A Further Investigation to Introducing the Equal Sign in China

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Fostering students' bidirectional conception of the equal sign (viewing the equal sign as indicating an equivalence of two sides rather than a 'show result' symbol) is challenging, and students' misconception of the equal sign is persistent. Some studies mention that in China, the pedagogical approach to introducing the equal sign supports students' development of bidirectional sense toward the equal sign. Built up on this body of literature, this paper further investigates the Chinese pedagogy from the researcher, students and the teacher's perspectives.

Forty years ago, Kieran (1981) foregrounded a prevalent students' misconception of the equal sign, as many students possessed a narrow conception that is treating the equal sign as a 'show result' symbol instead of a relational understanding of the equal sign (the equal sign indicates the equivalence of both sides). Students without relational understanding of the equal sign have difficulties in understanding the number sentences such as '3 = 3', and '3 + 2 = 4 + 1' (Kieran, 2004). Nowadays, this students' misconception of the equal sign is still widely documented (Blanton et al., 2018; Ralston & Li, 2022).

Literature has well argued that the relational understanding of the equal sign is a fundamental concept in students' transition from arithmetic to algebra (e.g. Carpenter et al., 2003). Without it, students will encounter difficulties learning algebra, such as equation solving. For instance, Blanton et al. (2018) showed that students with a narrow conception of the equal sign were not able to make sense of solving equations with unknowns on both sides (e.g., $3x + 2 = 2x + 1$), since they consider the equal sign should always be followed by a result carried out from the operation. There has been a great deal of research exploring pedagogical approaches to foster students' relational understanding of the equal sign (see below). In previous work (Sun, 2019), the author has illustrated how the equal sign has been introduced to Chinese students. Built on Sun (2019), this paper conducted a more thorough and detailed investigation of the Chinese pedagogical approach to the concept of the equal sign.

Literature Review

In a seminal research, Carpenter et al. (2003) showed a number sentences comparison activity that could support students' development of conception on the equal sign. By evaluating a number of pairs number sentences as true or false (e.g., 3 + 5 = 8, 8 = 3 + 5, 8 = 8, 3 + 5 = 5 + 3), Carpenter et al. (2003) demonstrated that students' conception that considering the equal sign as showing results could be suspended, as this task could draw students' attention to the number sentences on both sides of the equal sign. Therefore, it can provide students with an opportunity to have a bidirectional view on the equal sign. This pedagogical approach is evidenced to be effective by other algebra researchers (e.g. Knuth et al., 2016).

Alternatively, others researchers such as McNeil et al. (2015), proposed a pedagogy that is modifying conventional arithmetic operation practice. Traditionally, arithmetic operation questions are generally presented in a form such as '1 + 2 = __' , and this traditional form of practice might lead to students think the right side of the equal sign always displayed an answer carried out from the calculation on the left side, thus activating or reinforcing students' 'show result' conception of the equal sign (McNeil, 2008). In this sense, McNeil et al. (2015) recommended in everyday mathematics classroom, students should be exposed more to non-conventional forms of arithmetic operation practice, such as '__ = 1 + 2' and '3 = 1 + __'. By doing this, students' conception of the equal sign will not be impeded by a one-directional calculation pattern, rather, students' bidirectional

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view of the equal sign is possible to emerge. McNeil et al. (2015) showed this pedagogy took effect to improve students' relational understanding of the equal sign.

This study considers that while the starting points are different, both approaches mentioned above are essentially similar. Both pedagogies are to suspend students' one-directional conception of the equal sign by exposing them to a wide range of non-conventional forms of equation, so as to provide students with an opportunity to attend to the equivalence of both sides of the equal sign. In this sense, this study considers that diverting students from focusing on traditional forms of arithmetic operation could be a foundational way to develop relational understanding of the equal sign at an early stage.

While students' narrow conception of the equal sign is widely spread, this difficulty is not universal. The literature has shown the primary students in China commonly possess a robust relational understanding of the equal sign (Li et al., 2008). Li et al. (2008) concluded a few factors that contribute to Chinese students' success in understanding the equal sign, such as introducing the concept of the equal sign before students' exposure to arithmetic operation. Students understand the concept the equivalence by describing the quantities relationship (e.g. 'more than', 'the same', 'less than') in a concrete context before they are formally introduced to the equal sign. Following Li et al. (2008), Sun (2019) further explored the Chinese approach by analysing an official lesson plan used in the classroom through a theoretical lens of early algebra education. Echoing Li et al. (2008), Sun (2019) discovered that students in China learn the concept of the equal sign at the Kindergarten level, which is prior to that students learn arithmetic operation, so the interference of the 'show results' conception possibly brought by traditional forms of arithmetic operations can be reduced. Sun (2019) also identified other features of the Chinese approach to introduce the equal sign. First, the teaching sequence starts from students' own experience and informal solutions and continues to formal mathematical symbols. For instance, students learn the concept of the equality by comparing the quantities of two piles of concrete objects, and then students use own symbols to represent the equivalence before being exposed to the formal equal sign. Furthermore, the pedagogy emphasises both "left side is the same as the right side" and "right side is the same as the left side", which stresses a bidirectional conception of the equal sign. Finally, the lesson plan highlights the way of drawing an equal sign (i.e. two short horizontal lines with the same length). However, Sun (2019) is not an empirical study, so the feature exhibited in the lesson plan needs to be further investigated with empirical evidence. This study will examine how the pedagogy based on the lesson plan supports Chinese students' understanding of the equal sign.

Methodology

Methodological Approach

This research aims to explore how students learn the concept of the equal sign. Therefore, this study employs a qualitative case study, which has the affordance to provide fine-grained details of students' learning (Hamilton & Corbett-Whittier, 2012). With these details, research can have an in-depth understanding of what happened in the students' thinking in learning (Hamilton & Corbett-Whittier, 2012). Furthermore, according to Clarke (1998), to better understand students' learning, data sources that reflect different perspectives should be triangulated against each other. In this sense, three data sources are used for data collection and analysis in this study. The first one is the student interview, that reflects their own thinking and reasoning. The second data source is the teacher interview that reveals their opinions on students' learning and this pedagogy after the implementation. The third data source is students' work samples (the short quiz after the lesson), whereby this research could interpret their learning outcome.

Research Procedure

One kindergarten class with twenty-nine pre-school students (about 5.5 - 6 years old) and one teacher participated in this study. The kindergarten is located in Wuxi, JiangSu province. A lesson to introduce the equal sign as per an above-mentioned official lesson plan in Sun (2019) (see Figure 1) had been conducted to students (this study focuses on the equivalence part, so the inequality part will not be investigated). After the lesson, students' work samples were collected. Ten students and one teacher were interviewed. Interviewee students are selected as per ethical approvals (ten students and their carers consented the interview).

Activity 7: Are they equivalent? (Mathematics, Symbols)

Objective:

1. Learn to use '=' or '≠' to indicate the quantity relationship between two sets of numbers.
2. Be able to think about the problems proactively, be able to use the appropriate language to present the results of the activity.

Preparation:

1. 3-4 cards with concrete objects, one '=' card and one '≠' card
2. Children's graphic book

Activity:

1. Recognising the equality, understanding how to represent the equivalent relationship between two sets of numbers.
   a) Teacher shows students two cards with the same number of fruits, for instance, six apples and six pears. Let students count the numbers of apples and pears, and let them decide whether they have the same number of fruit.
   b) Teacher asks students what symbols they can put between two cards so other people can clearly see that they have the same number. After that, you can introduce the equal sign by emphasising two short horizontal lines must be at the same length.
2. Recognising the inequality, understand the inequality and how to represent it.
   a) Teacher shows students cards with different numbers of fruits, asks students whether two cards have the same number now. If two sides have different numbers, asks students what symbols they can put to indicate the two sides have different numbers. After that, showing students the sign '≠'.
   b) Teacher shows students several pairs of cards with a mixture of equivalent and non-equivalent relationships, and then asks students what sign they should put in-between and why.
3. Practicing " the left side and the right side is equivalent" in the children's graphic book.

Figure 1. The lesson plan for introduction of the equal sign.

Student interview started from asking students to describe the meaning of the equal sign. Then they were encouraged to explain their thinking on the work samples. The prompt question is "can you explain to me why you did xxx?" Finally, they were queried about the most impressive part of the lesson. The prompt question is "Which part(s) of this lesson help you learn and why?" Some follow up questions might be asked to further probe students' thinking. In teacher interview, they were asked about their understanding and/or opinion on students' learning and how the lesson supports students' conception of the equal sign. It is worth mentioning that the literature has shown the kindergartners are capable of verbally expressing their mathematical thinking in interview (Lenz, 2022).

Results and Discussion

Due to space limit, two students (Yang and Ming) and one teacher's (MS Q) data is reported here. Two students were chosen because their work samples and interview responses are typical as well as reflecting the diversity of data. The data will be presented in the following way. A snapshot of all twenty-nine students' work samples is shown first, to provide an overview of students' learning outcome, followed by an analysis of two students and the teacher interview.

Students Work Samples

The Figure 2 shows twenty-nine students work samples and Figure 3 showed Yang and Ming's work samples.

Figure 2. All students' work samples.

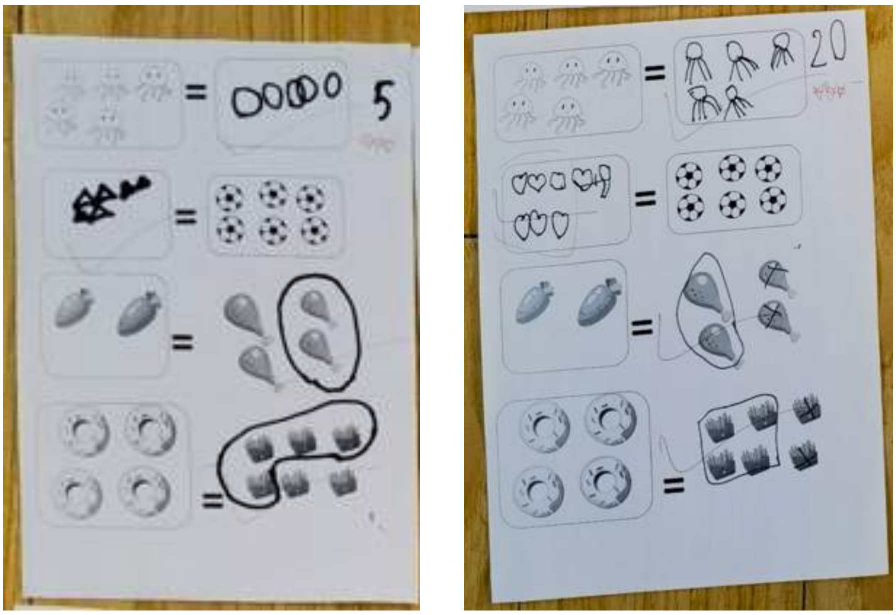

Figure 3. Yang (left) and Ming's (right) work samples.

The short quiz comprises four tasks. The first two are to match and draw the object, and the last two are to make two sides equivalent by circling the same amount of objects from the side with redundant items. It could be noted the missing parts of the first two tasks are shown at either right or left side of the equal sign. This tests students' bidirectional sense about the equal sign, referring to whether they understand the equal sign indicates the equivalence of both sides rather than from left to right. Almost all students completed four tasks correctly, except for a couple of students who filled the incorrect number of objects initially and made corrections afterward (e.g. Yang is one of these students, and his thinking will be elaborated below). This result tends to suggest that after the lesson most of students were able to develop a bidirectional view towards the equal sign, concurring with Sun (2019) in which a teacher stated 90% of his/her students could do a similar short quiz correctly. It is also worth mentioning that almost all students draw the missing objects which are not the same as the objects to be matched. This data could be interpreted as students focused on the quantity of objects on both sides of the equal sign instead of the objects per se. Therefore, it could be inferred that for students, four tasks are an informal way to represent a) 'five equals what number', b) 'what number equals six', c) 'two equals four takes away how many' and d), 'four equals six takes away how many'. These resemble the formal arithmetic operations a) 5 = __, b) __= 6, c) 2 = 4 - __ and d) 4 = 6 - __. In this sense, while students had not been exposed to formal arithmetic operations, their responses could be considered as being in line with the 'flexible operational level' of knowledge of the equal sign (students can accept the equation forms such as 'a = a' and 'a = b + c') suggested by Matthews et al. (2012). The 'flexible operational level' is a crucial step towards full relational understanding of the equal sign. Taken together, students' work samples evidence that students' conception of viewing the equal sign as an indication of the equivalence of the quantity on both sides has emerged.

Students' Interview

When Yang and Ming were asked to describe the meaning of the equal sign, Yang stated, "we use this to tell other people two sides as the same". Similarly, Ming said, "it shows left side and right side have the same number of things." These words "two sides" and "the same" appear to show that both students understand the equal sign as indicating the bidirectional equivalence. Ming explicitly pointed out the equivalence of the quantity. While Yang did not clearly refer to the quantity, his work sample shown above suggested he focused on the quantity, as discussed early.

Then two students' work samples are presented to them, and they were prompted to further explain their work. Yang's response is shown below,

> Interviewer: For the question 1 and 2, do you think any difference when the missing part appears on the right side of the equal sign and when it is at left side?
>
> Yang: No, we are just filling the same number of things.
>
> Interviewer: How do you know you need to fill the same number?
>
> Yang: Because of this sign [point to the equal sign]
>
> Interviewer: Ok, you have explained this sign to me just now, let see this, these are octopuses and footballs, why you just draw the circles and triangles for the missing part?
>
> Yang: It is easy to draw [laugh].
>
> Interviewer: It's ok two sides are not the same thing?
>
> Yang: Yes, I think is ok because the number are the same.
>
> Interviewer: I see. Let see question 3 and 4, you just circle the same number of things and keep the redundant things out. Do you have the other way to make two sides as the same, if you don't use circle.
>
> Yang: I can cross the more things.

Interviewer: Ok, that's right. What about if we do something from the left side, to make left side equal the right side?

Yang: um…oh… I can add more on left side.

The excerpt above confirmed Yang's bidirectional sense of the equal sign as he expressed left side and right side did not make difference. Yang also revealed that he did not have to match the shape of objects since only quantity mattered. This result corroborated this research's interpretation to students' work samples above, which is that students focused on quantities instead of objects per se. This research probed Yang's thinking further by prompting him to seek an alternative approach to make two sides equivalence for task 3 and 4. The data suggested that Yang did not understand the question asked, but following the prompt provided, he was able to find out that he can add more objects on left side instead of taking away things on right side. This tends to indicate that Yang accepts both forms 'a + b = c' and 'a = c - b' as legitimate, which showing Yang has achieved the flexible operational view towards the equal sign suggested by Matthews et al. (2012). It is worth mentioning that in the interview, Yang was encouraged to use numbers and symbols to represent task 1, and he was able to write '5 = 5', providing further evidence to his flexible operational thinking to the equal sign. Since students had not learnt the arithmetic operation yet, they were not asked to mathematically represent task 3 and 4 which addition and subtraction are involved. Therefore, this study does not seek the evidence to whether students achieve 'comparative relational level' understanding of the equal sign suggested by Matthews et al. (2012). Comparative relational level understanding of the equal sign refers to that students can use relational strategy to evaluate number sentences such as '67 + 86 = 68 + 85' without full calculation (Matthews et al., 2012).

In Ming's interview, he was first asked why he filled the wrong quantity of objects initially and corrected it. Ming responded,

I made a mistake on counting these hearts (the object drawn by Ming), and then the teacher showed me the number was not correct, so I counted again and change it.

Ming's words appear to suggest that his mistake is a result of miscounting rather than misunderstanding of the equal sign (his teacher, Ms Q, confirmed this interpretation, as will be seen later). Then Ming was also asked for task 1 and 2, whether there are any differences if the missing part is shown on left side or right side. Ming had a similar response as Yang, stating there were no difference and he explained,

As long as two sides has the same quantity, the equal sign works, it does not matter right side or left side.

Similar to Yang, Ming's explanation indicated that he had a clear bidirectional conception toward the equal sign. Next, the interviewer asked Ming why he drew the same octopuses for task 1 and draw the different things for task 2. The excerpt is shown below,

Ming: Because the octopuses are easy to draw but the footballs (objects in task 2) are more difficult to draw, so it is quicker to draw hearts.

Interviewer: Ok, so do you worry about one side is football and another side is heart they are different things.

Ming: No. I can draw the different things, as long as they have the same many of things.

Ming's words showed that, like Yang, he recognised that the equal sign is about showing the equivalence of the amount rather than the objects per se. Finally, Ming was also encouraged to find the alternative approach to make two sides in task 3 and task 4 equivalent. Ming provided a similar response as Yang: suggesting it can be done by drawing more objects at left sides. Finally, Ming's thinking was also further probed by requiring him to write a number sentence to represent the task 1, the excerpt is shown below,

Interviewer: Could you use mathematics to describe task 1, please?

Ming: Use mathematics to show? What do you mean?

> Interviewer: Like using numbers to represent the graph.
>
> Ming: Like five is the same as five?
>
> Interviewer: Can you write it down?
>
> (Ming wrote '5 is the same as 5')
>
> Interviewer: Can you use a symbol to replace the words here?
>
> Ming: oh, I can use the equal sign (then Ming wrote '5 = 5')

Unlike Yang who wrote '5 = 5' straightway, Ming needed more prompt to reach the answer. It might attribute to that he did not quite understand what he was required to do. Nevertheless, Ming was able to eventually write '5 = 5'. In this sense, together with the above mentioned alternative approach to task 3 and 4 provided by Ming, it could be argued, like Yang, Ming's understanding of the equal sign reached the flexible operational view suggested by Matthews et al. (2012).

Two students were asked which part of the activity helped them learn most. Yang revealed that the most impressive part was that Ms Q let them draw their own symbol to indicate the same quantity of two sides, and he explained that Ms Q let them do this first then introduce the formal the equal sign, so he had a strong impression that this is a symbol to show the equivalence of both sides. Ming provided a different response by suggesting that he thought the teacher's emphasis on that two short horizontal lines must be the same length when drawing the equal sign helped his understanding. Ming explained that the way of drawing the equal sign pressed a bidirectional sense of the equal sign to him. Ming and Yang's response made the connection between their conception of the equal sign and the learning activity, illustrating the important parts of the activity that help their learning. It is worth noting the way of drawing the equal sign and students' own construction of symbol are two well-mentioned parts of the activity in student interview. This finding tends to suggest these two parts took effect in supporting students' bidirectional conception of the equal sign. As will be seen next, in the teacher interview, the way of drawing equal sign is also mentioned.

Teacher Interview

In the interview with Ms Q, she first confirmed Yang and Ming's learning progress, as she stated "I believe both Yang and Ming understand the equal sign is to show two sides are equivalent quantitatively", and she followed, "Not only Yang and Ming, I think all other students also have the same understanding of the equal sign." These words echoed this research's interpretation to students' work samples and interviews, that all students were developing bidirectional conception to equal sign. Ms Q also mentioned Ming's mistake made in task 1, as she said, " some students like Ming had a mistake on counting, but they understand the concept of the equal sign." Ms Q also revealed her opinion about this pedagogical approach, as shown below,

> I think it is good to start with comparing the quantity of the real objects on cards such as apples and pears, it can give students an impression that on this lesson we focus on comparing the quantities of two sides. Then we let students put their own symbols to show this equivalence, I believe this step is important since it stars making connection between students' words such as 'the same number' and an abstract symbol. While not every student can draw '=' sign at this stage, they understand this symbol is to indicate the sameness. Lastly, I think emphasising the way to draw the equal sign, like two short horizontal lines with the same length, is most impressive to me, because it is making the equal sign that is an abstract mathematical symbol more sense to students, as they would think like "oh, these two short lines are the same, so that's why this one is an equal sign".

The except above tends to show that Ms Q highlighted the step in which students construct their own sign as she considers it progressively step students towards the abstract symbol. This finding echoes Sun (2019): in the teaching reflective journal, a teacher wrote that this step pressed to students a conception that the sign they were drawing represented the equivalence of amount on both sides. Furthermore, it appears that Ms Q values the way of drawing the equal sign the most. She considers this step is crucial for the students' sense making of the representation of formal symbol of the equal sign since students can visualise the meaning. This result also concurs with Sun (2019), in which the way of drawing the equal sign is reportedly to make students consider "it was very appropriate that the equivalent relationship was represented by two short horizontal lines with equal length" (p.55). It is worth mentioning that the effect of the way of drawing the equal sign is highlighted by both students and Ms Q, and this step appears has not been underscored by previous literature. Therefore, questions could be asked: what the value of the way of drawing the equal sign brings to students' learning and how this step supports students' understanding in detail. This might be worthy of further investigation in future research.

Conclusion

By gathering the data from different perspectives (students, the teacher and the researcher), this paper showcases how Chinese kindergarten students develop the bidirectional conception of the equal sign. The data suggests the pedagogical approach mentioned in Sun (2019) helps students grapple the bidirectional sense towards the equal sign and achieve the 'flexible operational level' of the equal sign. While students need to continue developing 'comparative relational level' understanding of the equal sign when they start doing arithmetic, this research argues the pedagogy introduced here provides a solid foundation for students' comprehensive relational understanding of the equal sign.

Furthermore, this study discovers that the emphasis on the way of drawing (two horizontal lines with the same length) of the equal sign is widely mentioned by students and the teacher as supporting students' understanding of the equal sign. As mentioned above, this step seemingly has not been noticed by previous literature, therefore how it contributes to students' understanding and so might warrant further investigation.

Finally, this study recognises that this is a small-scale study, so the effectiveness of pedagogical approach proposed in this research could be further examined in possible large-scale quantitative research.

References

Blanton, M., Otálora, Y., Brizuela, B. M., Gardiner, A. M., Sawrey, K. B., Gibbins, A., & Kim, Y. (2018). Exploring kindergarten students' early understandings of the equal sign. *Mathematical Thinking and Learning*, *20*(3), 167–201.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Heinemann.

Clarke, D. (1998). Studying the classroom negotiation of meaning: Complementary accounts methodology. In A. Teppo (Ed.), *Qualitative research methods in mathematics education*. Monograph number 9 of the *Journal for Research in Mathematics Education* (pp. 98-111+164-177). Reston, VA: National Council of Teachers of Mathematics. https://doi.org/10.2307/749949

Hamilton, L., & Corbett-Whittier, C. (2012). *Using case study in education research*. Sage.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*(3), 317–326.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it. *The Mathematics Educator*, *8*(1), 139–151.

Knuth, E., Stephens, A., Blanton, M., & Gardiner, A. (2016). Build an early foundation for algebra success. *Phi Delta Kappan*, *97*(6), 65–68.

Lenz, D. (2022). The role of variables in relational thinking: an interview study with kindergarten and primary school children. *ZDM–Mathematics Education*, *54*(6), 1181–1197.

Li, X., Ding, M., Capraro, M. M., & Capraro, R. M. (2008). Sources of differences in children's understandings of mathematical equality: Comparative analysis of teacher guides and student texts in China and the United States. *Cognition and Instruction*, *26*(2), 195–217.

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education*, *43*(3), 316–350.

McNeil, N. M. (2008). Limitations to teaching children 2+ 2= 4: Typical arithmetic problems can hinder learning of mathematical equivalence. *Child Development*, *79*(5), 1524–1537.

McNeil, N. M., Fyfe, E. R., & Dunwiddie, A. E. (2015). Arithmetic practice can be modified to promote understanding of mathematical equivalence. *Journal of Educational Psychology*, *107*(2), 423.

Ralston, N., & Li, M. (2022). Student conceptions of the equal sign: Knowledge trajectories across the elementary grades. *The Elementary School Journal*, *122*(3), 411–432.

Sun, J. (2019). Beyond the arithmetic operation: How the equal sign is introduced in Chinese classrooms. *Mathematical Association of Victoria annual conference proceedings*. Melbourne: Australia.