Learning to Notice Algebraically: The Impact of Designed Instructional Material on Student Thinking

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In this paper, we explore how students' algebraic noticing's and explanations changed across a two-year period with the introduction of designed instructional material. The data in this report is drawn from n=53 Year 7-8 students' responses to a free-response assessment task across two different years. Analysis focused on how students noticed and explained algebraic relationships in pairs of equivalent equations. Findings indicate that with the introduction of designed instructional material, there was a shift in student noticing of number properties to identify equivalence between pairs of equations. However, identifying the distributive property of multiplication and developing generalisations about the algebraic relationships remained challenging for students.

Both in New Zealand and internationally, there has been increased attention to algebra and relationships as key learning areas of mathematics in research studies and policy documents (MoE, 2007; Schifter, 2017). In the New Zealand context, middle school students (aged 10-13) are expected to generalise the properties of multiplication and division with whole numbers. Despite this expectation, teachers often focus considerable attention and time on teaching their students how to calculate (Schifter, 2017), and rarely give opportunities for learning that focuses on algebraic structures (Arcavi et al., 2017). This approach results in students developing an over-reliance or compulsion to calculate (Hunter et al., 2022, Arcavi et al., 2017) unless an algebraic intervention (Blanton et al., 2015) or an "algebrafying" of the classroom occurs (Blanton & Kaput, 2003). Supporting teachers to implement sound research-based instructional approaches that focus on algebraic structures is an important aspect of positioning teachers to move beyond teaching calculation and focus on the algebraic nature of number. Previous studies have focused on design experiments or professional development in relation to early algebra and teacher change (Blanton, et al., 2015; Blanton & Kaput, 2003). However, in this study we address a gap in the field by focusing on the introduction of designed instructional material for teachers to use in the classroom. Specifically, we address the following research question:

- How do student responses to an assessment item involving noticing, and explaining algebraic structures change after provision of designed instructional materials to teachers?

Literature Review

Understanding number properties, relationships and mathematical structure are vital elements of developing sound number sense, and the importance of this has been well documented across the last decades (e.g., Mason et al., 2009; Kaput, 2017; Carpenter et al, 2003). In the last ten years the field of early algebra has gained significant movement particularly in relation to a focus on algebraic thinking with young students, as opposed to the 'arithmetic-then-algebra' approach that is deeply institutionalized within educational structures (Kaput, 2017, p.5). An early algebraic thinking approach builds on students' natural ideas of patterning and relationships (Blanton et al., 2015) emphasising the complex kinds of mathematics that young students can achieve when provided with opportunity.

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 517–524). Newcastle: MERGA.

Two of the big ideas underpinning early algebraic understanding are proposed by Blanton et al. (2015, p.43) as being 'equivalence, expressions, equations, and inequalities' and 'generalized arithmetic'. These ideas involve developing an understanding of the equals sign and equivalence, the properties of number (commutative, associative, distributive, identity, inverses), and the ability to reason with the structure of expressions and equations rather than calculating an 'answer'. If we consider the missing number equation 7 + 3 = __ + 4, a student relying on calculation may first compute 7 + 3 = 10 and then reason that 6 + 4 is also 10 (Blanton et al., 2015, p.51). In contrast, a student drawing on an equivalence or compensation approach, or who notices a relationship between both sides of the equal's sign; may be drawing on a form of relational or structural thinking (Carpenter et al., 2003; Mason et al., 2009).

Mason et al., (2009, p.12) states that "attention to structure runs through the whole of mathematics, and that shifts of attention make a difference to how mathematics is seen". Developing structural awareness allows students to move from arithmetic (calculation) towards algebraic thinking (hence a "shift of attention"). Students who notice and understand mathematical structure are typically comfortable applying number properties to different situations, can form generalisations and are reported by teachers as being more engaged within the classroom (Carraher et al., 2008; Gronow et al., 2022: Mason et al., 2009). In contrast, students who do not attend to structure may view the equals sign as a command to carry out a calculation (Carpenter et al., 2023), and as a result are likely to find it significantly more difficult to reason with algebraic concepts in the future. Gronow et al., (2022) found that some teachers may even believe that "low ability" students do not have the capability to notice mathematical structure, however, it is well reported by researchers that students can do this from a young age (Blanton & Kaput, 2003; Carraher et al., 2008).

Instructional Materials

Instructional materials lie within the curriculum enactment process between the official and operation curriculum (Remillard & Heck, 2014). They refer to resources designed to support teachers with lesson instruction, and 'play a critical role in national education systems' (p.707). Within New Zealand, the MOE (2021) affirms that successful resources help teachers to understand what research is saying about effective teaching and how to put it into practice. However, for teachers to use instructional materials effectively, they must devote significant time and attention to develop a deep understanding of the mathematical concepts involved. Teachers must also hold the concept of 'explicitness' in the forefront of their minds whilst using the instructional materials, as this will ensure mathematical ideas are made clear to students (Leong et al., 2019). This concept of 'explicitness' is described by Selling (2016) as raising the collective awareness of the existence of mathematical concepts and practices and knowing why they are important in understanding mathematics. Mason et al., (2009) reports that it is not enough for teachers themselves to be aware of algebraic structure. They need to expect their students to justify and explain their actions using number properties that have been made explicit in the classroom. When a classroom has been 'algebrafied' and number properties and relational structure are made explicit to students through task design, mathematical practices, and substantive classroom conversations, then student outcomes show improvement (Blanton & Kaput, 2003). Whilst instructional materials can have considerable promise in supporting algebraic understanding, when used as a prescriptive tool by the teacher a surface level understanding may result for both teacher and students.

Methodology

The data and participants of this study are drawn from a larger ongoing research project focused on schools involved in a professional learning and development research initiative entitled Developing Mathematical Inquiry Communities (DMIC). In this paper, we draw on data from a qualitative case study involving middle school students and their responses to an assessment item administered in two different school years after a taught unit. Analysis of student responses was used to identify common themes with a specific focus on identifying what changes occur in students' algebraic noticing across a two-year period.

Participants and Setting

The participants were Year 7-8 students (aged 10-13) attending a low socio-economic middle school within New Zealand. All students present on the assessment day n=170 (2021) and n=157 (2022) completed a free-response assessment task (see data collection section). As this paper focuses on change over time, responses from Year 8 students in 2021, and Year 7 students in 2022 were removed along with any students who had left the school or did not complete both assessments due to Covid interruptions. This resulted in a cohort of n=53 students who completed an algebraic assessment task during both years. The cohort included students from the Pacific Nations (35%), Māori (30%), and NZ European (24%). It is important to emphasize that in the first year of the study, teachers individually designed a series of algebraic tasks to form a unit on number and algebra for their students. In the second year, the teachers utilized a research-based instructional unit provided within the DMIC PLD that was compiled and developed by the third author (see Figure 1). This instructional unit consisted of 14 contextualized and problematic tasks that drew students' attention to algebraic structures and relationships. Accompanying each task was information regarding big mathematical ideas (Randall, 2005), links to the New Zealand Curriculum, expected learning outcomes, and general notes that alerted teachers to important aspects regarding the teaching and learning of algebra as reported within research literature. Teachers were facilitated within the PLD to become familiar with this information and focus attention on the five teacher practices of anticipating, monitoring, selecting, sequencing, and connecting (Smith & Stein, 2018).

Task 7	Can you work together in your group to solve these number sentences? Make sure that you develop an explanation and justification. 189 + 25 = __ + 26 85 – __ = 75 – 28 674 + 56 – __ = 671 24 x 16 = 48 x __ 105 ÷ 15 = (45 ÷ 15) + (__ ÷ 15)
Big Ideas	Equations show relationships of equality between parts on either side of the equal sign. The properties of equality are: If the same real number is added or subtracted to both sides of an equation, equality is maintained; If both sides of an equation are multiplied or divided by the same real number (not dividing by 0), equality is maintained; Two quantities equal to the same third quantity are equal to each other.
Curriculum Links	**NA4-1:** Use a range of multiplicative strategies when operating on whole numbers. **NA4-8:** Generalise properties of multiplication and division with whole numbers. **NA4-7:** Form and solve simple linear equations.
Learning Outcomes Students will be able to:	Explain and justify relationships between numbers in an equation. Write statements of equivalence in words and using notation. Solve equivalence problems and explain and justify the solutions.

Mathematical language	Equivalent, equal sign.
Sharing back/Connect	Select student solution strategies that use relational reasoning. **Connect:** Ask students to generate conjectures related to the equivalence problems that build on the properties of equality.
Teacher Notes	• Students may initially treat the equals sign as an operator or indication to write the answer next. • Students also may compute each side to work out whether they are equal. • Notice students who use the relationships across the equals sign to see whether there is balance. • Highlight to the students to look across the equals sign and find the relationships between numbers to the left and the numbers on the right. • Notice students who use the relationships across the equals sign to see whether there is balance. • Highlight the students relational responses (e.g., noticing the + 2 – 2 relationships). • Press for use of arrows and notations to highlight the relationships.

Figure 1. Example task from the instructional unit.

Data Collection

Students were asked to complete a written free-response assessment task (see Figure 2) at the completion of the algebraic unit each year. This free-response task consisted of 12 individual equations that had equivalence to another equation through the distributive and associative number properties, or exponents. Under these equations were three prompts which encouraged students to describe and explain the number patterns, and to show if they work with other numbers (generalisation). The assessment task was launched by the teacher to ensure students knew what they were expected to do. Students then worked on the task individually within class time and were encouraged to explain and represent their thinking. The completed assessment tasks were collected, scanned by the research team, and stored securely.

76 x 15 = 37 + 43 + 40 + 36 = 99 ÷ 3 ÷ 3 =

7 x 86 = 99 ÷ 9 =

6^3= (70 x 5) + (70 x 10) + (6 x 10) + (6 x 5) =

37 + 40 + 36 + 43 = 12 x 22 = 6 x 6 x 6 =

(7 x 90) - (7 x 4) = 4 x 66 =

Look at the number sentences above.

- Describe what patterns you can find.
- Why do your patterns work?
- Do they work with other numbers?
- Will they always work? Explain and justify your thinking

Figure 2. Free-response assessment task (Hunter et al., 2022).

Data Analysis

Initially, student responses to the task were coded as either ‘not identifying’ or ‘identifying’ algebraic relationships. Responses coded as ‘not identifying’ were those in which students treated the task as a calculation exercise of individual equations. Responses coded as ‘identifying’ showed evidence of noticing or explaining one or more relationships between the six possible pairs of equations or items within the task. The samples coded as ‘identifying’ were then examined per item (equation pairs). Each item was assigned a code of 0-4 relating to the sophistication of the explanation given (see Table 1). Furthermore, items were coded as showing evidence of computational (C) or relational thinking (R) (Carpenter et al., 2003). For example, the student response “7 x 86 = (7 x 90) - (7 x 4). When you minus this (7 x 4) that will mean it will be 7 x 86. (90 - 4)” was coded as R3: explanation using relational thinking. The first and third author independently coded the samples. Any differences in coding were then discussed until a consensus was agreed on.

Table 1

Examples of how Students Explained 7 x 86 = (7 x 90) - (7 x 4) with Codes

Code	Example of Code
(N) Not Identifying	“7x86=602”
(0) No explanation	Drew an arrow between the two number sentences
(1) Calculation Only	(7 x 90) - (7 x 4) = 7 x 86 because 7 x 80 = 560, 7 x 6 = 42
(2) Low Level Explanation	“All I did to find the relationship between different equations is to find the answer using place value and see if the answers match up.”
(3) Explanation	“7 x 86 = (7 x 90) - (7 x 4). When you minus this (7 x 4) that will mean it will be 7 x 86. (90-4)”
(4) Partial Generalisation	“So you can do 90 x 7 - 28 to get 86 x 7. Examples: 7 x 4 = 28 so 7 x 3 = 7 x 4 - 7”

Results and Discussion

This section will focus on identifying and describing several changes that occurred in students' algebraic thinking. Results indicate that a significant shift of attention occurred within the way students view the mathematical equations between 2021 and 2022 with the introduction of the designed instructional material. As shown on Table Two, this involved a significant shift in students viewing individual equations as a command to calculate towards identifying similarities between pairs of equations by using structural or relational reasoning. In the second year of data collection, 90.6% of students were able to ‘identify’ relationships between pairs of equations, in contrast to 22.6% of students during the first year (an increase of 68%). Findings from the 2021 sample are consistent with previous research, in which many students used calculations (Hunter et al., 2022; Schifter, 2017). However, the 2022 data indicates that the introduction of the material supported a shift with students away from solely calculation towards noticing and considering relational structure (Mason et al., 2009).

Table 2

Percentage of Students Noticing Algebraic Relationships

	2021	2022	Change
Not identifying	77.4%	9.4%	-68%
Identifying	22.6%	90.6%	+86%

Table 3

Percentage of Students Identifying Algebraic Properties (all codes)

Item	2021	2022	Change
76 x 15 = *	5.7%	41.5%	+35.8%
37 + 43 + 40 + 36 = 37 + 40 + 36 + 43	15.1%	81.1%	+66%
99 ÷ 3 ÷ 3 = 99 ÷ 9	13.2%	60.4%	+47.2%
7 x 86 = (7 x 90) - (7 x 4)	3.8%	37.7%	+33.9%
6^3 = 6 x 6 x 6	22.6%	81.1%	+58.5%
12 x 22 = 4 x 66	7.5%	41.5%	+34%

*(70 x 10) + (70 x 5) + (6 x 10) + (6 x 5).

Also notable was a significant shift in the number of students identifying each pair of related equations (Table 3), with an increase from 33.9% of students in 2021 to 66% in 2022. The largest of these gains occurred in the number of students noticing the associative property of addition (+66%) and exponents (+58.5%), with 81.1% of the students being able to identify both these properties in 2022. Equation pairings involving the distributive and associative properties of multiplication also improved from 33.9% to 35.8%, however, this was a much smaller shift. Possible reasons for this could include that these number properties are more difficult for students to notice and explain (Hunter et al., 2022), or teachers themselves have a superficial understanding of these concepts and therefore do not make these explicit within their classrooms (Grownow et al., 2022; Mason et al., 2009).

Although results were generated for how students responded to each item, for the purpose of this paper, only 7 x 86 = (7 x 90) - (7 x 4) will be discussed in detail here (Table 4). This is because the item had the lowest percentage of change, so warrants closer inspection. In 2021, only two of the 53 students identified this relationship. This included one student who drew an arrow between the equations, and one student who gave a low-level explanation saying they both equal 602. In contrast, in 2022 there were 20 students who identified this relationship. Most commonly, students provided low-level explanations using calculation (n=6) or identified relational structure with no-response (n=7). These students drew arrows to show the equations were related or explained equality by relying on calculation means. This may indicate that whilst students' attention is shifting towards noticing relationships, explaining these relationships may take more time to develop. Or as described earlier, teachers may not be making these explanations explicit within their classrooms.

Table 4

Number of Students Per Code Identifying the Distributive Property of Multiplication

		No Response	Calculations Only	Low-Level	Explanation	Partial Generalisation
2021	C	0	0	1	0	0
	R	1	0	0	0	0
2022	C	2	2	6	0	1
	R	7	0	0	2	0

*C=students gave indication of calculating, R=students only drew on relational structure.

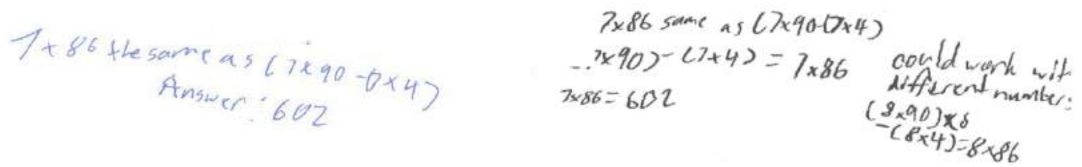
Examples of Specific Students

Three specific students' responses will be shown here to illustrate different ways in which change occurred between the years. This includes the two students who identified the item in 2021 (Table 5), and one who did not.

Student A responded in 2021 by drawing a line between 7 x 86 and (7 x 90) - (7 x 4) and gave no further explanation (see Figure 3). We cannot be sure about the reasons why the student connected these two equations, as they gave written explanations for other items. So, there is a chance these may have been connected by the process of elimination. In 2022, they provided supplementary evidence of calculations proving both are equal to 602.

Figure 3. Student A's responses.

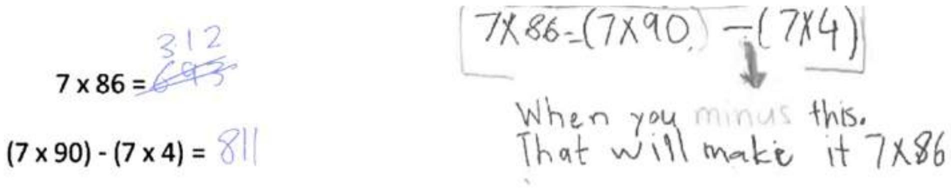
Student B's explanation in 2021 stated the equations were "the same as" and the "answer: 602", indicating they solved the equation to prove equality (see Figure 4). Interestingly, in 2022 they repeated the previous response, however, they also gave an example of generalising this property to another example "(8 x 90) - (8 x 4) = 8 x 86", becoming the only one of the n=53 students who gave a partial generalisation of the distributive property of multiplication. We can surmise that whilst this student is either still reliant or compelled to calculate to confirm equality, they do realise number properties can be generalised.



2021 2022

Figure 4. Student B's responses.

Student C is an example of one of the n=51 students who did 'not identify' this item in 2021 and attempted to answer each individual equation (Figure 5). In contrast, they were one of the two students in 2022 who were able to give a relational explanation in 2022. As seen in Figure 4, the student used their understanding of structure to show that both equations are equivalent to 7x86 without drawing on any calculative means.



2021 2022

Figure 5. Student C's responses.

In summary, the way in which each student noticed and reasoned with algebraic structure changed in different ways across the two-year period. This implies that the ways in which students develop an understanding of the distributive property is not constrained to a linear track of progression. They may in fact move back and forth between using calculation and relational means.

Conclusion

We aimed to identify how students noticing and explaining of algebraic structures changed after teachers were provided with designed instructional materials. In summary, the results from 2021 showed students had little awareness of algebraic structure, and focused on solving individual equations (calculating), rather than noticing relationships between equations. In contrast, a shift of attention was seen in 2022, with many more students noticing algebraic structure, especially regarding the associative property and exponents. This indicates that the focus on calculating can be shifted by teachers who take the time to expose students to algebraic thinking within their classrooms. We surmise that providing research-based instructional materials could be a useful tool and show considerable promise in educating teachers about mathematical structure. This in turn will help to 'algebrafy' classrooms, support student outcomes and support student outcomes.

Despite this large shift, many students still appeared to experience difficulty in identifying the distributive property. Additionally, it appeared that generalising number properties to other instances, an expectation of the New Zealand Curriculum for this age cohort, was challenging. Future research would be helpful to investigate whether teachers are making generalisations explicit within their classrooms and how they can be supported to do so. Moreover, beyond instructional materials, how can teachers be supported to develop sufficient mathematical content knowledge to teach generalisation and number properties in their classrooms. We aim to gather further data in 2023, to contribute to the understanding of how different students' algebraic reasoning develops over time.

References

Arcavi, A., Drijvers, P., & Stacey, K. (2017). *The learning and teaching of algebra: Ideas, insights, and activities.* Routledge.

Blanton, M., Stephens, A., Knuth, E., Gardiner, M., Isler, I., & Kim, S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education, 46*(1), 39–87.

Blanton, L., & Kaput, J. (2003). Developing elementary teachers' algebra eyes and ears. Teaching children mathematics, *10*(2), 70–77.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Heinemann.

Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM, 40*, 3–22.

Gronow, M., Mulligan, J., & Cavanagh, M. (2022). Teachers' understanding and use of mathematical structure. *Mathematics Education Research Journal, 34*, 215–240.

Hunter, J., Miller, J., Bowmar, A., & Jones, I. (2022). "It has the same numbers, just in a different order": Middle school students noticing algebraic structures within equivalent equations. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Research Group of Australasia* (pp. 138–145). Launceston: MERGA.

Kaput, J. J. (2017). 1 What is algebra? What is algebraic reasoning? In James J. Kaput, David W. Carraher, & Maria L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–18). Routledge.

Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (2019). Making things explicit using instructional materials: A case study of a Singapore teacher's practice. *Mathematics Education Research Journal, 31*, 47–66.

Selling, S. K. (2016). Making mathematical practices explicit in urban middle and high school mathematics classrooms. *Journal for Research in Mathematics Education*, *47*(5), 505–551.

Smith, M. S., & Stein, M. K. (2018). *5 practices for orchestrating productive mathematics discussion*. National Council of Teachers of Mathematics.

Ministry of Education (2007). *The New Zealand curriculum*. Wellington: Learning Media. https://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum

Randall, CC (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *NCSM Journal,* Spring-Summer.

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *Zdm*, *46*, 705–718.

Schifter, D. (2018). Early algebra as analysis of structure: A focus on operations. In Kieran, C. (Ed.), *Teaching and learning algebraic thinking with 5 to 12-year-olds. The global evolution of an emerging field of research and practice* (pp. 309-328). New York: Springer.