Use of Card Sorting Methodology to Characterise a Primary Teacher's Mathematical Knowledge for Teaching

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Many studies attempt to study teachers' mathematical knowledge for teaching (MKT) through instruments such as multiple-choice questions. Many of these instruments require considerable context information and often provide a static view of a teacher's knowledge. In this paper, we describe the design and use of a card-sorting instrument that can elicit a teacher's MKT. By illustrating how the card-sorting instrument was used to capture a primary teacher's knowledge about teaching division, we argue that card sorting, as part of a suite of other methods, can be a powerful approach to elicit teachers' knowledge in teaching. We conclude by highlighting possible directions in refining and developing card sorting instruments for future studies.

What mathematics teachers know about teaching and learning is an important attribute of teaching expertise. Even before Shulman's (1986) introduction of pedagogical content knowledge (PCK), and subsequently, the conceptualisation of mathematical knowledge for teaching (MKT) by Ball and Bass (2003), there had been efforts to investigate and measure teachers' knowledge (Hill et al., 2007). These early measurements of teachers' knowledge focused solely on mathematical content. However, as highlighted by Hill et al. (2007), teaching requires "more than the ability to do the mathematics in the school curriculum" (p. 125). Instead, Ball et al. (2008) argued that teaching needs knowledge about student errors, alternative algorithms, "rationales for procedures, meanings for terms, and explanations for concepts" (p. 398). Moreover, effective teaching requires coordination of appropriate representations (Shulman, 1987), selecting and using appropriate examples (Chick, 2009), as well as the sequencing and tailoring of content to specific profile of students (Shulman, 1987). Accessing and assessing these domains of knowledge often involve the use of observations, tasks and interviews, and test items, amongst others (Delaney et al., 2008; Hill et al., 2004; Hill et al., 2007; Li, 2007). Despite developments in assessing teachers' knowledge, many of these methods require substantial contextual information (Phelps & Howell, 2016), which may result in a rather limited view of a teacher's knowledge (Hill et al., 2008). We wonder about other alternative tasks that might be useful in uncovering a teacher's mathematical knowledge for teaching. One possible way is to consider the use of card sorting and sequencing tasks, commonly used to analyse participants' thinking and understanding of mathematical concepts as part of a task-based interview (Goldin, 2000). In this paper, we describe the design and use of a card-sorting instrument to elicit a teacher's knowledge in teaching and illustrate how the instrument was used to capture a primary teacher's knowledge about teaching division.

Conceptualizations of MKT

In this section, we begin by recapping the notion of mathematical knowledge for teaching before we examine how teachers' MKT can be assessed. Ball et al. (2008) built on on Shulman's PCK to delineate three subsets of knowledge: knowledge of content and student (KCS); knowledge of content and teaching (KCT); and knowledge of content and curriculum. In addition, Ball and her colleagues expanded the notion of subject-matter knowledge (SMK) by considering the mathematical content needed for teaching. To that end, they characterised the idea of specialised content knowledge (SCK) as the mathematical knowledge and skills that are uniquely required in the work of teaching in contrast to common content knowledge (CCK). this may include knowledge of less common algorithms, different representations of concepts, and explanations of rules and procedures (Ball et al., 2008). In addition, they also developed the provisional idea of *horizon*

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 155–162). Newcastle: MERGA.

content knowledge, which "is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum", and this may include "vision useful in seeing connections to much later mathematical ideas" (p. 403).

Common Methods for Assessing Teachers' MKT

Since Ball et al.'s (2008) description of MKT, researchers had explored different methods to measure teachers' MKT. However, the process of measurement is complex and challenging (Fauskanger, 2015; Hill et al., 2007). It is common for MKT to be assessed through pen and paper test such as multiple-choice items during teachers' examination (Hill et al., 2007) but such tests are usually focused on CCK and SCK (Hill et al., 2007). Other domains of MKT, such as knowledge of their students' common misconceptions and difficulties when learning a particular topic (KCS) or the knowledge as demonstrated by teachers' ability to select the appropriate materials and sequence their teaching examples (KCT) are generally not tested in these items.

With the aim of capturing a more comprehensive view of MKT, researchers have begun to explore the use of lesson observations, mathematical tasks which includes open ended questions, interviews, and teachers' responses to video clips of teaching (Fauskanger, 2015; Hill et al., 2007). Some of these more recent methods provide participants opportunities to explain a misconception or to produce alternative explanations of a mathematical concept (Fauskanger, 2015). For example, Roche and Clarke (2013) used division stories as a questionnaire item to assess teachers' MKT—participants had to name the form of division (partitive or quotative), drawing a picture and write a story problem to represent the form of division. Story problems are also used in the study conducted by Simon (1993) as one of their open-response problems to assess prospective elementary teachers' knowledge of division. In Simon's (1993) study, the participants were required to solve word problems, demonstrate the division algorithm, calculate remainders, and explain how they carried out long division as part of their responses. These methods have attempted to capture a more comprehensive view of teachers' MKT by capturing their pedagogical reasoning. However, the questions are often contextualised which limit teachers' responses and the nature of the questions also result in closed-ended and static responses which are unable to capture the depth of teachers' knowledge (Fauskanger, 2015). In this paper, we contribute to this ongoing effort by introducing another method—sorting and sequencing task—which has the potential to capture participants' MKT as demonstrated by their pedagogical reasoning and illustrate its use to elicit a teacher's MKT in the topic of division.

Sorting and Sequencing Tasks

In sorting and sequencing tasks, participants are presented with a set of cards, carefully designed according to the structure or properties of the concept, for them to sort the cards into different categories or to sequence the cards according to some criteria or preferences (Galant, 2013; Hillen & Malik, 2013). These cards can be printed with different representations such as words, pictures, graphs, and mathematical equations related to the concept. The representations can be related or can be classified according to similar properties. For example, in a set of sorting cards on the topic of division, the representations can be designed based on key ideas of division such as quotient, divisor, and remainder. As the participants sort or sequence their cards, they have to make connections and identify relationships among the structure and properties of the mathematical concept, which are indicators of the level of their conceptual knowledge (Eli et al., 2011). For instance, if a participant is able to sort the cards according to the structures or mathematical features of the concept, we can infer that the participant may have a well-connected set of SCK (Galant, 2013). To gain more insights into the participant's knowledge, card sorting and sequencing tasks can be carried out together with a semi-structured interview (Goldin, 2000) to elicit their explanations and justifications. Using the sorting and sequencing task as a basis for discussion, researchers may ask their participants about the 'why' and 'how' of their sorting and sequencing. For example, in our

study, participants may elaborate by describing how they will teach the concept based on students' profiles and highlight possible students' difficulties (KCS). Participants may also share the contexts that they will use in their teaching (KCT) and explain how they may make reference to the curriculum in their teaching. Hence, when compared to the current methods used to assess MKT, the use of sorting and sequencing tasks with semi-structured interviews can allow for more dynamic and open responses from participants, which can potentially elicit a more comprehensive perspective of a teacher's knowledge in the different domains of MKT. In the next section, we will describe how we designed a card sorting and sequencing instrument to capture teachers' MKT for the topic of division.

Design of our Card Sorting Instrument on Division

Division at the primary level is a difficult topic for teachers to teach and for students to learn (Graeber, 1993; Holland, 1942; Pope, 2012; Sellers, 2010). For teachers, key ideas of division such as partitive and quotative concepts of division (Graeber, 1993; Martin, 2009; Pope, 2012), and its relation to other arithmetic operations (Holland, 1942) may be challenging for some to grasp. For students, they may lack the prerequisite knowledge of division such as multiplication facts, and other operations and face difficulties when learning division because of their emerging multiplicative thinking which is developed on the basis of partitioning (Holland, 1942; Pope, 2012). Furthermore, the foundational ideas of division form the basis for students in later grade levels to make sense of operations involving fractions, which often pose challenges for many teachers and students (Lamon, 2012). Hence, it is imperative for mathematics educators to gain insights into teachers' knowledge of the key ideas of division and the difficulties that students' face during learning (Holland, 1942; Hopkins et al., 2009) so that teachers can be better supported to design instruction for this foundational topic.

To capture teachers' knowledge for teaching division, we designed a set of sorting cards on division expressions as part of a semi-structured interview to elicit teachers' different domains of MKT in the topic of division. When designing the cards, we first considered the key division concepts such as quotient, divisor, remainder in relation to the long division algorithms. We referred to one of the coursebooks (Collars et al., 2015) used in Singapore to teach mathematics at the primary levels when designing the cards. According to the sequence of the content in the coursebook, the concept of quotient and remainder is first introduced, followed by division without renaming and division with renaming. 'Division without renaming' consists of dividends in which the digit in every place value can be easily divided by the divisor. For example, in '42 ÷ 2', the 4 tens and 2 ones can both be divided by 2 exactly. On the other hand, 'Division with renaming' consists of digits in the dividends that need to be renamed. For example, in '42 ÷ 3', 4 tens is not divisible by 3, and there is a need to rename 42 as 3 tens and 12 ones as we work through the long division algorithm. Based on the key ideas of renaming and remainder, four categories are created—with remainder and no renaming, with remainder and renaming, no remainder and renaming and no remainder and no renaming—to form the basis of our sorting cards.

Next, we decided on the choice of numbers in the expressions by applying the theory of discernment and variation (Kullberg, 2017). For example, we used the four expressions '43 ÷ 2', '42 ÷ 2', '43 ÷ 3' and '42 ÷ 3' to provide opportunities for the participants to discern the idea of remainder ('43 ÷ 2' and '42 ÷ 2') and the concept of remainder ('42 ÷ 2' and '42 ÷ 3'). In addition, smaller numbers are chosen so that participants can focus on the concepts of division, instead on the calculation of the answers. '205' and '204' were included as students often have difficulties working out division algorithm with zero in the place holders (Holland, 1942) and students have difficulties with the placement of digit when attempting long division (Holland, 1942; Martin, 2009). These considerations in the designing of the sorting cards expressions aim to provide a basis for participants to elaborate on their knowledge in the teaching of division during the interviews. For

the semi-structured interview, we developed a protocol, as shown in Table 2, and provided guiding questions when participants needed more prompts to elicit their ideas. An example of guiding question asked is: What is the difference between '42 ÷ 3' and '42 ÷ 2'? We used this question to prompt our participants to explain the difference based on their understanding of the concept of remainder, renaming or quotative/partitive concept.

Table 1

Division Expressions Used in Sorting Cards

	With Remainder	No Remainder
Renaming	7 ÷ 3 21 ÷ 2 43 ÷ 2 54 ÷ 4 205 ÷ 2 483 ÷ 2	20 ÷ 2 8 tens ÷ 4 9 hundreds ÷ 3 6 ÷ 3 42 ÷ 2 204 ÷ 2 482 ÷ 2
No Renaming	17 ÷ 4 21 ÷ 4 43 ÷ 3 54 ÷ 5 205 ÷ 4 482 ÷ 3	16 ÷ 4 20 ÷ 4 42 ÷ 3 54 ÷ 3 204 ÷ 4 483 ÷ 3

Table 2

Key Questions Asked During Semi-structured Interviews

Sorting Task	How would you categorise the 25 division expressions? Why do you categorise them this way? Are you able to sort them further?
Sequencing Task	How would you sequence the categories in your teaching? Why did you sequence it this way?

When we worked with the teachers with this card sorting and sequencing task, we aimed to capture video and voice recordings of the teachers' sorting. To do so, we set up the interview venue as shown in Figure 1. Referring to the set-up, iPad 2 (recording device) was positioned overhead to capture the movement and the grouping of sorting cards and iPad 1 mainly act as a backup to capture a different view of the participants. During the tasks, we asked questions to understand participants' decisions behind their sorting and sequencing. During the interviews, we also took photos of the participants' key steps when they completed their sorting.

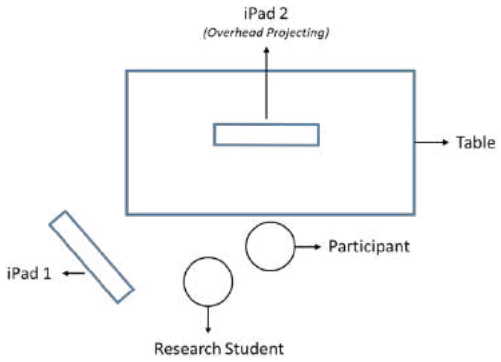

Figure 1. Set up of interview station.

Findings were developed from the transcripts of the video recordings of the card sorting and sequencing, voice recordings of the interviews, and the photographs of the different sorts and sequences as demonstrated by the participants. We analysed the card sort and sequence in chronological order and paid attention to the way the cards were sorted and sequenced. By making references to the interview for their sorting decisions, we matched the participants' understanding of the different domains of MKT using the descriptions developed by Ball et al. (2008). We also compared the ideas of division mentioned by the participants with the key ideas in extant literature related to the teaching of division (Graeber, 1993; Martin, 2009; Pope, 2012). To confirm or refute our assessment of the participant's understanding, we also referred to the coursebook used by the teacher and the syllabus document.

What did the Cards Tell Us About Eleanor's MKT?

In this section, we will illustrate the use of our instrument by describing and analysing the responses of Eleanor, one of our participants, to the sorting and sequencing task. Eleanor is a Primary teacher with four years of teaching experience, a typical representative of an experienced teacher in Singapore. For Eleanor, only the first author was involved in the data collection. Through her responses, we characterised some domains of her MKT, in particularly, her KCT, KCS, and SCK.

Eleanor's Sorting and Sequencing of Cards

We first present Eleanor's first sorting and further sorting as shown in Figure 2. When presented with the 25 cards, Eleanor first sorted the cards into nine groups according to divisors (2, 3, 4, and 5), followed by the number of digits in the dividends. Referring to Figure 2, we note that Groups 1 and 2 contain expressions involving 2 as a divisor; Group 3, 5 and 6 are expressions with 3 as a divisor; and Group 7 and 8 are expressions with 4 as a divisor. Furthermore, we see that Group 4 comprised expressions with dividends in words and Group 9 consists of an expression with divisor 5. When Eleanor was asked to further sort her expressions, she separated Group 7 into Groups 7a and 7b as shown in Figure 2, according to the number of digits in the quotients. Looking at the expressions in the two sub-groups, we note the 1-digit and 2-digit quotients in Group 7a and Group 7b respectively. For the sequencing task, Eleanor explained that she would start with Groups 3 and 7 as the expressions are linked to the standard multiplication facts covered in Primary Two. However, after the first author prompted her that some expressions in Groups 3 and 7 will result in remainders, she retracted and highlighted that she would begin with smaller number, and introduce expressions which answers have no remainder first, followed by expressions which answers consists of remainder without remainder. She also sequenced the cards according to the number of digits in the dividend in an increasing order.

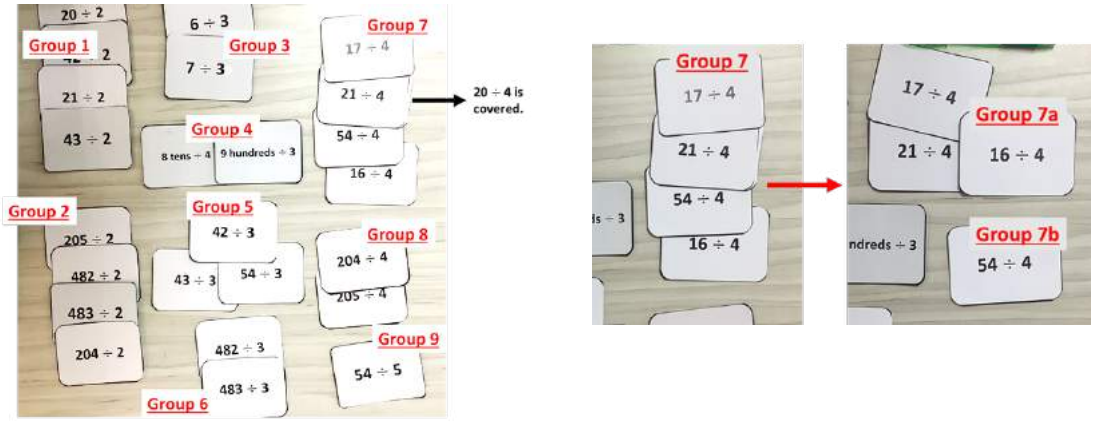

Figure 2. Eleanor's 1st sorting and further sorting.

Eleanor's KCT and KCS

We observe that Eleanor would think about the sequencing of the expressions according to the different structures of the division expressions (KCT). In particular, she was cognisant of the number of digits in the dividends, i.e., a 3-digit number divided by 1-digit number, and 2-digit number divided by 1-digit number. When asked to explain her sorting for Groups 1 and 2, she elaborated:

> So, I [pointed at Group 1] look at it, the first, the first one is 2-digit. This is 3-digit [Group 2]. Because this one will lead to perform the algorithm. You know the long division thing. So, it's another time of skills also. For example, this one. This [Group 1] is quite straight forward.

Here, we see that Eleanor noticed the number of digits in the dividends and quotients and how students might respond to the different division problems (KCS). For instance, she noted that students would need to use long division ("algorithm") to find the answer for 3-digit dividends. Similarly, Eleanor emphasised on the importance of using smaller numbers because it would be easier for the students, and it is also easier for teachers to represent the information using other representation (KCT):

> I think I will start with this [group 3] because it's easier. Lesser number for them…Like what you said, I will bring in the simple numbers so that they are able to see. And, also easier for us to show visuals also because it is [a] smaller number.

Besides focusing on the number of digits in the dividends or quotients, Eleanor also demonstrated her KCS by identifying some of her students' difficulties when learning division:

> I realise that they are just memorizing the steps. That's why I always try to change it in such a way that they understand them, so I always use terms like how many 3 gets into this? Rather than what is divided by what. Because divided by what, sometimes they are also weak in the times table.

In the explanation above, Eleanor's pointed out that her students may be weak in their "times table" (multiplication facts) and some may have memorised the algorithm without understanding. She went on to explain that students were confused with the steps in the division algorithm (KCS) and provided explanation on how she tried to shift students away from rote memorisation (KCT). Instead of reading out the steps of the algorithm during teaching, she would attempt to explain the steps in a way that is meaningful to the students (KCT).

Eleanor also added that another common difficulty that students experienced when learning division was the concept of remainder (KCS). She shared that she used stories and the concept of equal grouping to teach the remainder concepts to help students to overcome this difficulty (KCT):

> Because normally right when you tell them like a story, like you share, then later on, you all want to be, let's say you all got a box of chocolate right, you all share, I give you one, then in the end, got left one. Do you put in either of the group or not? Then they will tell you 'no', must be equal. Then I say, this is remainder.

Hence, we can see that Eleanor had demonstrated several aspects of KCS and KCT because she not only understood students' learning difficulties in the long division algorithm but also highlighted how to help students overcome them.

Eleanor's SCK

Eleanor's specialised content knowledge was also revealed during the card sort task. When asked about the difference between '42÷3' and '42÷2', she replied that "the number of groups are different". The first author continued to prompt and ask her how she would teach the division algorithm. For 42 ÷ 2, Eleanor explained that she would get them to think about equal grouping and asked the students ''like how many 2s can get into 4?', which is measurement model of division. Her response suggests that she was aware of the idea of division as equal grouping (measurement model). For '42 ÷ 3', she explained: "So, you have 3 groups, so they realise that only 1 group. That means 10 in each group". This was puzzling to us because "10 in each group" suggests an equal sharing concept of division. She then continued to explain that she would ask the pupils "how many 3s can get into 40?", which signify an equal grouping idea of division. Although she had wanted to use equal groupings to explain both questions, her use of two different models of division suggest that Eleanor might be confused about these two models of division. Again, the card sort task provided opportunities for us to understand a teacher's SCK and made it possible to identify the point of confusion that a teacher might have.

Concluding Remarks

In this paper, we introduced the card sorting methodology as an alternative novel method to capture teachers' MKT and illustrated its design and use with a teacher, Eleanor, in the context of a division topic. As opposed to existing methods for measuring MKT, our card sort instrument together with the interview can potentially capture a wider and more dynamic view of a teacher's MKT. As seen in the case of Eleanor, we were able to capture her thought processes on how she sort and sequence her cards, and hence characterised some of the domains of her MKT. We believe that our card sort instrument on division can be refined by reviewing the division expressions selected, improving the interview procedures and analytical approaches, and possibly including other tasks to capture a more comprehensive view of a teachers' MKT. At this stage, how the method can provide a more usable snapshot of a teacher's MKT for a given topic remains unclear. How the instrument can be further refined and used will be an interesting area for research.

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