Emergent Division Thinking on Entry to School

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This paper contributes to the research literature describing young children's thinking about partitive and quotitive contexts of division. A task-based interview was conducted with 98 5-6-year-old children who had not received any formal instruction about division at school. Three main findings are reported: knowledge of division was exhibited by many children who could solve both partitive and quotitive problems; a range of emergent thinking was found in children's responses to tasks; and a substantial proportion of the children could reason about division with remainders.

Introduction

Young children's number knowledge is a predictor of future mathematical success (Peter-Koop & Scherer, 2012). Research has established children's knowledge of addition and subtraction prior to school; yet there is little research about early multiplication and division knowledge. This paper reports the results of task-based interviews with 98 children who were 5-6 years old, in their first year of school and had not been formally taught division.

Australian and international research has established that young children engage with a range of mathematical concepts and processes prior to starting school (e.g., Aubrey, 1993; Carpenter et al., 1993; Gervasoni & Perry, 2015). Several research studies have reported young children's knowledge of multiplication and division prior to school (e.g., Bicknell et al., 2016; Carpenter et al. 1993; Desforges & Desforges, 1980; Frydman & Bryant, 1988). Recently Cheeseman and Downton (2021) found that young children think multiplicatively much earlier than is reflected in the research literature. More specifically, some children can imagine and draw equal group structures and, in doing so, recognize composite units; others could also enumerate the composite units. In addition, children's emergent ideas of division have been underestimated (Cheeseman & Downton, 2021).

As part of the Multiplication and Division Investigations (MULDI) research project we developed a task-based MULDI Division clinical interview. We used identical division questions to those presented in the pencil and paper test reported by Cheeseman and Downton (2021). Preliminary results indicate that even children as young as 4 years of age can use everyday equipment to solve division problems in ways that reveal their emergent concepts of division (Cheeseman et al., 2022). The aim of this paper is to present the findings of the larger data set and to question the assumption made by many educational authorities and teachers that young children have no early multiplicative concepts—by this term we mean rudimentary ideas of multiplication and division. The research question we posed was:

- What thinking about division concepts do young children have before they are formally taught division at school?

Theoretical Background

Three themes emerged from a review of the literature relevant to our study: partitive and quotitive division; young children's intuitive strategies; and unequal grouping situations. We acknowledge that much of the literature reviewed was dated, highlighting a gap in research concerning young children's intuitive ideas of division thinking prior to instruction.

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Partitive and Quotitive Division

The sematic structure of division word problems can be interpreted and represented in two different ways, as explained by Greer (1992, p. 176):

> Dividing the total by the number of groups to find the number in each group is called partitive division, which corresponds to the familiar practice of equal sharing […]. Dividing the total by the number in each group to find the number of groups is called quotitive division.

A key difference between the partitive and quotitive problems is that the action is different. In partitive division the action is one of sharing, whereas in quotitive division the action is forming equal groups. Previous studies advocate that young children are more successful on partitive than quotitive division due to their limited experience with quotitive division prior to formal school (e.g., Correa et al., 1998; Kouba, 1989; Squire & Bryant, 2002). However, from their study Squire and Bryant (2002) concluded that both partitive and quotitive experiences are important in young children's conceptual development of division concepts. In a more recent study, Ching and Wu (2021) reported that 5-6-year-olds could recognise and reason about multiplicative relationships in partitive and quotitive problems, and that explicit instruction is not a prerequisite for understanding division.

Three studies that investigated young children's approaches to partitive and quotitive division (Carpenter et al., 1993; Kouba, 1989; Mulligan & Mitchelmore, 1997) used different contexts and numbers (dividend, divisor) for each task. The contexts were familiar to the children, for example, sitting at tables, sharing toys or cakes. Some of the numbers included (e.g., 6, 3; 8, 2; 12, 3; 15, 3; 16, 2; 18, 3; 20, 4) were within the children's number experiences. Unlike the other studies, Kouba (1989) used the same numbers across the partitive and quotitive word problems as a way to control one of the variables. Of note, children in these studies had some experience of division, and two studies extended to children aged 6 and 7 (Kouba, 1989; Mulligan & Mitchelmore, 1997). In all of these studies it was reported that the children used different strategies for each division type.

Young Children's Strategy Use

In studies examining how four and five year old children solved division tasks, it was found that most children used direct modelling of the situation using one-to-one correspondence or dealing out (e.g., Blevins-Knabe, 1988; Carpenter et al., 1993; Davis & Pitkethly, 1990; Desforges & Desforges, 1980; Frydman & Bryant, 1988). Although children as young as three and four could share a quantity of 12 equally between 2, 3, or 4 recipients, Frydman and Bryant (1998) found that only 41% of four year olds were able to articulate the number in each shared set and recognise the equivalence of the quantities without having to count each set. Other researchers found that for the partitive division tasks children estimated the number of items to put in each group and tested it out, or used trial and error sharing into groups of items and adjusting (Carpenter et al., 1993; Kouba, 1989; Mulligan & Mitchelmore, 1997). In contrast, strategies used to solve the quotitive division tasks included direct modelling of the situation, double count—keeping a running count of the items in the groups, at the same time as they counting out the items to form groups.

Unequal Grouping Situations

Three of the aforementioned studies (Blevins-Knabe, 1988; Carpenter et al., 1993; Desforges & Desforges, 1980) included unequal grouping situations. Four strategies were evident in the Desforges and Desforges (1980) study: asking for one or more to make the shares equal; removing the excess to make the shares equal; breaking the remainder into two or three to equalise the shares; looking puzzled about what to do with the remainder; one to one sharing; and sharing of the dividend and ignoring the remainder (3.6 to 4.6 year old students). This last strategy was consistent with those used by 4 year olds in the Blevins-Knabe (1988) study. Common strategies used by the middle and older cohorts included asking for an extra item to make up fair shares, indicating their concern to make the sharing fair, and removing the excess (Desforges & Desforges, 1980). In contrast, Carpenter et al. (1993) included one division with remainder problem (19 children how many cars needed if 5 children could fit in each car) in their study. They found that more than half of the 5 year-olds could successfully solve the problem using materials. Findings from these studies suggest that young children can engage with division situations involving remainders with varying degrees of success. However, no recent studies have examined young children's performance on both partitive and quotitive division problems involving remainders.

In summary, some of these studies provided the same context and materials (e.g., cookies and dolls) for each interview task and the same dividend (e.g., 12) but varied the divisor (e.g., 2, 3, 4) (Blevins-Knabe, 1988; Davis & Pitkethly, 1990; Desforges & Desforges, 1980; Frydman & Bryant, 1988). Doing so provided the children with structure and consistency of materials and dividend. These studies focused on partitive division only. In contrast, Carpenter et al. (1993) took more of a problem-solving approach, used different contexts, dividends, and divisors (15, 3; 20, 4; 19, 5), and provided counters and paper and pencil for the children to use. Unlike these studies, we used different contexts in our study, included the same dividend and divisor for pairs of partitive and quotitive division tasks, and two tasks involved remainders. We were interested in the thinking the children exhibited as they solved the different tasks.

Research Method

To investigate young children's intuitive ideas of division we interviewed 98 young children (5 years 4 months to 6 years 8 months) using a task-based interview. This form of interview involves the interviewee interacting with the interviewer as well as with carefully designed mathematical tasks (Goldin, 2000). In the present study, tasks were constructed to elicit children's developing division knowledge and problem-solving behaviours. According to Maher and Sigley (2020), task-based interviews are useful for investigating "existing knowledge, growth in knowledge, and [students'] representations of particular mathematical ideas, structures, and ways of reasoning" (p. 821). Our focus in the present research was on children's informal knowledge of division, their representation of ideas, structures, and ways of reasoning. Our interview protocols used a structured script to ensure reliability, replicability, and generalisability of data collected by seven interviewers.

Data Collection—Interviewing Children

A sample of children, who were in their first year of school in Australia and had not formally been introduced to division, was obtained. There was a representation of schools across the Government, Catholic, and Independent sectors, providing data from 15 classes across 13 schools. Each child was interviewed for approximately 20 minutes by an experienced interviewer who was formerly a classroom teacher. The interviewers were trained to use the protocol. They were asked to read the script and to repeat, or to clarify, but not to explain the problems. Each interview task included equipment for children to manipulate. A video record documented the events, and the interviewer noted each child's responses on an interview record sheet for later analysis.

Data Analysis

To summarise the data, each interviewer entered their interview results on a formatted spreadsheet and compiled a summary of their personal reflections. All video recordings were uploaded to a central digital file accessible only to the interviewers and researchers. Researchers met with the interview team to discuss their shared insights. The third author managed the data for analysis by collating a master file and deidentifying the data in the approved university ethics process (ID 18827). Each video was viewed by Author 3 to check the accuracy of the database. The thinking strategies children demonstrated for each problem were identified and categorised. When children's actions or words required interpretation, the three authors discussed the responses and consensus was reached about the appropriate category to apply. In an interactive way the categories were refined and applied to the data. To check for reliability the researchers then double coded 20% of the data with an 83% inter-rater reliability.

Findings and Discussion

Our findings will detail the knowledge of division that the children's responses revealed, and the range of thinking young children displayed. Some findings related to children's partial understandings will also be outlined. Finally, children's knowledge of remainders will be raised, and children's application of partitive and quotitive thinking will be described.

In the analysis of the results of the interviews, item facility was calculated. The analyses are summarised in Table 1. As can be seen, the proportion of children correctly answering each question varied. The most difficult problem was 22 ÷ 4, presented in a quotitive context of 22 children, to be seated 4 at a table; 40% of the children reached the correct solution. The easiest problem for the children involved 12 ÷ 3, where the partitive context was of 12 candies shared equally between the 3 jars; a 90% success rate was achieved. In response to each task-based interview question between 40% and 90% of students correctly solved problems involving ideas of division. Therefore, the most important finding from the analysis of the interview results was that many young children have emergent understandings of division prior to formal school instruction. The finding that explicit instruction is not a prerequisite for understanding division echoes the argument of Ching and Wu (2021).

Table 1

Proportion of Students Correctly Answering the Interview Question

Worded problem	Correct (n = 98)
These are twelve apples. Three apples fit in a bag. How many bags do you need to carry all apples home?	58%
There are seven socks in the drawer. How many pairs can you put together?	46%
Here are 22 children. Four sit at a table. All the children want to sit down. How many tables do you need?	41%
These are twelve candies. Share all of them out equally between the three jars. How many candies go in each jar?	90%
These are seven donuts. Share all of them out equally between the two children. How many will each child get?	71%
Four children want to play cards. 22 playing cards are on the table. Share them out equally between the four children. How many cards will each child get?	43%

Range of Knowledge

It would be misleading to claim that all young children on entry to school have informal or emergent ideas about division. We observed a range of children's thinking about division. In Table 2 we present the distribution of correct solutions achieved by individual children. As can be seen, 20 children answered no questions or only one question correctly. In contrast, 14 children answered all six problems correctly. The box around the figures indicates the 64 children who showed some partially correct thinking about division contexts.

Table 2

Distribution of Students Achieving Correct Solutions

Number of correct solutions achieved by each child	0	1	2	3	4	5	6
Total number of children in each category (n = 98)	5	15	10	14	19	21	14

Looking at the extremes of achievement brought to mind two children we interviewed who exemplified children who knew little about division and those who knew a lot. Johnny (pseudonyms are used) was able to share 12 candies (glass beads) between 3 jars by placing the candies into each jar one by one. This was the only problem he answered correctly. We came to know a little about Johnny, and that his prior-to-school opportunities to learn were limited. He had not attended kindergarten. His number knowledge was limited to counting to 10 and he could not say how old he was. However, he knew he was "the number after 5" and counted aloud to work out that he was 6 years old.

In contrast, Georgie, who was in the same school and the same class, could answer every task correctly. She was a sophisticated thinker who could share 12 ÷ 3 candies mentally using a known fact. Georgie could also recognise that 7 socks could not be put into pairs and visualised that one sock would be left over. These two children illustrate the extremes of division thinking we observed in our study.

Emergent Thinking

While the two children described above exemplify the range of thinking about division in their responses to the interview questions, there were 64 children who exhibited partial understanding of division concepts. We consider these children are emergent division thinkers. In Table 2 the numbers of children in each category of correct solutions (2 to 5) provide evidence that the largest proportion of children in the sample group had partial understandings of division concepts. We have chosen to describe such partial understanding as *emergent thinking*. For example, in Table 3 the categories of response (thinking strategies) for the task for which children were asked to share 7 donuts between two children are listed, and the percentage of children responding to each is shown. We argue that each of these categories represents some correct thinking. The category that may not be self-explanatory is the second category in Table 3 for which 11 children made two plates of 3½ donuts (playdoh) and answered "four" when asked, how many will each child get? Our reasoning is that these children could divide partitively, they could recognise equal shares, but they did not know how to count the pieces (fractional parts) when they saw four pieces on the plate.

Table 3

Categories of Strategic Thinking for Seven Donuts Shared Between Two Children

Thinking Strategy	Percent of correct responses
Answered "three for each and one left over"	48%
Made 3 ½ and said "four"	12%
Said "three and a half"	5%
Said "split one donut in half"	6%

Their division thinking was partially correct. In a future publication we will report a detailed examination of the characteristics of emergent thinking. However, we make some general observations from an examination of our results. Young children's division concepts seemed to be reliant on children forming and recognising equal groups. Whether it was putting the same number in each group, such as was required for the quotitive thinking to put 3 apples into each bag, or the partitive thinking required to place candies equally into three jars, children had to recognise and construct equal groups to divide successfully.

Our finding concurs with earlier research that the acquisition of equal group structure is important to understanding multiplication and division (Clarke et al., 2002; Killion & Steffe, 2002). We found that knowledge of fair shares also influenced the knowledge of division concepts children were able to display in problem solving. In addition, the contexts that we anticipated would be familiar and meaningful to young children were not universally common, and the interview results were dependent on children's general life experiences. For example, as Amy told us, the socks problem was easy for her because her Mum gave her that job—to sort their socks and put them into pairs. Whereas some children did not know what a "pair of socks" was. These responses drew our attention to the importance of everyday life experiences where young children *mathematise* their world (van Oers, 2013) and develop informal mathematical concepts—in this case, division.

Remainders

We found that young children recognised that division into equal sized groups was not always possible, that is, in situations involving remainders. For example, 71% of children correctly answered 7 ÷ 2 (see Table 1) when we accepted as correct any answer that showed an awareness that one donut of the seven donuts could not be shared equally between two children. The actions of the children indicated that they thought about the situation carefully. For example, it was clear from Georgie's solution—"three each and one for a spare" that she understood the remainder of one. Answering the same problem, Adriana worried about the extra donut. She placed three on each plate and tried several different ways of adding the extra donut to either of the two plates unsuccessfully. It seemed to us that she was testing whether the specific object (donut) mattered. She replaced one donut with another three times before she volunteered, "You have to do nothing with it." It seemed that this was the first time Adriana had been confronted with a sharing situation involving a remainder.

Four of the six interview problems involved remainders; the easiest one to solve was the donut problem (71%). This context for 7 ÷ 2 was easier than the context of pairing socks, which 46% of the children solved correctly. The two contexts for 22 ÷ 4 were correctly achieved by 41% and 43% (see Table 1) of the children. We noted that a substantial proportion of 5-6-year-olds were aware of the need to make equal shares and that some objects would be left over after the equal shares were made.

Our findings expand upon the earlier research (Blevins-Knabe, 1988; Carpenter et al., 1993; Desforges & Desforges, 1980) in that this study involved both partitive or quotitive division situations where each way of conceptualising division was contextualised using the same division operation. In addition, the problems posed in this study involved remainders.

Partitive and Quotitive Contexts

An overview of the correct responses in Table 1 shows that many young children in this study could think about division problems solved in both partitive and quotitive contexts. While this finding concurs with research conducted decades ago (e.g., Carpenter et al., 1993; Kouba, 1989), it is unique in that the same children were asked to think about the same division calculation on the same occasion. In the present study, 68 children (see Table 1) could successfully interpret both partitive and quotitive division situations. We query the claim found in the literature that dealing is a natural way for young children to make equal groups (Davis & Pitkethly, 1990). It seems, based on our experience, that children can deal out the cards one by one. They continue until the cards are exhausted and consider the job done. When playing cards the children paid no attention to whether the dealt groups were equal. Johnny was more interested in playing with the cards. He asked, "How many should each person get?" Ann responded with, "I am sure that they would love whatever you give them." When the dealing was finished he announced that, "This lady has 7. She has more." Johnny dealt the cards and was unconcerned about forming unequal groups.

Georgie, when asked, "How many each?" counted the piles of cards and said, "6, 5, 6, 5". Asked, "Is that equal?" Georgie thought for a few seconds before saying, "Five each and two spare". Thereby she demonstrated that she knew about equal groups and what to do with amounts that were not divisible. Initially she had thought the process of dealing cards was completed when the cards were dealt out. As a result of the 22 cards divided between 4 people task, we learned that cards are a natural dealing context for these children but they can be a distraction.

In summary, we identified three major findings from our study.

1. Knowledge of division was exhibited by 95% of children in this study. The children had not been formally taught division at school at the time of interview. This result contributes to the literature by revealing 5-6-year-old children could solve both partitive and quotitive problems using informal mathematical knowledge.
2. A range of emergent thinking was found in the young children who were studied. This is an original finding as no research has been found that has considered the emergence of division concepts in young children's thinking.
3. Division involving reasoning about remainders was demonstrated by more than 40% of the children who realised that some objects could not be used to make equal groups. No earlier studies have investigated both partitive and quotitive division involving remainder thinking with young children.

Conclusion

The limitation of this investigation is that it used a relatively small number of worded problems to assess division concepts. However, in the study young children's thinking about division was investigated, with some thought-provoking results. Our findings indicate that some young Australian children have emerging ideas of division in their first year of school, and before they are formally taught division. Unlike earlier studies, problems in our study were numerically matched (same dividend, same divisor) and presented in both a partitive and quotitive context to elicit children's thinking. Another distinguishing feature of this study was the inclusion of situations involving remainders. While earlier researchers reported the strategy use of children, they did not report children's emergent thinking. Thus, the findings of our study extend the research in this field. We think there is much to be added to the body of scholarly knowledge about key ideas in early division reasoning, as it is still an under-researched field of study. How children's informal knowledge of division productively develops into formal concepts is yet to be fully understood even though it has major implications for teaching early childhood mathematics.

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