathematics Teacher Education and Development*, *20*(2), 133–150.

Goos, M., Ní Ríordáin, M., Lane, C., & Faulkner, F. (2019). Impact of a national professional development program on the beliefs and practices of out-of-field teachers of mathematics. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice* (pp. 316–323). MERGA.

Hattie, J. (2009). *Visible learning synthesis of over 800 meta-analyses relating to achievement.* Routledge.

Hine, G. S. C. (2015). Strengthening pre-service teachers' mathematical content knowledge. *Journal of University Teaching and Learning Practice*, *12*(4), 1–14.

Labone, E., & Long, J. (2016). Features of effective professional learning: A case study of implementation of a system-based professional learning model. *Professional Development in Education*, *42*(1), 54–77.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 585). Newcastle: MERGA.

# Building Capability for Prep-12 Teachers in Mathematical Inquiry Pedagogies

Rhonda Horne
*Department of Education Queensland*
rhonda.horne@qed.qld.gov.au

Linda Carroll
*Department of Education Queensland*
linda.carroll@qed.edu.au

Paulina Sliedrecht
*Department of Education Queensland*
paulina.sliedrecht@qed.qld.gov.au

Rose Wood
*Department of Education Queensland*
rose.wood@.qed.qld.gov.au

Mathematical inquiry is a process of solving ill-structured problems that significantly rely on mathematics in the solution process (Makar 2012). The pedagogy of inquiry underpins the Australian Curriculum: Mathematics and is encouraged in mathematics teaching across all phases of schooling in Queensland. In the senior schooling phase, a required assessment Problem-Solving and Modelling Task (PSMT) is completed by students in their mathematics course. This assessment task is designed to evaluate a student's ability to respond to an investigative mathematical scenario or stimulus (QCAA, 2021). In most cases, the key features of this task provides a response that addresses the real-life application of mathematics, using technology and diagrams. The Queensland Department of Education has built a Prep to Year 12 approach to developing students' problem solving skills with a focus on building teacher capability in inquiry pedagogies in the P -10 phase of schooling.

This short communication will present the department's strategies that support teachers in adopting inquiry pedagogies. The 'M in STEM' professional learning suite was developed to support the P-12 mathematics pedagogy across a range of topics including Problem-solving—Inquiry, developed in collaboration with a leading academic researcher. Each topic provides a self-paced two hour professional development module.

Mathematical guided inquiries (MGIs) for Prep to Year 9 were developed using the inquiry pedagogy model (Allmond et al., 2010) and are situated in real world problems. For state-wide use appropriate scaffolding was provided to support teachers (Debritz & Horne 2013). The MGIs are being updated to align with the Australian Curriculum: Mathematic (v9) and surface the proficiencies and mathematical processes. Both these resources are shaping mathematical inquiry pedagogy across the state and are supporting the connections and continuities and in students mathematical learning journey from Prep to Year 12.

## References

Australian Curriculum, Assessment and Reporting Authority. (2023). *The Australian curriculum: Mathematics*. Melbourne: Author. https://www.australiancurriculum.edu.au/

Allmond, S., Wells, J., & Makar, K (2010). *Thinking through mathematics: Engaging students in mathematical inquiry*. Melbourne: Curriculum Press.

Debritz, C., & Horne, R (2013). Guided inquiry as a model for curriculum resources in mathematics. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning. Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia (MERGA)*. Melbourne: MERGA.

Makar, K. (2012). The pedagogy of mathematical inquiry. In R. Gillies (Ed.), *Pedagogy: New developments in the learning sciences* (pp. 1–23). Nova Science.

Queensland Curriculum and Assessment Authority (2021). *Problem-solving and modelling task*. QCAA. https://www.qcaa.qld.edu.au.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 586). Newcastle: MERGA.

# Exploring Using the Empty Number Line for Enhancing Pre-Service Teachers' Mental Computations

Tarryn Lovemore
*Nelson Mandela University*
tarryn.lovemore@mandela.ac.za

This short communication shares the preliminary stages of a study exploring pre-service mathematics teachers' (PSTs) use of the empty number line (ENL) as a strategy for performing and teaching mental computations. As part of a broader South African project—Mental Mathematics Work Integrated Learning (MM-WiL) (Graven & Venkat, 2023)—the intention of this study will be to answer the research question:

- In what ways can the empty number line enhance mathematics PSTs' confidence and competence in performing and teaching mental computation strategies?

Evidence from international and local studies shows that the mathematics knowledge and number sense of many PSTs globally is limited (e.g., Aktaş & Özdemir, 2017; Bowie et al., 2019). Such limitation consequently results in negatively affecting learners. The study is, therefore, framed by Ball et al.'s (2008) Mathematical Knowledge for Teaching. One useful tool to support number sense development and mental calculation strategies is the empty number line. The ENL is a visual representation, with no numbers or unit markers, used for supporting mental computations of addition and subtraction. Bobis (2007) and van den Heuvel-Panhuizen (2008) caution against using the ENL rigidly as a procedural tool, but rather encourage its use as a flexible strategy. The MM-WiL programme's use of the ENL is thus appropriate for supporting participating PSTs' mathematical knowledge.

In my presentation I will share some guiding literature on the ENL from leading authors working from Australian, Dutch and South African perspectives. I will outline aspects of my intended mixed methods research design for the study. I will include in the design collecting pre- and post-test data from my participating PSTs, plus their responses to in-depth interviews and their written reflections on the various micro-teaching opportunities they will have been exposed to as part of the MM-WiL project. As this study is in its initial stages, I welcome engagement with fellow researchers on the research design.

## References

Aktaş, M. C., & Özdemir, T. E. (2017). An examination of the number sense performances of preservice elementary school mathematics teachers. *European Journal of Education Studies, 3*(12). https://doi.org/ 10.5281/zenodo.1117088

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. https://doi.org/ 10.1177/0022487108324554

Bobis, J. (2007). The empty number line: A useful tool or just another procedure? *Teaching Children Mathematics, 13*(8), 410–413. http://www.jstor.org/stable/41198983

Bowie, L., Venkat, H., & Askew, M. (2019). Pre-service primary teachers' mathematical content knowledge: An exploratory study. *African Journal of Research in Mathematics, Science and Technology Education, 23*(3), 286–297. https://doi.org/10.1080/18117295.2019.1682777

Graven, M., & Venkat, H. (2023, January 17–19). Incorporating attention to mental mathematics in pre-service teacher education: The mental mathematics work-integrated learning project. *Symposium presented at the 31st annual conference of the Southern African Association for Research in Mathematics, Science and Technology Education.* Bloemfontein.

Van den Heuvel-Panhuizen, M. (2008). Learning from "Didactikids": An impetus for revisiting the empty number line. *Mathematics Education Research Journal, 20*(3), 6–31. http://www.merga.net.au/node/41?volume=20&number=3

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 587). Newcastle: MERGA.

# Mathematical Thinking in Primary School Students: The Relative Contribution of Student and Teacher Characteristics

Erin Mackenzie
*Western Sydney University*
e.mackenzie@westernsydney.edu.au

Nathan Berger
*Western Sydney University*
n.berger@westernsydney.edu.au

Matt Thompson
*Western Sydney University*
matt.thompson@westernsydney.edu.au

Kathryn Holmes
*Western Sydney University*
k.holmes@westernsydney.edu.au

The development of a strong foundation in primary mathematics is viewed as a critical outcome of schooling (Goos & Kaya, 2020), contributing to students' future achievement in mathematics (Siegler et al., 2012). While there are many factors that may influence children's achievement in mathematics, their enjoyment of mathematics and their teacher's pedagogical decisions when teaching mathematics are of particular interest, due to their malleability (and opportunity for intervention). Whether the relative importance of these factors in predicting children's mathematical thinking changes over the primary years is relatively underexplored, however.

In this study, we examined three waves of data from the Kindergarten cohort of the Longitudinal Study of Australian Children. The data were collected in 2006, 2008, and 2010, when the children were in Stage 1, 2, and 3 respectively. In each wave, children ($n$ = 4464) were asked about their enjoyment of learning mathematics, and each child's teacher completed a measure of the child's level of mathematical thinking. Teachers also reported their approach to teaching mathematics, use of ability groupings to teach mathematics, and teaching self-efficacy.

Regression models predicting mathematical thinking were conducted for each Stage. Findings suggest that student gender was not a significant predictor of mathematical thinking in any Stage. In all Stages, there was a positive relationship between children's enjoyment of mathematics and their mathematical thinking. Similarly, teacher self-efficacy was positively related to children's mathematical thinking in all Stages. All other teacher variables differed in their relationship with children's mathematical thinking across the Stages. Teachers' frequency of use of ability groupings to teach mathematics was positively related to children's mathematical thinking in Stage 1, but not Stage 2 or 3. Conversely, teachers' emphasis on talking about and solving mathematical problems (rather than on learning rules, facts, and procedures) was positively related to children's mathematical thinking in Stage 3, but not Stage 1 or 2. These findings emphasise the importance of children's enjoyment of mathematics and teacher self-efficacy, while also pointing to differences in how teaching strategies predict mathematical thinking across the primary years, offering opportunities for targeted intervention strategies.

## References

Goos, M., & Kaya, S. (2020). Understanding and promoting students' mathematical thinking: A review of research published in ESM. *Educational Studies in Mathematics, 103*, 7–25. https://doi.org/10.1007/s10649-019-09921-7

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23*(7), 691–697. https://doi.org/10.1177/0956797612440101

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 588). Newcastle: MERGA.

# Differentiating Instruction in Junior Secondary Mathematics: A Resource Perspective

Andrew Marks
*The University of Queensland*
a.marks@uqconnect.edu.au

Jana Visnovska
*The University of Queensland*
j.visnovska@uq.edu.au

When the question of supporting diverse students in classrooms is raised, the practice of differentiating instruction is often offered as an answer. By adopting a "resource" approach to teaching (Remillard, 2005), we demonstrate a considerable lack of support that teachers face when addressing diversity through differentiation. Specifically, we report on how we explore the extent to which instructional resources available to teachers currently aim to equip them for making differentiated instruction a reality in their classrooms. We foreground the construction of an analytical framework for this kind of teaching resource analysis, and illustrate its use, in the process demonstrating that current resources indeed differ considerably in terms of the differentiation support they provide to the teacher.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*(2), 211–246.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 589). Newcastle: MERGA.

# Creating Dialogic Spaces: A Comparative Study of Ground Rules

Kevin Martin
*Cambridge University*
kcm38@hughes.cam.ac.uk

Nigel Calder
*University of Waikato*
nigel.calder@waikato.ac.nz

This short communication reports on a comparative case study that is in the analysis stage. It involves the examination of the ground rules that were used to facilitate a dialogic space in two discrete and diverse research studies. The first one was the learning process that took place as Year 5 & 6 children learnt to code with *ScratchMaths* as part of their mathematics programmes. The second one was the learning process undertaken as crop farmers in rural east Africa looked to develop their practice through meetings and various other communications. The intention was to compare the development of general ground rules to see if there were common actions or principles that might indicate that they are important for the establishment of ground rules in dialogic spaces in general.

Researching and understanding the nature of dialogic space has become increasingly important in many areas of education in the last 15 years, including in mathematics education. Hence a better understanding of the central elements of dialogic space, such as ground rules, is valuable and of interest to the mathematics education community, and the education community in general. A comparative case study was undertaken with a focus on the perceptions and interpretations of key participants, and the actions and processes that occurred. The two projects were set in two discrete contexts, but their key purpose, to facilitate learning, was aligned. Some initial aspects of ground rules were collaboratively identified, with both studies then independently analysed to identify emerging themes related to these ground rules. Through several iterations of analysis, including cross analysis, and co-construction through the full research team, the themes were adapted and data from both projects were considered and ascribed to the appropriate theme as supporting evidence.

While that process is still currently not fully completed there are several key elements that are emerging and that seem to be important. These are related to: developing the processes for interaction and communication; developing trust between participants; developing respectful dialogue; teacher roles; and facilitating collaborative work and the co-construction of meaning. It is not the purpose of the research to establish key elements that are critical or might ensure that a dialogic space will emerge, as the aims of mathematics education research are usually context centric and eclectic. However, if there are key common attributes and processes that are evident in, and common to, these two diverse research projects, it does suggest that they might be important for other mathematics education work when establishing dialogic space and research, and perhaps for teaching, learning and educational research in general.

## Acknowledgements

Imogen Casebourne & Cynthia James, Cambridge University, and Kate Rhodes, University of Waikato, are also members of the research team.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 590). Newcastle: MERGA.

# Maths in the Play World of Kindergarten

Nadine Meredith
*Catholic Schools Parramatta Diocese*
nmeredith@parra.catholic.edu.au

Alison Busuttil
*Catholic Schools Parramatta Diocese*
abusuttil2@parra.catholic.edu.au

Play is a promising setting for early mathematics education that incorporates the children's perspective into teaching (Ginsburg, 2006). Recent research has indicated that the early years is a time to engage children in a range of mathematical ideas to develop their mathematical capability (MacDonald, 2018). However, tensions can arise for teachers of the early years of primary school between providing opportunities for mathematical play experiences versus more formal guided instruction lessons. Teachers in the following study did not seek to choose one approach or the other, but rather investigated whether planned play experiences at the start of the day in Kindergarten (Foundation year) contributed to students using their mathematical ideas later in the day (or week) during their daily mathematics lesson.

A group of Kindergarten teachers, and their school leaders, from nine schools in the New South Wales Catholic Schools Parramatta Diocese engaged in professional learning. This learning focussed on developing young learners' oral language and inquiry skills in the early years of school through play. In parallel with this learning, two of these nine schools were part of the Exploring Mathematical sequences of Connected, Cumulative and Challenging tasks (EMC$^3$) Research Project (Sullivan et al., 2020). While regularly supporting one of these schools during the EMC$^3$ project, the teachers combined the pedagogical approaches of mathematics inquiry and mathematical talk into the children's play experiences. As Helenius et al. (2016) noted, "a teacher's active participation in the play could contribute to children learning more about mathematics" (p. 154).

The teachers regularly reflected on their role during the children's play using the continuum of play-based learning (Pyle & Danniels, 2017), in particular inquiry play, collaboratively designed play, and playful learning. The lead teacher noted a Kindergarten teacher's insight that, "When you play with them, you get a better understanding of their maths knowledge." They discovered there were more opportunities to notice, explore, and talk about mathematics, and help support their Kinder students' mathematical development, than they initially anticipated. Through the teachers' active participation in the Kindergarten children's play world, they were able to gain deeper insights about their students' understanding of early mathematical concepts, mathematical thinking, and mathematical development in a playful way. Research related to this study is continuing.

## References

Ginsburg, H. P. (2006). Mathematical play and playful mathematics: A guide for early education. In D. G. Singer, R. M. Golinkoff, & K. Hirsh-Pasek (Eds.), *Play = learning: How play motivates and enhances children's cognitive and social-emotional growth* (pp. 145–165). Oxford University Press.

Helenius, O., Johansson, M. L., Lange, T., Meaney, T., Riesbeck, E., & Wernberg, A. (2016). When is young children's play mathematical? In T. Meaney, O. Helenius, M. L. Johansson, T. Lange, & A. Wernberg (Eds.), *Mathematics education in the early years* (pp. 139–156). Springer International Publishing.

MacDonald, A. (2018). *Mathematics in early childhood education*. Oxford University Press.

Pyle, A., & Danniels, E. (2017). A continuum of play-based learning: The role of the teacher in play-based pedagogy and the fear of hijacking play. *Early Education and Development, 28*(3), 274–289.

Sullivan, P., Bobis, J., Downton, A., Hughes, S., Livy, S., McCormick, M., & Russo, J. (2020). Ways that relentless consistency and task variation contribute to teacher and student mathematics learning. In A. Coles (Ed.), *For the learning of mathematics monograph 1:Proceedings of a symposium on learning in honour of Laurinda Brown* (pp 32–37). FLM Publishing Association.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 591). Newcastle: MERGA.

# Interdisciplinary Mathematics and Science (IMS): Data Modelling of Plant Growth in Grade 2

Joanne Mulligan
*Macquarie University*
joanne.mulligan@mq.edu.au

Peta White
*Deakin University*
peta.white@deakin.edu.au

Russell Tytler
*Deakin University*
russell.tytler@deakin.edu.au

Melinda Kirk
*Deakin University*
melinda.kirk@deakin.edu.au

The Interdisciplinary Mathematics and Science (IMS) 3-year longitudinal project[a,b] (https://imslearning.org/) is developing and investigating a pedagogical approach that implements integrated learning sequences where students' representational systems in the two subjects can support measurement and data modelling (Tytler et al., 2023). The model of interdisciplinarity is illustrated through a case study of a learning sequence on plant growth with Grade 2 classes based on Lehrer and Schauble's (2004) approach. The guided inquiry pedagogy involves students in engaging in concepts that sit at the intersection of the two disciplines (estimating, measuring, representing and explaining growth of plant height and depth, changes over time, and organising and structuring data to best display results). We describe the pedagogy used by teachers to support mathematics learning, measurement and data representation. Data sources comprised student artefacts, interviews with students, and pre- and post-test results of student's conceptual understanding and mathematical representations. Grade 2 students' data collection and recording of measures, their predictions and representations of plant growth demonstrated their impressive and emerging mathematical and scientific knowledge through structured inquiry.

## References

Lehrer, R., & Schauble, L. (2004). Modeling natural variation through distribution. *American Educational Research Journal*, *41*(3), 635–679.

Tytler, R., Prain, V., Kirk, M., Mulligan, J. T., Neilsen, C., Speldewinde, C., White, P., & Xu, L. (2023). Characterising a representation construction pedagogy for integrating science and mathematics in the primary school. *International Journal of Science and Mathematics Education*, *21*, 1153–1175.

[a]Australian Research Council Discovery Grant No. DP180102333 *Enriching mathematics and science learning: an interdisciplinary approach.*

[b]Research Team: Russell Tytler (Deakin University), Joanne Mulligan (Macquarie University), Peta White, Lihua Xu, Vaughan Prain, Chris Nielsen, Melinda Kirk, Chris Speldewinde, (Deakin University), Richard Lehrer, Leona Schauble, (Vanderbilt University).

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 592). Newcastle: MERGA.

# Talking Maths: Analysing Classroom Discourse and Student Talk

Carol Murphy
*University of Tasmania*
carol.murphy@utas.edu.au

Tracey Muir
*University of Tasmania*
tracey.muir@utas.edu.au

We report on initial stages of the pilot phase of our Australian Research Council funded project: Talking Maths: Bridging the gap through talk in Early Years mathematics. The study aims to address the gap in mathematical performance in Australia in relation to socioeconomic status (SES) by focusing on language and learning in mathematics. We report on our early work with teachers in designing and evaluating an intervention intended to support students win engaging in productive talk in small group work.

We will share some initial analysis of student talk based on students' linguistic choices when engaging in mathematical tasks with their peers and in levels of communication during individual interviews with researchers. Our early analysis suggests a wide range of language and literacy skills and we consider the implications for young students' access to the discourse in mathematics classrooms, including responding to questions such as why and how.

Initial reflections by teachers on their professional development, based on Clarke and Hollingsworth's (2002) interconnected model will be explored, alongside task design. We will also share methodological experiences of using the Cambridge Dialogue Analysis Scheme for teacher-student classroom discourse (Hennessy et al., 2020).

## References

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*(8), 947–967.

Hennessy, S., Howe, C., Mercer, N., & Vrikki, M. (2020). Coding classroom dialogue: Methodological considerations for researchers. *Learning, Culture and Social Interaction, 25*, 100404

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 593). Newcastle: MERGA.

# An Autoethnographic Intervention to Improve Own Teaching Practices and Student Learning: An Innovative Approach

Sitti Maesuri Patahuddin
*University of Canberra*
sitti.patahuddin@canberra.edu.au

This paper explores the concept of autoethnographic intervention as a tool for improving teaching practices and student learning. It involves conducting self-reflection and introspection to gain insights and make targeted improvements in teaching. Autoethnographic intervention is an extension of the autoethnographic method (Douglas & Carless, 2013; Ellis & Bochner, 2000; Reed-Danahay, 2021), specifically focusing on the implications for one's own professional development. It emphasises the journey of improving one's own teaching practices, tackling challenges, and striving for a deeper understanding of how to engage pre-service teachers (PSTs) effectively. The use of different approaches, such as writing reflections, seeking feedback, and engaging in discussions with colleagues, allows for a comprehensive exploration of teaching experiences. This paper explores three snapshots of the autoethnographic intervention in teaching practices to illustrate the implementation of this innovative approach in the context of teaching primary mathematics.

The first snapshot discusses the use of a "CONTINUE, STOP, and START Doing" survey as a reflective practice tool. The survey, administered anonymously to PSTs, allows the teacher-educator to strike a balance between their expectations and the expectations of the students. The feedback received from the survey helps identify areas for improvement, such as segmenting recorded lectures, adjusting quiz formats, and providing breaks during tutorials. Self-reflection plays a crucial role in interpreting the survey results and understanding how to improve teaching practices to enhance PSTs' learning experiences. The second snapshot focuses on the benefits of recorded lectures, especially in the context of remote teaching due to COVID-19. The autoethnographic intervention prompts the educator to consider the impact of their teaching on PSTs' learning. To address challenges in online teaching, the educator meticulously scripts their talks, pays attention to grammar, and prepares multiple representations of mathematical concepts. This process not only improves communication skills and content pedagogical knowledge but also results in positive feedback from PSTs. The autoethnographic intervention proves effective in transforming dissatisfaction into satisfaction in subsequent teaching sessions. The third snapshot highlights the use of data to improve learning outcomes. Recognising the cultural differences between the educator and PSTs, the autoethnographic intervention prompts the exploration of strategies to engage students who may have missed pre-recorded lectures. By involving students in a workshop activity and collecting data about the pre-recorded lectures using a pictorial graph, the educator facilitates discussions and enhances understanding among PSTs. This approach creates an open and safe space for learning and encourages reflection on pedagogical strategies.

## References

Douglas, K., & Carless, D. (2013). A history of autoethnographic inquiry. In S. H. Jones, T. Adams, & C. Ellis (Eds.), *Handboook of autoethnography* (pp. 84–106). Left Coast Press.

Ellis, C., & Bochner, A. (2000). Autoethnography, personal narrative, reflexivity. Researcher as subject. In N. Denzin, & Y. Lincoln (Eds.), *The Sage handbook of qualitative research* (pp. 733– 768). Sage.

Reed-Danahay, D. (Ed.). (2021). *Auto/ethnography: Rewriting the self and the social*. Routledge.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 594). Newcastle: MERGA.

# "I Did it in My Head": Investigating Children's Mathematical Thinking

Kate Quane
*University of South Australia*
kate.quane@unisa.edu.au

Carolyn Bruhen
*University of South Australia*
carolyn.bruhen@unisa.edu.au

Mathematical thinking has multiple definitions, including a means to "describe mathematical growth" (Rasmussen et al., 2005, p. 52) and as a function of mathematical processes and operations (Burton, 1984). Liljedahl (2021) describes mathematical thinking as "messy" requiring risk-taking (p. 72), and "difficult" (p. 87). Mathematical thinking can be described as an invisible and individualized "self-sustaining process" that is a form of self-communication (Sfard, 2008, p. 81). Liljedahl (2021) much publicized work on thinking classrooms in mathematics makes the association between thinking and learning. Given this sentiment, we seek to understand the processes that students engage when thinking mathematically with the work of de Bono (1971) proving instrumental in describing thinking processes.

This paper reports on research conducted in a small regional South Australian School where the entire school population (N = 36) was divided into two classes. The multi-phase research used participants' drawings, their written descriptions, and interviews about their drawings. Children were withdrawn from class to complete a drawing and semi-structured interview. A drawing prompt was read to each childing, outlining the requirement for children to "draw themselves doing mathematics" with further instructions stating that children needed to include their face and that the focus of the drawing could be any aspect of mathematics (Quane et al., 2021).

Our findings highlight that there were varying degrees of invisible thinking from students from not being able to articulate their thinking "I can't really describe it" or "I don't really know how to explain it", to students who gave an indication that they were thinking but providing little or no description or explanation. Additionally, students used a range of "porridge words" (de Bono, 1971) to describe their mathematical thinking. Further analysis of the occurrence of invisible thinking revealed that students use invisible thinking in several ways which when examined closely provide clues to educators regarding how to support students to improve how they communicate their mathematical thinking.

## References

Burton, L. (1984). Mathematical thinking: The struggle for meaning. *Journal for Research in Mathematics Education JRME*, *15*(1), 35–49. https://doi.org/10.5951/jresematheduc.15.1.0035

de Bono, E. (1971). *Practical thinking: 4 ways to be right; 5 ways to be wrong; 5 ways to understand*. Penguin Books Ltd.

Liljedahl, P. (2021). *Building thinking classrooms in mathematics: Grades K–12 14 teaching practices for enhancing learning*. Corwin: A SAGE company.

Quane, K., Chinnappan, M., & Trenholm, S. (2021). Draw yourself doing mathematics: Developing an analytical tool to investigate the nature of young children's attitudes towards mathematics. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-021-00399-2

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 51–73. https://doi.org/10.1207/s15327833mtl0701_4

Sfard, A. (2008). *Thinking as communicating human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 595). Newcastle: MERGA.

# Orchestrating Mediational Means in Solving a Mathematical Problem

Sally-Ann Robertson
*Rhodes University*
s.a.robertson@ru.ac.za

Mellony Graven
*Rhodes University*
m.graven@ru.ac.za

Working as we do in an interdisciplinarity space, the first author being involved in second language teaching and learning, the second author a mathematics educator, we see MERGA 45's theme: 'Weaving mathematical education research from all perspectives' as aligning well with our shared focus on the ways in which teachers and their learners use language and other semiotic modes in making mathematical meaning. Research highlights the need to provide learners with opportunities to participate in mathematical talk and to engage in what Mercer (2000) termed 'interthinking' around the mathematical ideas they encounter. Research also shows, however, that learners often struggle to verbally articulate their thinking. Walshaw and Anthony (2008, p. 254), for instance, observed that many children "were decidedly ill at ease [about sharing] their thinking with others". While this may apply even in contexts where learners are proficient users of the language of teaching and learning, it applies even more so where learners are in a second language teaching and learning environment and are still in the process of gaining proficiency in that additional language. In either situation, teachers need to take on the responsibility of helping to induct their learners into appropriate ways of participating in mathematical dialogue.

There is growing recognition that communication almost never happens via words alone: it is intrinsically multimodal. We locate our own work within a broadly socio-cultural framework whereby we accept the view that learning is a social process of co-construction requiring mediation from more knowledgeable others (Vygotsky, 1978). Venkat and Askew (2018) identify four key mediational means for primary mathematics teaching: (1) tasks and example spaces, (2) artifacts, (3) inscriptions, and (4) talk (in which they include gesture). We focus particularly on the latter three in our highlighting of some of the ways in which these three mediational means were orchestrated in the course of a primary after-school mathematics club session. We explore how the problem-solving task used in the session was mediated through a combination of these means, enabling the club members to successfully navigate their way towards solving the mathematical challenge set them. In our presentation we will share our multimodal analysis of selected transcripts taken from the full transcription of the club session videotape. The session was conducted primarily in English, and our analysis illuminates the ways in which the club facilitator (second author) built in the added support of artifacts, inscriptions, gesturing, and translanguaging to help mediate the linguistic challenges facing the club members, none of whom were first language users of English nor yet particularly adept at engaging in mathematical talk. We drew on multimodal and second language acquisition analytic frames to examine how the orchestration of the mediational means enabled interthinking.

## References

Mercer, N. (2000). *Words and minds: How we use language to think together*. Routledge.

Venkat, H., & Askew, M. (2018). Mediating primary mathematics: Theory, concepts, and a framework for studying practice. *Educational Studies in Mathematics*, *97*(1), 71–92.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Harvard College.

Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classrooms. *Review of Educational Research*, *78*(3), 516–551.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 596). Newcastle: MERGA.

# How Do Primary Pre-Service Teachers Use Feedback and Reflection Cycles to Plan Rich Mathematics Learning Experiences?

Shaileigh Rowtcliff
*Flinders University of South Australia*
shaileigh.rowtcliff@flinders.edu.au

Timna Garnett
*Flinders University of South Australia*
timna.garnett@flinders.edu.au

Reflective practice is of utmost importance in teacher education as deep-transformative reflection is shown to improve life-long learning, professional practice and ultimately transform practice (AITSL, 2013). Research suggests that pre-service teachers struggle with the feedback and reflection cycle lacking justification for their thoughts, thus creating the need for more structured reflection throughout the teaching degree (Killic & Dogan, 2021). Cavanaugh (2021) along with Killic and Dogan (2022) in their recent publications on preservice teacher noticing, reflection and feedback, states that oral and written reflection after the lesson helps preservice teachers to review what happened providing them opportunities to develop a plan for future action. This also ties into the notion of receiving feedback to 'feedforward' (Price, 2010), allowing pre-service teachers to analyse their teaching and connect theory to practice, becoming more aware of any pre-conceived ideas.

Our study, conducted with 2nd year pre-service teachers, aimed to explore how a feedback and reflection cycle could influence lesson planning in a primary mathematics curriculum studies topic. There were three cycles of feedback each prompting reflection and adaptation or refinement of the lesson including: 1) a written lesson plan with feedback from lecturer, 2) present the lesson plan to peers with feedback from peers and lecturer, and 3) teach the lesson to children on campus with feedback from classroom teacher. An action research (AR) model is used to assess the culture and work of the academy which focuses on assessing teaching through systematic reflection and evaluation to inform … innovative teaching practices (Harvey & Jones, 2021, p. 173). In this short communication, a discussion of the preliminary results will be offered. The findings suggest that detailed written feedback using descriptive rubrics (Brookhart and Chen, 2015) partnered with verbal feedback is worthwhile in helping PSTs plan rich mathematics learning experiences. Preliminary findings also suggest structured writing prompts like the 4R framework is somewhat useful in helping PSTs reflect on feedback.

## References

AITSL (2021), Australian professional standards for teachers. https://www.aitsl.edu.au/docs/default-source/default-document-library/download-lead-level-descriptors.pdf [accessed 10.08.2022]

Brookhart, S., & Chen, F. (2015). The quality and effectiveness of descriptive rubrics, *Educational Review*, *.67*(3), 343–368).

Choy, S., Dinham, J., Yim, J., and Williams, P. (2021). Reflective thinking practices among pre-service teachers: Comparison between Malaysia and Australia. *Australian Journal of Teacher Education*, *46*(2).

Cavanagh, M. (2021). Pre-service teacher reflection and feedback using an online video platform during professional experience. *Australian Journal of Teacher Education*, *46*(2).

Harvey, M., & Jones, S. (2021). Enabling leadership capacity for higher education scholarship in learning and teaching (SOTL) through action research. *Educational Action Research*, *29*(2), 173–190.

Kilic, H. and Dogan, O. (2022). Preservice teachers' noticing in action and in reflection, *International Journal of Science and Mathematics Education, 20*, 345–366.

Price, M., Handley, K., Millar, J., and O'Donovan, B. (2010), Feedback: All that effort, but what is the effect? *Assessment and Evaluation in Higher Education*, *35*(3), 277–289.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 597). Newcastle: MERGA.

# A Middle School Student's Concept of Equivalent Fractions: Misconception or Transitional Conception?

Jaehong Shin
*Korea National University of Education*
jhshin@knue.ac.kr

Soo Jin Lee
*Korea National University of Education*
sjlee@knue.ac.kr

It has been well documented in many previous studies that students have difficulty understanding the concept of equivalent fractions that two different fractions can represent the same quantity (Behr et al., 1984; Kamii & Clark, 1995; Wong, 2010). They struggle to translate between pictorial and symbolic representations for equivalent fractions (Wong & Evans, 2007). Even in the early years of secondary school, some students exhibited whole number thinking about fractions despite their procedural competence with fractions (Pearn & Stephens, 2004). Understanding of fraction equivalence must not be reduced to the mastery of the procedure: multiplying or dividing the numerator and denominator of a fraction by the same number (Ni, 2001).

The data reported here were collected as part of the larger project to understand the relationship between students' arithmetic and algebraic knowledge. While conducting a retrospective analysis of our clinical interviews with middle school students, we found that Hyun (the seventh-grade student)'s unexpected answers and problem-solving behaviours were partly due to the quantitative structure related to his use of the equivalent fraction algorithm. (e.g., His multiplying the numerator and denominator of $\frac{1}{5}$ by 3 to make $\frac{3}{15}$ is associated with his pictorial representation of $\frac{3}{15}$ as the composition of three $\frac{1}{5}$s).

In this short communication, we report on Hyun's (mis)understanding of equivalent fractions and how the concept of equivalent fractions influenced his problem-solving activities across various problem contexts during the interview. We will also discuss whether his fraction concept should be considered as a misconception to be removed and replaced with the expert (teacher)'s view or a *transitional conception* that arose as a result of his sense-making of those specific problem contexts and "can be used productively to move toward a subsequent conception, a refined version of the original conception" (Moschkovich, 1998, p. 169).

## References

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323–341.

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *Journal of Mathematical Behavior, 14*(4), 365–378.

Moschkovich, J. N. (1999). Students' use of the x-intercept as an instance of a transitional conception. *Educational Studies in Mathematics, 37*(2), 169–197.

Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence. *Contemporary Educational Psychology, 26*(3), 400–417.

Pearn, C., & Stephens, M. (2004). Why you have to probe to discover what year 8 students really think about fractions. In I. Putt, R. Faragher, & M. McLean (Eds.), *Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia* (pp. 430–437). MERGA.

Wong, M. (2010). Equivalent Fractions: Developing a pathway of students' acquisition of knowledge and understanding. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 673–680). MERGA.

Wong, M., & Evans, D. (2007). Students' conceptual understanding of equivalent fractions. In J. Watson, & K. Beswick (Eds.), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (pp. 824–833). MERGA.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 598). Newcastle: MERGA.

# Is It Merely A 'Drill'? A Lesson Learnt from Chinese Mathematics 'Drill Practice'

Jiqing Sun
*Deakin University*
jiqing.sun@deakin.edu.au

Mathematics education in China is always perceived as 'drill practice' focused, which means students usually do an extensive amount of practice to master the content learned. In this sense, sometimes it is criticized as heavily 'procedural' orientated teaching and learning. However, when closely looking at these so-called 'drill-based' questions, it could be noticed many of them are well crafted to address students' conceptual understanding, flexible thinking, and reasoning. For instance, in algebra, many questions expose students to the structure flexibilities that are critical for mathematics learning at the senior level. Another example is that judging whether a statement or a procedure is correct or not is a very popular type of practice, which is supportive to the development of students' conceptual understanding. In this short communication, the presenter will show a range of examples of questions in the Chinese mathematics exercise book and discuss the rich pedagogical opportunities behind them, which might be directly applied to Australian secondary mathematics classrooms.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 599). Newcastle: MERGA.

# A Study of the Mathematics Experiences of Students with Down Syndrome in Australian Primary Schools

Matt Thompson
*Western Sydney University*
matt.thompson@westernsydney.edu.au

Catherine Attard
*Western Sydney University*
c.attard@westernsydney.edu.au

Kathryn Holmes
*Western Sydney University*
k.holmes@westernsydney.edu.au

Often, mathematics is viewed as a problem for individuals with Down syndrome, with many being developmentally "behind their class peers in attainment of mathematics curriculum milestones" (Faragher & Clarke, 2020, p. 122). Faragher and Gil Clemente (2019) challenge this position by viewing mathematics as a solution. This solution is grounded in the potential opportunities that mathematics has in allowing individuals with Down syndrome to engage with the world around them.

For students with Down syndrome, the exposure to positive mathematics experiences from an early age is crucial, to foster a 'want' or desire from the child with Down syndrome to engage with the subject. Developing the appropriate numeracy skills needed to be able to function in, contribute to and make sense of the world in which they live is crucial for this population of students to experience independence, develop a sense of purpose and function as a member of wider society.

With the emergence of a heightened focus on inclusive education, students with Down syndrome are in a more advantageous position to be exposed to and experience the mainstream mathematics curriculum when they are in a mainstream classroom (Faragher & Clarke, 2020). However, there does appear to be a lack of evidence of how best to teach mathematics to students with Down syndrome (Faragher & Clarke, 2020).

In this session, we outline a project that aims to generate new insight into how best to provide a positive mathematics education for students with Down syndrome. A multiple case study approach will be used in which each case will be representative of the Down syndrome student ($n$=6), the teaching team around the student as well as the students' parents/caregivers. Additionally, six individuals with Down syndrome and their parents/caregivers who have recently graduated from school will also be interviewed to explore if their experiences with mathematics at school have had consequences for how they engage and participate in a post-school setting.

## References

Faragher, R., & Clarke, B. (2020). Inclusive practices in the teaching of mathematics: Some findings from research including children with Down syndrome. *Mathematics Education Research Journal*, *32*(1), 121–146. https://doi.org/10.1007/s13394-019-00294-x

Faragher, R., & Gil Clemente, E. (2019). Emerging trends in mathematics education for people with Down syndrome: Current research and future directions. *International Journal of Disability, Development, and Education*, *66*(2), 111–118. https://doi.org/10.1080/1034912X.2019.1579891

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 600). Newcastle: MERGA.

# A Conceptual Classification of Mathematical Symbols: Encompassing a Student's Stroke Order of Mathematical Symbols in Semiotic Resources by Unpacking Written Signs

Daiki Urayama
*Hiroshima University*
d-urayama@hiroshima-u.ac.jp

A wide variety of modalities (e.g., Edwards & Robutti, 2014) exist in classrooms, including discursive, embodied, and material signs and artifacts, each with different affordances, offering different possibilities for students' actions. For instance, since students think and do mathematics by "drawing" mathematical symbols using writing tools, drawing these integrated symbols reflects the result of selecting the most significant and rational affordances of these modalities in line with their mathematical thinking.

The significance of symbolic processes for mathematics education lies in the use of symbols, which is ubiquitous in the field of mathematics. Semiotics has the potential to be a powerful theoretical lens for studying diverse topics in mathematics education research (Presmeg et al., 2018), and Duval's cognitive semiotics (e.g., Duval, 2006), a leading study of the relationships between symbols and cognition, uses the concept of the registers of semiotic representations in relation to semantics. However, insufficient research results have been obtained on the pragmatic aspects of symbolic use specific to students' drawings, which are subjective cognitive actions performed by students.

The cognitive action of drawing acted upon from a specific modality is therefore considered to be selective and subject-dependent. From this viewpoint, it can be hypothesized that the student's cognitive structure is represented in their drawing. In this study, the discussion will initially focus on a conceptual classification of mathematical symbols, such as sigma and integral, in terms of the conceptual structures of them as well as a potential stroke order that can come into play when students solve mathematical tasks. Additionally, a stroke order can be regarded as one of the semiotic resources and results in a construct of a system of signs, the semiotic bundle (Radford & Sabena, 2015), that is used as a methodological tool to analyze students' interpretations.

## References

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*, 103–131.

Edwards, L., & Robutti, O. (2014). Embodiment, modalities, and mathematical affordances. In Edwards, L. et al. (Eds.), *Emerging perspectives on gesture and embodiment perspectives in mathematics*. (pp. 1–3). Information age Publishing.

Presmeg, N., Radford, L., Roth, M., & Kadunz, G. (2018). Introduction. In Presmeg, N., Radford, L., Roth, M., & Kadunz, G. (Eds.), *Signs of signification; Semiotics in mathematics education research*. (pp. 1–15). Cham, Switzerland: Springer.

Radford, L., & Sabena, C. (2015). The question of method in a Vygotskian semiotic approach. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 157–182). Dordrecht: Springer Netherlands.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 601). Newcastle: MERGA.

# Teacher-Parent Partnerships in the Post-Covid Context

Pamela Vale
*Rhodes University*
pamela.vale@ru.ac.za

Mellony Graven
*Rhodes University*
m.graven@ru.ac.za

It is without question that children whose home environment is supportive of their school learning experience an academic advantage. Muir (2011) writes that parents are influential in whether or not their children experience success in mathematics and Wadham et al. (2022) argue that "schools and families should work collaboratively to achieve shared goals for children's mathematics learning" (p. 679). Despite evidence to indicate the value of parental involvement, in pre-Covid South Africa it was found that very few teachers assigned homework (Graven, 2018). Some reasons included that parents were perceived as being unable to support the homework, or would do the homework themselves (Graven, 2018). Darragh and Franke (2022) also report that pre-pandemic research revealed that "mathematics homework is often unsuccessful or stressful for both parents and children" (p. 1521). In contrast, our research conducted during Covid-19 showed that teachers recognised the critical importance of engaging with parents and many reported very positive experiences of doing so (Vale & Graven, 2022; 2023).

In this research, we ask: How are mathematics teachers engaging with parents in the post-Covid context? Early grade (Grades R, 1 and 2) mathematics teachers (38) completed a questionnaire that probed for responses about how they are engaging with parents and the practice of giving homework in the post-Covid context. The results indicate that most are continuing to engage with parents (32/38), saying "we now have a common understanding of the importance of working together" (W, Q2) and "parents started to engage with their children which is a golden reward for us" (AC, Q2). It is concerning however, that there is evidence of a strand of narrative re-emerging that is indicative of difficulties: "[homework] is a no go at public schools" and "it has changed…parents no longer show interest" (Z, Q2). We need to learn from the data how to assist teachers in continuing to reach families. We learn of some successful measures, like the teacher who has "facilitated workshops with parents to equip them with materials that will assist at home" (C, Q3) and several who "used WhatsApp to explain how to 'teach'" (J, Q5). In contexts where the "community is very illiterate and poor" (X, Q5), this is clearly challenging, but we learn much from this data about how to productively move forward in encouraging teacher-parent partnerships.

## References

Darragh, L., & Franke, N. (2022). Lessons from Lockdown: Parent perspectives on home-learning mathematics during COVID-19 lockdown. *International Journal of Science and Mathematics Education, 20*, 1521–1542.

Graven, M. (2018). Teacher perceptions of the successes and challenges of a mathematics homework drive for primary learners. In P. Webb, & N. Roberts (Eds.), *The pedagogy of mathematics: Is there a unifying logic?* (pp. 161–175). Real African Publishers.

Vale, P., & Graven, M. (2022). Teacher views of parent roles in continued mathematics home learning. In N. Fitzallen, C. Murphy, V. Hatisaru,, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 514–521). Launceston: MERGA.

Vale, P., & Graven, M. (2023). Strategies implemented by South African teachers to ensure continuing mathematics education during COVID-19. *ZDM Mathematics Education, 55*, 163–176.

Wadham, B., Darragh, L., & Ell, F. (2022). Mathematics home-school partnerships in diverse contexts. *Mathematics Education Research Journal, 34*, 679–699.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 602). Newcastle: MERGA.

# Change in Primary Students Algebraic Functional Thinking

Bridget Wadham
*Massey University*
b.wadham@massey.ac.nz

Emily Pearce
*Massey University*
e.pearce1@massey.ac.nz

Within Level Two of the New Zealand Curriculum (MOE, 2007) students aged 7-9 years are expected to 'Find the rule for the next member in a sequential pattern'. Despite this requirement, many teachers continue to place importance on developing number knowledge at the expense of developing deep understandings across the content strands of algebra. This results in algebra being an area of weakness for New Zealand students and a declining number of students selecting algebra pathways in high school. Researchers are now at an increasing rate, recognizing the importance of providing young students with the opportunity to explore and develop functional thinking from the start of their formal schooling journey (Chimoni et al., 2018), with the view that this will aid the struggles and lack of success many students face when they encounter algebra for the first time in high school (Blanton et al., 2017).

We will present the initial findings of a study focused on how students' functional reasoning changes over a five-week algebraic patterning teaching unit. The student participants (n=85) aged 8-9 years old completed a pre and post open-response task (based on the linear function 4x+1=y). Drawing on Stephens et al., (2017) framework for Levels of Sophistication and the New Zealand Curriculum (Ministry of Education, 2007) students' responses were coded and analysed to identify common themes. A key finding shows a shift from mainly pre-structural and variational thinking (pre-unit) to differing levels of correspondence thinking (post-unit). Other key themes evident include the remediation of the assumption of proportionality, changes in representations to structured diagrams to show variables, and the use of natural language to explain and apply emerging functional rules.

## References

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2017). A progression in first-grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics*, *95*, 181-202.

Chimoni, M., Pitta-Pantazi, D., & Christou, C. (2018). Examining early algebraic thinking: Insights from empirical data. *Educational Studies in Mathematics*, *98*, 57-76.

Ministry of Education (2017). *New Zealand curriculum*. Wellington: Learning Media. https://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum

Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning*, *19*(3), 143-166.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 603). Newcastle: MERGA.

# 'Chalk and Talk' and the Development of 21st Century Skills in the Secondary Mathematics Classroom

Ben Zunica
*The University of Sydney*
benjamin.zunica@sydney.edu.au

The 'chalk and talk' method has been and is still a dominant method for teaching secondary mathematics. 'Chalk and talk' involves talking aloud to students while writing the mathematical narrative on a display device such as a whiteboard (Artemeva & Fox, 2011). While the 'chalk and talk' method has some desirable characteristics, it is opposed to the development of 21st century skills, which are important to possess for the current generation of students due to the considerable changes that have occurred in society and workplaces in the early 21st century. There is a tension here for teachers. 'Chalk and talk' has characteristics that make it an attractive and viable teaching method. However, if teachers predominantly use 'chalk and talk', they are hampered in developing the skills students need to prosper in the global and technological world of today. Thus, this study is guided by two research questions:

- Why is 'chalk and talk' a preferred instruction method for current mathematics teachers?
- What do teachers suggest could be changed in their professional lives to incentivise the use of alternative teaching methods?

To assist in answering the research questions, four secondary mathematics teachers were interviewed and observed teaching classes. Three of the four frequently used 'chalk and talk', with at least 80% of observed lessons utilizing the 'chalk and talk' method in a similar way to that outlined above. However, one participant (Daniel) used it less frequently, at only 40%. For participants who largely used 'chalk and talk', three common reasons were given as to why they used it frequently. The first reason given was that students felt that they were doing 'real mathematics' only in 'chalk and talk' classes. Second, these participants often felt time pressured, and preparing a 'chalk and talk' class requires less time compared to alternative methods. Third, assessments chiefly consisted of pen and paper tests meaning that the 'chalk and talk' method was most helpful in preparing students for assessment items.

Considering the reasons for their use of 'chalk and talk', all the participants suggested changes to assessment methods, professional learning programs and the culture of mathematics faculties are required to make alternative methods more viable, reducing the dominance of 'chalk and talk'. Based on the suggestions of the participants, along with the experiences of Daniel, recommendations are made for change at both the school and government level to assist mathematics teachers in lowering their use of 'chalk and talk'. Additionally, these recommendations are designed to assist teachers in creating classrooms that are more conducive to the development of 21st century skills, equipping students for the society and workplaces of today.

## References

Artemeva, N., & Fox, J. (2011). The writing's on the board: The global and the local in teaching undergraduate mathematics through chalk talk. *Written Communication, 28*(4), 345–379. https://doi.org/10.1177/0741088311419630

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 604). Newcastle: MERGA.